ALGEBRAIC WEAK FACTORISATION SYSTEMS II: CATEGORIES OF WEAK MAPS

JOHN BOURKE AND RICHARD GARNER

Abstract. We investigate the categories of *weak maps* associated to an algebraic weak factorisation system (AWFS) in the sense of Grandis–Tholen [14]. For any AWFS on a category with an initial object, cofibrant replacement forms a comonad, and the category of (left) weak maps associated to the AWFS is by definition the Kleisli category of this comonad. We exhibit categories of weak maps as a kind of "homotopy category", that freely adjoins a section for every "acyclic fibration" (=right map) of the AWFS; and using this characterisation, we give an alternate description of categories of weak maps in terms of spans with left leg an acyclic fibration. We moreover show that the 2-functor sending each AWFS on a suitable category to its cofibrant replacement comonad has a fully faithful right adjoint: so exhibiting the theory of comonads, and dually of monads, as incorporated into the theory of AWFS. We also describe various applications of the general theory: to the generalised sketches of Kinoshita–Power–Takeyama [22], to the two-dimensional monad theory of Blackwell–Kelly–Power [4], and to the theory of dg-categories [19].

1. Introduction

This paper continues the authors' ongoing investigations into the *algebraic weak factorisation systems* of [14, 10, 28, 1]. An algebraic weak factorisation system (henceforth AWFS) is a weak factorisation system in which each map f is equipped with a factorisation $f = Rf \cdot Lf$, in such a way that the assignations $f \mapsto Lf$ and $f \mapsto Rf$ become the actions on objects of an interacting comonad L and monad R on the arrow category. This extra structure allows algebraic weak factorisation systems to do things that mere weak factorisation systems cannot; for example, the first paper in this series [6] constructed an AWFS on the category of *quasicategories* [16, 17, 26] whose (algebraically) fibrant objects are quasicategories with finite limits; and another whose (algebraic) fibrations are the Grothendieck fibrations of quasicategories.

In this paper, we study a further aspect of the theory of AWFS which is specifically enabled by the algebraic perspective. Any weak factorisation system on a category $\mathcal{C}$ with an initial object gives rise to a notion of "cofibrant replacement" by factorising the unique maps out of the initial object, and in the algebraic case, this cofibrant replacement underlies a comonad Q. For suitable choices of

Date: December 23, 2014.

2000 *Mathematics Subject Classification.* Primary: 18A32, 55U35.

The first author acknowledges the support of the Grant agency of the Czech Republic, grant number P201/12/G028. The second author acknowledges the support of an Australian Research Council Discovery Project grants DP110102360 and DP130101969.

AWFS, the Kleisli category of Q—wherein maps $A \rightsquigarrow B$ are maps $QA \to B$ in the original category—will equip $\mathcal{C}$ with a usable notion of *weak map*. For example:

(i) There is an AWFS on the category of tricategories [13] and strict morphisms (preserving all structure on the nose) for which $\mathbf{Kl}(\mathsf{Q})$ comprises the tricategories and their trihomomorphisms (preserving all structure up to coherent equivalence); see [11].

(ii) If T is an accessible 2-monad on a complete and cocomplete 2-category, then there are AWFS on the category $\mathsf{T}\text{-}\mathbf{Alg}_s$ of T-algebras and strict morphisms such that the corresponding $\mathbf{Kl}(\mathsf{Q})$ is the category $\mathsf{T}\text{-}\mathbf{Alg}_p$ of T-algebras and algebra pseudomorphisms, respectively the category $\mathsf{T}\text{-}\mathbf{Alg}_l$ of T-algebras and lax algebra morphisms; see Section 5 below.

(iii) If T is a sufficiently cocontinuous dg-monad on a cocomplete dg-category, then there is an AWFS on $\mathsf{T}\text{-}\mathbf{Alg}_s$, the category of T-algebras and strict morphisms, for which $\mathbf{Kl}(\mathsf{Q})$ is the category $\mathsf{T}\text{-}\mathbf{Alg}_w$ of T-algebras and homotopy-coherent algebra morphisms; see Section 6 below.

Guided by these examples, we are led to define the *category of (left) weak maps* $\mathbf{Wk}_\ell(\mathsf{L}, \mathsf{R})$ of an AWFS as the Kleisli category of its cofibrant replacement comonad. (Dually, we have categories of *right* weak maps associated to fibrant replacement monads, but this plays only a minor role here.) While the value of the construction is apparent from the applications listed above, the abstract role it plays is less obvious; an important objective of this paper is to clarify this.

Our first main result, Theorem 10, characterises the category of weak maps as a kind of "homotopy category". Recall that the *homotopy category* of a Quillen model category $\mathcal{C}$ is the category $\mathcal{C}[\mathcal{W}^{-1}]$ obtained by freely inverting each weak equivalence. Similarly, the category of weak maps of an AWFS (L, R) arises by "freely splitting each R-map"; thus, for each R-algebra structure $\boldsymbol{f}$ on a morphism $f\colon A \to B$, we adjoin a section $\boldsymbol{f}^*$ for f, subject to certain coherence axioms.

Given this characterisation, we may associate a category of weak maps to an AWFS even in the absence of an initial object—by *defining* it in terms of the universal role it fulfils. Our second main result, Theorem 11, exploits this to give (in the presence of pullbacks) a second construction of categories of weak maps, wherein morphisms are equivalence-classes of spans with left leg an R-algebra; this is the analogue of Gabriel and Zisman's representation [9] of morphisms in a localisation $\mathcal{C}[\mathcal{W}^{-1}]$ as equivalences classes of spans with left leg in $\mathcal{W}$.

Our next main result, Theorem 13, turns the universal property of the category of weak maps into that of a 2-*adjunction* between 2-categories of AWFS and of comonads on categories with finite coproducts. The left adjoint sends an AWFS to its cofibrant replacement comonad. The right adjoint sends a comonad $\mathsf{P}\colon \mathcal{C} \to \mathcal{C}$ to the P-*split epi* AWFS on $\mathcal{C}$, whose R-algebras are maps of $\mathcal{C}$ equipped with a retraction in $\mathbf{Kl}(\mathsf{P})$; this is an AWFS with cofibrant replacement comonad P and is in fact universal among such. This 2-adjunction exhibits the 2-category of comonads as a full, reflective sub-2-category of the 2-category of AWFS, so that the theory of AWFS fully incorporates that of comonads—and dually, monads.

The results described above provide not only an abstract characterisation of categories of weak maps, but also a useful tool for calculating them. We illustrate this in the final two sections of the paper by using our results to prove the claims made in (ii) and (iii) above. We deal with (ii) in Section 5: thus, for a suitable 2-monad $\mathsf{T}\colon \mathcal{C} \to \mathcal{C}$, we define AWFS on $\mathsf{T}\text{-}\mathbf{Alg}_s$ whose categories of weak maps are respectively $\mathsf{T}\text{-}\mathbf{Alg}_p$ and $\mathsf{T}\text{-}\mathbf{Alg}_l$. These AWFS will be obtained by projectively lifting the AWFS on $\mathcal{C}$ for *retract equivalences* and for *left adjoint left inverse* functors. In the pseudo case, the lifted AWFS on $\mathsf{T}\text{-}\mathbf{Alg}_s$ is part of a model structure on $\mathsf{T}\text{-}\mathbf{Alg}_s$, described in [24, Theorem 4.5].

Finally, in Section 6 we turn to the case (iii) of dg-monads. Given a suitable dg-monad $\mathsf{T}\colon \mathcal{C} \to \mathcal{C}$ on a dg-category, we define an AWFS on $\mathsf{T}\text{-}\mathbf{Alg}_s$ whose category of weak maps comprises T-algebras and their *homotopy-coherent* morphisms. For example, when $\mathcal{A}$ is a small dg-category and T is the monad on $\mathbf{DG}^{\mathrm{ob}\,\mathcal{A}}$ whose algebras are dg-modules over $\mathcal{A}$, we obtain the category of dg-modules and homotopy-coherent transformations described in [18, Example 6.6]. The AWFS in question is again obtained by projectively lifting from $\mathcal{C}$ and is part of a model structure on $\mathsf{T}\text{-}\mathbf{Alg}_s$ of the kind constructed in [8]. Of particular note is the fact that its cofibrant replacement comonad is precisely the classical *bar construction* [7, X, §6]; this clarifies the universal role that this construction fulfils. In future work we will describe a generalisation of these results from dg-monads to dg-operads using a *dendroidal* [27] analogue of the bar construction, closely related to the bar–cobar construction for operad algebras [25, Chapter 11].

2. Background material on algebraic weak factorisation systems

In this preliminary section we summarise from [6, §2–3] those aspects of the theory of algebraic weak factorisation systems necessary for the present paper. We will say enough so as to make our treatment self-contained for someone familiar with the basic notions; for a full treatment, with proofs, we refer the reader to [6] and the sources cited therein.

Before we begin, a brief note on foundational matters. We fix a Grothendieck universe κ, and call arbitrary sets *large* and ones in κ, *small*. **Set** and **SET** denote the categories of small and large sets, while **Cat** and **CAT** are the 2-categories of internal categories in **Set** and **SET**; in particular, objects of **CAT** are *not* assumed to be locally small. By a *double category*, we mean an internal category in **CAT**, and we write **DBL** for the 2-category of double categories and internal functors and internal natural transformations between such.

2.1. **Algebraic weak factorisation systems and their morphisms.** An algebraic weak factorisation system on $\mathcal{C}$ begins with a *functorial factorisation*: a functor $\mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{3}}$ from the category of arrows to that of composable pairs which is a section for the composition map $\mathcal{C}^{\mathbf{3}} \to \mathcal{C}^{\mathbf{2}}$. The action of this functor at an object f or morphism $(h,k)\colon f \to g$ of $\mathcal{C}^{\mathbf{2}}$ is depicted as on the left or right in:

$$f = X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y \qquad\qquad \begin{array}{ccccc} X & \xrightarrow{Lf} & Ef & \xrightarrow{Rf} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle E(h,k)} & & \downarrow{\scriptstyle k} \\ W & \xrightarrow{Lg} & Eg & \xrightarrow{Rg} & Z \end{array} .$$

From these data we obtain endofunctors $L, R\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$, together with natural transformations $\epsilon\colon L \Rightarrow 1$ and $\eta\colon 1 \Rightarrow R$ with respective f-components:

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ {\scriptstyle Lf}\downarrow & & \downarrow{\scriptstyle f} \\ Ef & \xrightarrow{Rf} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{Lf} & Ef \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Rf} \\ B & \xrightarrow{1} & B \end{array}. \tag{2.1}$$

An *algebraic weak factorisation system* (L, R) on $\mathcal{C}$ is a functorial factorisation as above, together with natural transformations $\Delta\colon L \to LL$ and $\mu\colon RR \to R$ making $\mathsf{L} = (L, \epsilon, \Delta)$ and $\mathsf{R} = (R, \eta, \mu)$ into a comonad and a monad respectively. The monad and comonad axioms, together with the form (2.1) of η and ϵ, force the components of Δ and μ at f to be as on the left and right in

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ {\scriptstyle Lf}\downarrow & & \downarrow{\scriptstyle LLf} \\ Ef & \xrightarrow{\Delta_f} & ELf \end{array} \qquad \begin{array}{ccc} Ef & \xrightarrow{\Delta_f} & ELf \\ {\scriptstyle LRf}\downarrow & & \downarrow{\scriptstyle RLf} \\ ERf & \xrightarrow{\mu_f} & Ef \end{array} \qquad \begin{array}{ccc} ERf & \xrightarrow{\mu_f} & Ef \\ {\scriptstyle RRf}\downarrow & & \downarrow{\scriptstyle Rf} \\ B & \xrightarrow{1} & B \end{array},$$

and imply moreover that the middle square is the component at f of a natural transformation $\delta\colon LR \Rightarrow RL$. The final axiom for an AWFS is that this δ should constitute a *distributive law* in the sense of [3] of L over R.

We now turn to morphisms between AWFS. As with other kinds of algebraic structure borne by categories (for example monoidal structure) various kinds of morphisms arise—strict, pseudo, lax and oplax—and in the case of AWFS it is the lax and oplax ones that play the central role. A *lax morphism* between AWFS (L, R) and $(\mathsf{L}', \mathsf{R}')$ on categories $\mathcal{C}$ and $\mathcal{D}$ is given by a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a natural family of maps α_f rendering commutative the left hand square in:

$$\begin{array}{ccccc} & & FA & & \\ & {\scriptstyle L'Ff}\swarrow & & \searrow{\scriptstyle FLf} & \\ E'Ff & & \xrightarrow{\alpha_f} & & FEf \\ & {\scriptstyle R'Ff}\searrow & & \swarrow{\scriptstyle FRf} & \\ & & FB & & \end{array} \qquad \begin{array}{ccc} E'Ff & \xrightarrow{\alpha_f} & FEf \\ {\scriptstyle E'(\gamma_A, \gamma_B)}\downarrow & & \downarrow{\scriptstyle \gamma_{Ef}} \\ E'Gf & \xrightarrow[\beta_f]{} & GEf \end{array},$$

and such that the induced $(\alpha, 1)\colon R'F^{\mathbf{2}} \to F^{\mathbf{2}}R$ are respectively a lax monad morphism $\mathsf{R} \to \mathsf{R}'$ and a lax comonad morphism[1] $\mathsf{L} \to \mathsf{L}'$ over $F^{\mathbf{2}}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{D}^{\mathbf{2}}$. A *transformation* $(F, \alpha) \Rightarrow (G, \beta)$ between lax morphisms is a natural transformation $\gamma\colon F \Rightarrow G$ rendering commutative the square above right for each $f\colon A \to B$ in $\mathcal{C}$. Algebraic weak factorisation systems, lax morphisms and transformations form a 2-category $\mathbf{AWFS}_{\text{lax}}$.

Dually, there is a 2-category $\mathbf{AWFS}_{\text{oplax}}$ whose 1-cells are the *oplax morphisms* of AWFS—for which the components α_f point in the opposite direction to above—and whose 2-cells are transformations between them. Both of these 2-categories have an evident forgetful 2-functor to $\mathbf{CAT}$, and restricting $\mathbf{AWFS}_{\text{oplax}} \to \mathbf{CAT}$ to the fibre over some category $\mathcal{C}$ yields the category $\mathbf{AWFS}(\mathcal{C})$ of AWFS and

[1] In the terminology of [31] a *monad functor* and a *comonad opfunctor*.

AWFS morphisms on $\mathcal{C}$; while restricting $\mathbf{AWFS}_{\text{lax}} \to \mathbf{CAT}$ to the fibre over $\mathcal{C}$ yields the *opposite* category $\mathbf{AWFS}(\mathcal{C})^{\text{op}}$.

Given AWFS (L, R) and $(\mathsf{L}', \mathsf{R}')$ on categories $\mathcal{C}$ and $\mathcal{D}$, and an adjunction $F \dashv G \colon \mathcal{C} \to \mathcal{D}$, there is a bijection between extensions of G to a lax AWFS morphism and extensions of F to an oplax AWFS morphism obtained by taking mates; this is the AWFS incarnation of the *doctrinal adjunction* of [20]. The functoriality of this correspondence is usefully expressed as an identity-on-objects isomorphism of 2-categories ${\mathbf{AWFS}_{\text{radj}}}^{\text{coop}} \cong \mathbf{AWFS}_{\text{ladj}}$, where $\mathbf{AWFS}_{\text{radj}}$ is defined identically to $\mathbf{AWFS}_{\text{lax}}$ except that its 1-cells come equipped with chosen left adjoints, and where $\mathbf{AWFS}_{\text{ladj}}$ is defined from $\mathbf{AWFS}_{\text{oplax}}$ dually. Further details on doctrinal adjunction for AWFS are in Section 2.10 of [6].

2.2. **Double-categorical semantics.** Given an AWFS (L, R) on $\mathcal{C}$ we can consider the Eilenberg–Moore categories $\mathsf{L}\text{-}\mathbf{Coalg}$ and $\mathsf{R}\text{-}\mathbf{Alg}$ of coalgebras and algebras over $\mathcal{C}^{\mathbf{2}}$; these are thought of as providing the respective left and right classes of the AWFS. An R-algebra $\boldsymbol{f} = (f, p) \colon A \to B$ is a morphism $f \colon A \to B$ equipped with algebra structure $p \colon Rf \to f$, while an algebra morphism

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle \boldsymbol{f}}\downarrow & & \downarrow{\scriptstyle \boldsymbol{g}} \\ B & \xrightarrow[v]{} & D \end{array} \tag{2.2}$$

is a commuting square in $\mathcal{C}$ compatible with the algebra structures on $\boldsymbol{f}$ and $\boldsymbol{g}$; similar notation and conventions will be used for L-coalgebras. The connection with the two classes of a weak factorisation system is made on observing that

- Each map $f \colon A \to B$ has a factorisation $f = \boldsymbol{R}f \cdot \boldsymbol{L}f$ into a (cofree) L-coalgebra followed by a (free) R-algebra;
- Each commuting square (2.2) wherein $\boldsymbol{f}$ is an L-coalgebra and $\boldsymbol{g}$ an R-algebra has a *canonical* diagonal filler: see Section 2.4 of [6].

It follows that each AWFS has an underlying weak factorisation system whose left and right classes are the retracts of L-coalgebras and R-algebras respectively.

The right class of a weak factorisation system contains the identities and is closed under composition. Correspondingly, in an AWFS, each identity map $1_A \colon A \to A$ bears a *unique* R-algebra structure $\mathbf{1}_A \colon A \to A$; while if $\boldsymbol{g} \colon A \to B$ and $\boldsymbol{h} \colon B \to C$ are R-algebras, then the composite underlying map $h \cdot g$ admits an R-algebra structure $\boldsymbol{h} \cdot \boldsymbol{g} \colon A \to C$, *uniquely* determined by the requirement that the canonical lifts against $\boldsymbol{h} \cdot \boldsymbol{g}$ should be obtained by first lifting against $\boldsymbol{h}$ and then against $\boldsymbol{g}$. The uniqueness just noted implies that composition of R-algebras is associative and unital, and that it is compatible with R-algebra maps: meaning that, for any $f \colon A \to B$, there is an R-algebra map $(f, f) \colon \mathbf{1}_A \to \mathbf{1}_B$, and that if $(a, b) \colon \boldsymbol{g} \to \boldsymbol{g}'$ and $(b, c) \colon \boldsymbol{h} \to \boldsymbol{h}'$ are maps of R-algebras, then so too is $(a, c) \colon \boldsymbol{h} \cdot \boldsymbol{g} \to \boldsymbol{h}' \cdot \boldsymbol{g}'$.

We may express this composition and its associated coherences concisely by saying that R-algebras form a *double category* $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$ whose objects and horizontal arrows are those of $\mathcal{C}$, and whose vertical arrows and squares are the R-algebras and the maps thereof. There is a forgetful double functor $U^{\mathsf{R}} \colon \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$

into the double category of commutative squares in $\mathcal{C}$, which, displayed as an internal functor between internal categories in **CAT**, is as on the left in:

$$\begin{array}{ccc} \mathsf{R}\text{-}\mathbf{Alg} & \xrightarrow{U^{\mathsf{R}}} & \mathcal{C}^{\mathbf{2}} \\ {\scriptstyle d}\downarrow\uparrow{\scriptstyle i}\downarrow{\scriptstyle c} & & {\scriptstyle d}\downarrow\uparrow{\scriptstyle i}\downarrow{\scriptstyle c} \\ \mathcal{C} & \xrightarrow[1]{} & \mathcal{C} \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{V_1} & \mathcal{C}^{\mathbf{2}} \\ {\scriptstyle d}\downarrow\uparrow{\scriptstyle i}\downarrow{\scriptstyle c} & & {\scriptstyle d}\downarrow\uparrow{\scriptstyle i}\downarrow{\scriptstyle c} \\ \mathcal{A}_0 & \xrightarrow[V_0=1]{} & \mathcal{C} \end{array}\ .$$

Note that this internal functor has object component *an identity* and arrow component a *faithful functor*. By a *concrete double category over* $\mathcal{C}$, we mean a double category $\mathbb{A}$ and double functor $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$, as on the right above, with these same two properties. In working with general concrete double categories, we may reuse our notation for R-$\mathbb{A}\mathbf{lg}$, writing $\boldsymbol{f}\colon A \to B$ to denote a vertical arrow of $\mathbb{A}$ with $V(\boldsymbol{f}) = f\colon A \to B$, and using (2.2) to depict a square of $\mathbb{A}$ which is sent by V to $(u, v)\colon f \to g$ in $\mathcal{C}^{\mathbf{2}}$; this is meaningful by fidelity of V_1.

Example 1. Given a category $\mathcal{C}$, we write $\mathbf{SplEpi}(\mathcal{C})$ for the category of split epimorphisms therein: objects are pairs of a map $g\colon A \to B$ of $\mathcal{C}$ together with a section p of g, while morphisms $(g, p) \to (h, q)$ are serially commuting diagrams:

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle g}\downarrow\uparrow{\scriptstyle p} & & {\scriptstyle h}\downarrow\uparrow{\scriptstyle q} \\ B & \xrightarrow[s]{} & D \end{array}\ . \tag{2.3}$$

Split epimorphisms compose—by composing the sections—so that we have a double category $\mathbb{S}\mathbf{plEpi}(\mathcal{C})$ which is concrete over $\mathcal{C}$ via the double functor $\mathbb{S}\mathbf{plEpi}(\mathcal{C}) \to \mathbb{S}\mathbf{q}(\mathcal{C})$ which forgets the sections.

Example 2. A *lali* (left adjoint left inverse) in a 2-category $\mathcal{C}$ is a split epi $(g, p)\colon A \to B$ such that $g \dashv p$ with identity counit; note that this is a *property* of (g, p), rather than extra structure, since the unit of an adjunction is determined by the two functors and the counit. A morphism of lalis is simply a morphism of split epis; commutativity with the adjunction units is automatic. Since split epis and adjoints compose, so too do lalis; thus—writing $\mathcal{C}_0$ for the underlying category of $\mathcal{C}$—lalis in $\mathcal{C}$ form a concrete sub-double category $\mathbb{L}\mathbf{ali}(\mathcal{C})$ of $\mathbb{S}\mathbf{plEpi}(\mathcal{C}_0)$. A *retract equivalence* in $\mathcal{C}$ is a lali whose unit is invertible; these form a sub-double category of $\mathbb{L}\mathbf{ali}(\mathcal{C})$.

2.3. **Pullback stability.** We now record the analogue for AWFS of the pullback-stability of the right class of a weak factorisation system—which is expressed in terms of a property of the double category of algebras.

We call a functor $F\colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}}$ a *discrete pullback-fibration* if, for every $\boldsymbol{g} \in \mathcal{A}$ over $g \in \mathcal{C}^{\mathbf{2}}$ and every pullback square $(h, k)\colon f \to g$, there is a unique arrow $\varphi\colon \boldsymbol{f} \to \boldsymbol{g}$ in $\mathcal{A}$ over (h, k), and this arrow is cartesian with respect to F; we call a concrete double category $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ *pullback-stable* just when $V_1\colon \mathcal{A}_1 \to \mathcal{C}^{\mathbf{2}}$ is a discrete pullback-fibration. In elementary terms, this asserts that for any vertical map $\boldsymbol{g}\colon A \to B$ in $\mathbb{A}$ and pullback square in $\mathcal{C}$ as on the right in:

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle \exists! \boldsymbol{f}}\downarrow & & \downarrow{\scriptstyle \boldsymbol{g}} \\ B & \xrightarrow[v]{} & D \end{array} \qquad \longmapsto \qquad \begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[v]{} & D \end{array}$$

there exists a unique vertical map $\boldsymbol{f}$ over f making (u, v) into a cartesian square of $\mathbb{A}$ as on the left. The cartesianness expresses that (u, v) *detects* squares of $\mathbb{A}$—meaning that for any vertical arrow $\boldsymbol{h}$, a commutative square $(r, s)\colon h \to f$ in $\mathcal{C}$ lifts to a square $(r, s)\colon \boldsymbol{h} \to \boldsymbol{f}$ of $\mathbb{A}$ just when the composite $(ru, tv)\colon h \to g$ lifts to one $(ru, tv)\colon \boldsymbol{h} \to \boldsymbol{g}$. It follows from [6, Proposition 8] that the concrete double category of algebras of an AWFS is always pullback-stable.

2.4. **Structure and semantics.** Each lax morphism $(F, \alpha)\colon (\mathcal{C}, \mathsf{L}, \mathsf{R}) \to (\mathcal{C}', \mathsf{L}', \mathsf{R}')$ of AWFS induces a lifting of $\mathbb{S}\mathbf{q}(F)\colon \mathbb{S}\mathbf{q}(\mathcal{C}) \to \mathbb{S}\mathbf{q}(\mathcal{D})$ to a morphism of concrete double categories as to the left in

$$\begin{array}{ccc} \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} & \xrightarrow{\bar{F}} & \mathsf{R}'\text{-}\mathbb{A}\mathbf{lg} \\ {\scriptstyle U^{\mathsf{R}}}\downarrow & & \downarrow{\scriptstyle U^{\mathsf{R}'}} \\ \mathbb{S}\mathbf{q}(\mathcal{C}) & \xrightarrow[\mathbb{S}\mathbf{q}(F)]{} & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{array} \qquad \begin{array}{ccc} \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} & \overset{\bar{F}}{\underset{\bar{G}}{\rightrightarrows}}\ {\scriptstyle \Downarrow \bar{\alpha}} & \mathsf{R}'\text{-}\mathbb{A}\mathbf{lg} \\ {\scriptstyle U^{\mathsf{R}}}\downarrow & & \downarrow{\scriptstyle U^{\mathsf{R}'}} \\ \mathbb{S}\mathbf{q}(\mathcal{C}) & \overset{\mathbb{S}\mathbf{q}(F)}{\underset{\mathbb{S}\mathbf{q}(G)}{\rightrightarrows}}\ {\scriptstyle \Downarrow \mathbb{S}\mathbf{q}(\alpha)} & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{array} \ . \tag{2.4}$$

Similarly, each 2-cell of $\mathbf{AWFS}_{\text{lax}}$ induces a lifted horizontal transformation as on the right; and in this way we obtain the *semantics 2-functor*

$$(\text{–})\text{-}\mathbb{A}\mathbf{lg}\colon \mathbf{AWFS}_{\text{lax}} \to \mathbf{DBL}^{\mathbf{2}} \ .$$

The following result, which combines Proposition 2 and Theorem 6 of [6], shows that AWFS and their morphisms may be characterised purely in terms of this double-categorical algebra semantics. In its statement, a concrete double category $\mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ is said to be *right-connected* just when each vertical arrow $\boldsymbol{f}\colon A \to B$ of $\mathbb{A}$ can be completed to a square:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \boldsymbol{f}}\downarrow & & \downarrow{\scriptstyle \mathbf{1}_B} \\ B & \xrightarrow{1_B} & B \end{array} \ . \tag{2.5}$$

Theorem 3. *The 2-functor* $(\text{–})\text{-}\mathbb{A}\mathbf{lg}\colon \mathbf{AWFS}_{\text{lax}} \to \mathbf{DBL}^{\mathbf{2}}$ *is* 2*-fully faithful, and the concrete* $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ *is in its essential image just when:*

(i) The functor $V_1\colon \mathcal{A}_1 \to \mathcal{C}^{\mathbf{2}}$ *on vertical arrows and squares is strictly monadic;*
(ii) $\mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ *is right-connected.*

Remark 4. For the most part, the means by which Theorem 3 was established in [6] will not concern us; however, there is one key point we will require later.

Lemma 1 of [6] asserts that, for any AWFS (L, R) on $\mathcal{C}$, the commutative square

$$
\begin{array}{ccc}
Ef & \xrightarrow{\Delta_f} & ELf \\
& & \big\downarrow{\scriptstyle \boldsymbol{R}Lf} \\
\big\downarrow{\scriptstyle \boldsymbol{R}f} & & Ef \\
& & \big\downarrow{\scriptstyle \boldsymbol{R}f} \\
B & \xrightarrow[1]{} & B\ .
\end{array}
\tag{2.6}
$$

is a square in the double category $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$, thus a morphism of R-algebras. This provides the means by which the *comultiplication* of the comonad L may be recovered from the *double categorical* structure of $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$—as by freeness of $\boldsymbol{R}f$, the map $(\Delta_f, 1)\colon \boldsymbol{R}f \to \boldsymbol{R}f \cdot \boldsymbol{R}Lf$ is the unique R-algebra map whose precomposition with the unit $f \to Rf$ is $(LLf, 1)\colon f \to Rf \cdot RLf$.

Example 5. For any category $\mathcal{C}$, the concrete double category $\mathbb{S}\mathbf{plEpi}(\mathcal{C})$ of Example 1 is right-connected; whence by Theorem 3, it will be the double category of algebras of an AWFS on $\mathcal{C}$ whenever $U\colon \mathbf{SplEpi}(\mathcal{C}) \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic. We may identify U with $\mathcal{C}^j\colon \mathcal{C}^{\mathcal{S}} \to \mathcal{C}^{\mathbf{2}}$, where $\mathcal{S}$ is the *free split epi*:

$$
\begin{array}{ccccc}
1 & \xrightarrow{m} & 0 & & \\
& {\scriptstyle 1}\searrow & \big\downarrow{\scriptstyle e} & \searrow{\scriptstyle me} & \\
& & 1 & \xrightarrow{m} & 0
\end{array}
\tag{2.7}
$$

and where $j\colon \mathbf{2} \to \mathcal{S}$ is the evident inclusion. Thus U strictly creates colimits, and so will be strictly monadic whenever $\mathcal{C}$ is cocomplete enough to admit left Kan extensions along j. Using the Kan extension formula one finds that only binary coproducts are required; the free split epi $\boldsymbol{R}f$ on $f\colon A \to B$ is $\langle f, 1\rangle\colon A + B \to B$ with section ι_B.

Example 6. For any 2-category $\mathcal{C}$, the concrete double category $\mathbb{L}\mathbf{ali}(\mathcal{C})$ of Example 1 inherits right-connectedness from $\mathbb{S}\mathbf{plEpi}(\mathcal{C}_0)$; whence by Theorem 3, it will be the double category of algebras of an AWFS on $\mathcal{C}$ whenever $U\colon \mathbf{Lali}(\mathcal{C}) \to (\mathcal{C}_0)^{\mathbf{2}}$ is strictly monadic. Now, we may identify U with restriction

$$\mathbf{2\text{-}CAT}(j, \mathcal{C})\colon \mathbf{2\text{-}CAT}(\mathcal{L}, \mathcal{C}) \to \mathbf{2\text{-}CAT}(\mathbf{2}, \mathcal{C})$$

along the inclusion $j\colon \mathbf{2} \to \mathcal{L}$ of $\mathbf{2}$ into the *free lali* $\mathcal{L}$—which has the same underlying category as the free split epimorphism $\mathcal{S}$ of (2.7) and a single non-trivial 2-cell $1 \Rightarrow me$; so as before, monadicity obtains whenever $\mathcal{C}$ is cocomplete enough to admit left Kan extensions along j. In this case the necessary colimits are *oplax colimits* of arrows; see [6, Section 4.2]. On inverting the non-trivial 2-cell of $\mathcal{L}$, it becomes the free *retract equivalence* $\mathcal{L}^g$, and from this, we obtain for any sufficiently cocomplete 2-category $\mathcal{C}$ the AWFS on $\mathcal{C}_0$ whose algebras are retract equivalences.

Example 7. A basic way of obtaining new weak factorisation systems from old is by *projective lifting* along a functor. In the algebraic context, the construction of projective liftings is simplified by Theorem 3. Given an AWFS (L, R) on $\mathcal{D}$ and functor $F\colon \mathcal{C} \to \mathcal{D}$ one can form the pullback of double categories to the left in:

$$\begin{array}{ccc} \mathbb{A} & \longrightarrow & \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \\ {\scriptstyle V}\downarrow & \lrcorner & \downarrow{\scriptstyle U^{\mathsf{R}}} \\ \mathbb{S}\mathbf{q}(\mathcal{C}) & \xrightarrow{\mathbb{S}\mathbf{q}(F)} & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{A}_1 & \longrightarrow & \mathsf{R}\text{-}\mathbf{Alg} \\ {\scriptstyle V_1}\downarrow & \lrcorner & \downarrow{\scriptstyle U^{\mathsf{R}}} \\ \mathcal{C}^{\mathbf{2}} & \xrightarrow{F^{\mathbf{2}}} & \mathcal{D}^{\mathbf{2}} \end{array} \tag{2.8}$$

It is easily verified that the concrete $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ so obtained satisfies all the hypotheses of Theorem 3 except possibly for the existence of a left adjoint to $V_1\colon \mathcal{A}_1 \to \mathcal{C}^{\mathbf{2}}$, as on the right; so whenever this adjoint can be shown to exist, $\mathbb{A}$ will comprise the algebra double category of an AWFS $(\mathsf{L}', \mathsf{R}')$ on $\mathcal{C}$, the *projective lifting* of (L, R) along F. Conditions ensuring the existence of this adjoint are described in Propositions 13 and 14 of [6].

2.5. **A reformulation of right-connectedness.** As should be clear from Theorem 3 above, right-connected concrete double categories play an important role in the theory of AWFS. So far, we have described such double categories and the maps between them in terms of a full sub-2-category of $\mathbf{DBL}^{\mathbf{2}}$—but in fact we can do better. The right-connectedness of the concrete $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ is easily seen to be equivalent to the property that the codomain functor $c\colon \mathcal{A}_1 \to \mathcal{A}_0$ is left adjoint left inverse for the identities functor $i\colon \mathcal{A}_0 \to \mathcal{A}_1$; we will say that $\mathbb{A}$ is *right-connected* if this is the case. Similarly, the fidelity of $V_1\colon \mathcal{A}_1 \to \mathcal{C}^{\mathbf{2}}$ is equivalent to $\mathbb{A}$ being *locally preordered*, in the sense that squares are determined by their boundaries.

In fact any right-connected locally posetal $\mathbb{A}$ arises in this way; the corresponding $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ has $\mathcal{C} = \mathcal{A}_0$, and $V_1\colon \mathcal{A}_1 \to \mathcal{A}_0^{\mathbf{2}}$ the functor classifying the composite natural transformation $d\eta\colon d \Rightarrow c\colon \mathcal{A}_1 \to \mathcal{A}_0$. If $\mathbb{B}$ is also right-connected and locally posetal, then any double functor $F\colon \mathbb{A} \to \mathbb{B}$ will as in Example 2 commute with the units of the adjunctions $c_{\mathbb{A}} \dashv i_{\mathbb{A}}$ and $c_{\mathbb{B}} \dashv i_{\mathbb{B}}$; whence F extends to a concrete double functor between the associated concrete double categories. The same argument pertains to 2-cells, giving that:

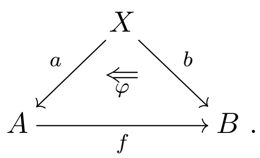
Lemma 8. *The domain* 2*-functor* $\mathbf{DBL}^{\mathbf{2}} \to \mathbf{DBL}$ *restricts to a* 2*-equivalence between the full sub-*2*-category of* $\mathbf{DBL}^{\mathbf{2}}$ *on the right-connected concrete double categories, and the full sub-*2*-category of* $\mathbf{DBL}$ *on the right-connected locally posetal double categories.*

3. Categories of weak maps

We now turn to the central object of study of this paper: the categories of *weak maps* associated to an algebraic weak factorisation system. It turns out that these may be defined in a number of different ways—the first and simplest of which is given with reference to *cofibrant replacement* comonads or *fibrant replacement* monads.

3.1. **Weak maps as Kleisli categories.** If (L, R) is an AWFS on $\mathcal{C}$, then its comonad L restricts to a comonad on each coslice category $X/\mathcal{C}$; in particular, if $X = 0$ is initial in $\mathcal{C}$, then we obtain a comonad Q on $0/\mathcal{C} \cong \mathcal{C}$, which we may call the *cofibrant replacement comonad* of (L, R). We define the category of *left weak maps* $\mathbf{Wk}_\ell(\mathsf{L}, \mathsf{R})$ to be the co-Kleisli category of Q. Dually, if $\mathcal{C}$ has a terminal object, then the monad R on $\mathcal{C}^{\mathbf{2}}$ induces a *fibrant replacement monad* on $\mathcal{C}$, whose Kleisli category is the category $\mathbf{Wk}_r(\mathsf{L}, \mathsf{R})$ of *right weak maps.*

Examples 9. (i) Let $\mathcal{C}$ be any 2-category admitting the AWFS for lalis of Example 2 above. For each $X \in \mathcal{C}$, this AWFS induces a *coslice* AWFS on $X/\mathcal{C}$ by projectively lifting, as in Example 7, along the forgetful functor $X/\mathcal{C} \to \mathcal{C}$. Direct calculation shows that the category of left weak maps for this AWFS is the lax coslice $X/\!/\mathcal{C}$, wherein morphisms $(A, a) \rightsquigarrow (B, b)$ are lax triangles



By starting from the AWFS for retract equivalences rather than that for lalis, we obtain instead the *pseudo* coslice category.

(ii) In [11] was constructed an AWFS on $\mathbf{Bicat}_s$, the category of small bicategories and strict morphisms—those preserving composition and identities on the nose—whose associated left weak maps are the *homomorphisms* of bicategories—those preserving composition and identities up to coherent isomorphism. [11] also exhibits trihomomorphisms between tricategories as weak maps, and, more generally, shows that for any *globular operad* $\mathcal{O}$ in the sense of [2], there is an AWFS on the category of $\mathcal{O}$-algebras and strict $\mathcal{O}$-algebra maps whose category of left weak maps provides a suitable notion of "weak morphism" of $\mathcal{O}$-algebras.

(iii) Let $\mathcal{C}$ be a category with pullbacks, and $\mathcal{M}$ a class of monomorphisms therein which contains all isomorphisms and is stable under composition and pullback. Suppose moreover that there is an *classifying $\mathcal{M}$-map*—an $\mathcal{M}$-map $t \colon 1 \rightarrowtail U$ which is terminal in the category of $\mathbf{M}_{\mathrm{pb}}$ of $\mathcal{M}$-maps and pullback squares—and that t is exponentiable in $\mathcal{C}$. Under this assumption, it was shown in [6, Section 4.4] that there is an AWFS on $\mathcal{C}$ whose category of L-coalgebras is $\mathbf{M}_{\mathrm{pb}}$, and whose fibrant replacement monad is the *partial $\mathcal{M}$-map classifier monad* of [30, Chapter 3]:

$$\mathsf{F} = \mathcal{C} \xrightarrow{\Pi_t} \mathcal{C}/U \xrightarrow{\Sigma_U} \mathcal{C} \ .$$

The associated category of right weak maps is the *$\mathcal{M}$-partial map category*, whose morphisms $A \rightsquigarrow B$ are isomorphism-classes of spans $A \leftarrowtail A' \to B$ with left leg in $\mathcal{M}$, and whose composition is by pullback.

(iv) In Section 5 we will see that, if T is an accessible 2-monad on a complete and cocomplete 2-category $\mathcal{C}$, then there are AWFS on $\mathsf{T}\text{-}\mathbf{Alg}_s$, the category of T-algebras and *strict* algebra morphisms, whose categories of left weak maps have as morphisms the *pseudo* or *lax* algebra morphisms.

(v) In Section 6 we will see that, if $\mathcal{C}$ is a cocomplete dg-category and T a sufficiently cocontinuous dg-monad thereon, then there is an awfs on $\mathsf{T}\text{-}\mathbf{Alg}_s$, the category of T-algebras and strict morphisms whose category of weak maps is the category of *homotopy-coherent* T-algebra morphisms. For example, if $\mathcal{A}$ is a small dg-category and $\mathcal{D}$ a cocomplete one, then there is an AWFS on the functor category $[\mathcal{A}, \mathcal{D}]$ for which the left weak maps are *homotopy-coherent natural transformations* between dg-functors in the sense of [32, §3].

3.2. **The universal property of the category of weak maps.** Our first main result expresses that the category of left weak maps arises by "freely splitting each R-map in $\mathcal{C}$". Of course, there is a dual result for right weak maps—but henceforth we leave it to the reader to formulate such dual cases.

Theorem 10. *Let* (L, R) *be an* AWFS *on a category* $\mathcal{C}$ *with an initial object. There are isomorphisms of categories*

$$\mathbf{CAT}(\mathbf{Wk}_\ell(\mathsf{L},\mathsf{R}), \mathcal{D}) \cong \mathbf{DBL}(\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}, \mathbb{S}\mathbf{plEpi}(\mathcal{D})) \ , \tag{3.1}$$

2-natural in $\mathcal{D}$*, that mediate between extensions of a functor* $F \colon \mathcal{C} \to \mathcal{D}$ *through the co-Kleisli category* $\mathbf{Wk}_\ell(\mathsf{L},\mathsf{R})$ *as on the left below, and liftings of* F *to a concrete double functor as on the right.*

$$\begin{array}{ccc} & \mathbf{Wk}_\ell(\mathsf{L},\mathsf{R}) & \\ {}^{\text{cofree}}\nearrow & & \searrow^{G} \\ \mathcal{C} & \xrightarrow{\quad F \quad} & \mathcal{D} \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} & \xrightarrow{H} & \mathbb{S}\mathbf{plEpi}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathbb{S}\mathbf{q}(\mathcal{C}) & \xrightarrow[\mathbb{S}\mathbf{q}(F)]{} & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{array} \ . \tag{3.2}$$

Proof. It is easy to see that $\mathbb{S}\mathbf{plEpi}(\text{–}) \colon \mathbf{CAT} \to \mathbf{DBL}$ preserves all 2-dimensional limits; in particular we have $\mathbb{S}\mathbf{plEpi}(\mathcal{D}^{\mathbf{2}}) \cong \mathbb{S}\mathbf{plEpi}(\mathcal{D})^{\mathbf{2}}$, so that to obtain isomorphisms of categories (3.1) it suffices to exhibit ones of underlying *sets*. As $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$ and $\mathbb{S}\mathbf{plEpi}(\mathcal{D})$ are right-connected and locally posetal, this is equivalent by Lemma 8 to giving a bijection, natural in $\mathcal{D}$, between extensions G and liftings H as in (3.2).

Suppose first that we are given a lifting of F to a double functor as on the right. For each $B \in \mathcal{B}$, let $!_B \colon 0 \to B$ denote the unique map; then for any R-algebra $\boldsymbol{f} \colon A \to B$, the map $(!_A, 1_B) \colon !_B \to f$ in $\mathcal{C}^{\mathbf{2}}$ induces an R-algebra morphism as on the left below. Applying (3.2) yields a morphism of split epis in $\mathcal{D}$ as on the right (observing that the underlying map of $\boldsymbol{R}!_B$ is the counit $\epsilon_B \colon QB \to B$).

$$\begin{array}{ccc} QB & \xrightarrow{\varphi_{\boldsymbol{f}}} & A \\ {\scriptstyle \boldsymbol{R}!_B}\downarrow & & \downarrow{\scriptstyle \boldsymbol{f}} \\ B & \xrightarrow[1]{} & B \end{array} \qquad\qquad \begin{array}{ccc} FQB & \xrightarrow{F\varphi_{\boldsymbol{f}}} & FA \\ {\scriptstyle F\epsilon_B}\downarrow\uparrow{\scriptstyle \alpha_B} & & {\scriptstyle Ff}\downarrow\uparrow{\scriptstyle s} \\ FB & \xrightarrow[1]{} & FB \end{array} \tag{3.3}$$

By the upwards commutativity we have $s = F\varphi_{\boldsymbol{f}} \cdot \alpha_B$, so that the lifting (3.2) is uniquely determined by giving the α_B's. These maps are the components of a natural transformation $\alpha \colon F \to FQ$ with $F\epsilon \cdot \alpha = 1 \colon F \to FQ \to F$; and since applying (3.2) to the algebra square (2.6) below left yields the one below right:

$$
\begin{array}{ccc}
QB & \xrightarrow{\Delta_B} & QQB \\
 & & \big\downarrow {\scriptstyle \boldsymbol{R}L!_B = \boldsymbol{R}!_{QB}} \\
{\scriptstyle \boldsymbol{R}!_B}\big\downarrow & & QB \\
 & & \big\downarrow {\scriptstyle \boldsymbol{R}!_B} \\
B & \xrightarrow[1]{} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
FQB & \xrightarrow{F\Delta_B} & FQQB \\
 & & {\scriptstyle F\epsilon_{QB}}\big\downarrow\big\uparrow{\scriptstyle \alpha_{QB}} \\
{\scriptstyle F\epsilon_B}\big\downarrow\big\uparrow{\scriptstyle \alpha_B} & & FQB \\
 & & {\scriptstyle F\epsilon_B}\big\downarrow\big\uparrow{\scriptstyle \alpha_B} \\
FB & \xrightarrow[1]{} & FB
\end{array} ,
\tag{3.4}
$$

we conclude that also $\alpha Q \cdot \alpha = F\Delta \cdot \alpha \colon F \to FQ \to FQQ$. Thus $\alpha \colon F \to FQ$ exhibits F as a coalgebra for the comonad $(-) \cdot \mathsf{Q}$ on the functor category $[\mathcal{C}, \mathcal{D}]$; and by [31, §5], to give such a coalgebra structure is equally to give an extension of F to a functor $\mathbf{Wk}_\ell(\mathsf{L}, \mathsf{R}) = \mathbf{Kl}(\mathsf{Q}) \to \mathcal{D}$. The process just described is clearly natural in $\mathcal{D}$; so to complete the proof, it suffices to show that any coalgebra $\alpha \colon F \to FQ$ arises from a double functor in this way.

For this, given a coalgebra α, we must show that the lifting (3.2) which sends an R-algebra $\boldsymbol{f} \colon A \to B$ to the split epi $(Ff, F\varphi_{\boldsymbol{f}} \cdot \alpha_B) \colon FA \to FB$ is well-defined. The only point that needs checking is that composition of algebras is preserved; thus given also $\boldsymbol{g} \colon B \to C$, we must show that $F\varphi_{\boldsymbol{f}} \cdot \alpha_B \cdot F\varphi_{\boldsymbol{g}} \cdot \alpha_C = F\varphi_{\boldsymbol{gf}} \cdot \alpha_C$. The left-hand side is equal to $F\varphi_{\boldsymbol{f}} \cdot FQ\varphi_{\boldsymbol{g}} \cdot \alpha_{QC} \cdot \alpha_C = F\varphi_{\boldsymbol{f}} \cdot FQ\varphi_{\boldsymbol{g}} \cdot F\Delta_C \cdot \alpha_C$ by naturality and the comultiplication axiom for α. So it suffices to show that $\varphi_{\boldsymbol{f}} \cdot Q\varphi_{\boldsymbol{g}} \cdot \Delta_C = \varphi_{\boldsymbol{gf}}$. For this, we consider the following diagram.

$$
\begin{array}{ccccccc}
QC & \xrightarrow{\Delta_C} & QQC & \xrightarrow{Q\varphi_{\boldsymbol{g}}} & QB & \xrightarrow{\varphi_{\boldsymbol{f}}} & A \\
 & & \big\downarrow{\scriptstyle \boldsymbol{R}L!_C} & & \big\downarrow{\scriptstyle \boldsymbol{R}!_B} & & \big\downarrow{\scriptstyle \boldsymbol{f}} \\
{\scriptstyle \boldsymbol{R}!_C}\big\downarrow & & QC & \xrightarrow{\varphi_{\boldsymbol{g}}} & B & \xrightarrow[1]{} & B \\
 & & \big\downarrow{\scriptstyle \boldsymbol{R}!_C} & & \big\downarrow{\scriptstyle \boldsymbol{g}} & & \big\downarrow{\scriptstyle \boldsymbol{g}} \\
C & \xrightarrow[1]{} & C & \xrightarrow[1]{} & C & \xrightarrow[1]{} & C
\end{array} .
$$

Each small square is an R-algebra map, whence the large rectangle is too; but since this rectangle precomposes with the unit $!_C \to R!_C$ to yield $(!_A, 1_C) \colon !_C \to gf$, it must by freeness of $\boldsymbol{R}!_C$ be the map $(\varphi_{\boldsymbol{gf}}, 1)$. Comparing domain-components yields that $\varphi_{\boldsymbol{f}} \cdot Q\varphi_{\boldsymbol{g}} \cdot \Delta_C = \varphi_{\boldsymbol{gf}}$ as required. □

3.3. **Weak maps as weighted colimits.** In order to define the category of left weak maps above, we were forced to assume the existence of an initial object; and while this is scarcely a restriction in practice, it is nonetheless a little inelegant. Theorem 10 suggests a way of removing this restriction: we *redefine* the category of left weak maps of an AWFS (L, R) to be any category which, as in (3.1), represents the 2-functor $\mathbf{DBL}(\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}, \mathbb{S}\mathbf{plEpi}(-)) \colon \mathbf{CAT} \to \mathbf{CAT}$. We shall adopt this broader definition henceforth; the problem now becomes one of determining when $\mathbf{Wk}_\ell(\mathsf{L}, \mathsf{R})$ exists. Theorem 10 tells us that this is so whenever $\mathcal{C}$ has an initial object; what we now show is that this is in fact *always* so: $\mathbf{Wk}_\ell(\mathsf{L}, \mathsf{R})$ can be computed, under no restrictions on $\mathcal{C}$, as a certain weighted colimit in $\mathbf{CAT}$.

To see this, let Δ_2 be the subcategory

$$[0] \;\substack{\xrightarrow{\;\delta_1\;}\\ \xleftarrow{\;\sigma_0\;}\\ \xrightarrow[\;\delta_0\;]{}}\; [1] \;\substack{\xrightarrow{\;\delta_2\;}\\ \xrightarrow{\;\delta_1\;}\\ \xrightarrow[\;\delta_0\;]{}}\; [2]$$

of the simplicial category, where $[n] = \{0 \leqslant \cdots \leqslant n\}$ and δ and σ are the usual coface and codegeneracy operators. Identifying each internal category in $\mathbf{CAT}$ with its truncated nerve gives a full embedding $\mathbf{DBL} \to [\Delta_2^{\mathrm{op}}, \mathbf{CAT}]$ of 2-categories; it follows that we may view the defining isomorphism (3.1) as one

$$\mathbf{CAT}(\mathbf{Wk}_\ell(\mathsf{L},\mathsf{R}),\mathcal{D}) \cong [\Delta_2^{\mathrm{op}}, \mathbf{CAT}](\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}, \mathbb{S}\mathbf{plEpi}(\mathcal{D}))\ .$$

Now, the 2-functor $\mathbb{S}\mathbf{plEpi}(-)\colon \mathbf{CAT} \to [\Delta_2^{\mathrm{op}}, \mathbf{CAT}]$ sends a category $\mathcal{D}$ to the truncated simplicial diagram as to the left of

$$\mathcal{D} \;\substack{\xleftarrow{\;d\;}\\ \xrightarrow{\;i\;}\\ \xleftarrow[\;c\;]{}}\; \mathcal{D}^{\mathcal{S}} \;\substack{\xleftarrow{\;p\;}\\ \xleftarrow{\;m\;}\\ \xleftarrow[\;q\;]{}}\; \mathcal{D}^{\mathcal{S}} \times_{\mathcal{D}} \mathcal{D}^{\mathcal{S}} \qquad\qquad \mathbf{1} \;\substack{\xrightarrow{\;d\;}\\ \xleftarrow{\;i\;}\\ \xrightarrow[\;c\;]{}}\; \mathcal{S} \;\substack{\xrightarrow{\;p\;}\\ \xrightarrow{\;m\;}\\ \xrightarrow[\;q\;]{}}\; \mathcal{S} +_{\mathbf{1}} \mathcal{S}$$

where $\mathcal{S}$ is the free split epi as in (2.7). We may see this diagram as induced by homming into $\mathcal{D}$ from the cosimplicial diagram $S\colon \Delta_2 \to \mathbf{CAT}$ right above; in other words, we have $\mathbb{S}\mathbf{plEpi}(-) \cong \mathbf{CAT}(S, 1)\colon \mathbf{CAT} \to [\Delta_2^{\mathrm{op}}, \mathbf{CAT}]$. Since $\mathbf{CAT}(S, 1)$ has a left adjoint sending $X \in [\Delta_2^{\mathrm{op}}, \mathbf{CAT}]$ to the weighted colimit $S \star X$, we conclude that we have a 2-representation

$$\mathbf{CAT}(S \star \mathsf{R}\text{-}\mathbb{A}\mathbf{lg}, \mathcal{D}) \cong \mathbf{DBL}(\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}, \mathbb{S}\mathbf{plEpi}(\mathcal{D}))\ ,$$

so that the category of left weak maps of the AWFS (L, R) may be constructed under no assumptions on $\mathcal{C}$ as the weighted colimit $S \star \mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$.

Unfolding the description, we find that for a right-connected double category $\mathbb{A}$, the colimit $S \star \mathbb{A}$ may be obtained by first taking a coinserter $v\colon \mathcal{A}_0 \to \mathcal{B}$ of $c, d\colon \mathcal{A}_1 \rightrightarrows \mathcal{A}_0$, with universal 2-cell $\theta\colon vc \Rightarrow vd$, and then taking two coequifiers making θ a section of $vd\eta\colon vd \Rightarrow vc$ and imposing the cocycle condition $\theta m = \theta p \cdot \theta q\colon vcq \Rightarrow vdp$. Thus $S \star \mathbb{A}$ is the category obtained from $\mathcal{A}_0$ by freely adjoining a morphism $\boldsymbol{a}^*\colon X \to A$ for each $\boldsymbol{a}\colon A \to X$ in $\mathbb{A}$ satisfying $a \cdot \boldsymbol{a}^* = 1$ and $(\boldsymbol{b} \cdot \boldsymbol{a})^* = \boldsymbol{a}^* \cdot \boldsymbol{b}^*$ and $u \cdot \boldsymbol{a}^* = \boldsymbol{b}^* \cdot v$ for all $(u, v)\colon \boldsymbol{a} \to \boldsymbol{b}$ in $\mathbb{A}$.

Note that applying this construction to a locally small category may yield one that is no longer locally small; by contrast, the construction of categories of weak maps via cofibrant replacement always preserves local smallness. This is analogous to the fact that that the homotopy category of a locally small model category is again locally small, while the localisation of a category at an arbitrary collection of morphisms need not be so.

3.4. **Weak maps as spans.** We have already mentioned the analogy between the weak map construction, which freely adds sections for R-algebras, and the better known construction of the *localisation* $\mathcal{C}[\mathcal{W}^{-1}]$ of a category $\mathcal{C}$ at a class of morphisms $\mathcal{W}$: here, rather than freely adding a section to each map, we add an inverse. Under certain hypotheses [9], the morphisms of $\mathcal{C}[\mathcal{W}^{-1}]$ from A to B can be viewed as equivalence classes of spans $A \leftarrow X \to B$ whose left leg belongs to $\mathcal{W}$. We now show that under similar hypotheses, the category of left weak maps can also be described in such terms.

Let $\mathcal{C}$ be a category with pullbacks and $U\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ a pullback-stable concrete double category over $\mathcal{C}$. We begin by forming a bicategory $\mathbf{Span}(\mathbb{A})$ that enhances the usual bicategory of spans in $\mathcal{C}$. Objects are those of $\mathcal{C}$, while morphisms from A to B are $\mathbb{A}$-*spans*: that is, spans

$$A \xleftarrow{\boldsymbol{a}} X \xrightarrow{f} B \tag{3.5}$$

in $\mathcal{C}$ whose left leg has the structure of a vertical arrow of $\mathbb{A}$. The 2-cells $(\boldsymbol{a}, f) \to (\boldsymbol{b}, g)$ are $\mathbb{A}$-*span maps*: that is, span maps

$$\begin{array}{ccccc} & & X & & \\ & {}^{\boldsymbol{a}}\swarrow & \downarrow{\scriptstyle r} & \searrow^{f} & \\ A & \xleftarrow[\boldsymbol{b}]{} & Y & \xrightarrow[g]{} & B \end{array} \tag{3.6}$$

in $\mathcal{C}$ for which $(r, 1)\colon \boldsymbol{a} \to \boldsymbol{b}$ is a square of $\mathbb{A}$. The identity morphism at A is $(\mathbf{1}, 1)\colon A \to A$; while the composite of $(\boldsymbol{a}, f)\colon A \to B$ and $(\boldsymbol{b}, g)\colon B \to C$ is obtained by first composing the underlying spans in $\mathcal{C}$ by pullback:

$$\begin{array}{ccccccccc} & & & & X \times_B Y & & & & \\ & & & {}^{\boldsymbol{p}}\swarrow & & \searrow^{q} & & & \\ & & X & & & & Y & & \\ & {}^{\boldsymbol{a}}\swarrow & & \searrow^{f} & & {}^{\boldsymbol{b}}\swarrow & & \searrow^{g} & \\ A & & & & B & & & & C \end{array} \tag{3.7}$$

and then using the fact that $U\colon \mathcal{A}_1 \to \mathcal{C}^{\mathbf{2}}$ is a discrete pullback-fibration to induce a structure of vertical arrow on p as displayed; this now allows us to define the composite $(\boldsymbol{b}, f) \cdot (\boldsymbol{a}, f)$ to be $(\boldsymbol{a} \cdot \boldsymbol{p}, g \cdot q)$. The remaining aspects of the bicategory structure—2-cell composition, associativity and unit constraints, and coherence axioms—are as for the usual bicategory of spans. The additional facts to be established are that various span maps are in fact $\mathbb{A}$-span maps, and here one makes full use of the fact that $U\colon \mathcal{A}_1 \to \mathcal{C}^{\mathbf{2}}$ is a discrete pullback-fibration.

Our present interest lies not in the above bicategory, but in a quotient of it. [2] Each bicategory $\mathcal{C}$ gives rise a category $\pi_1\mathcal{C}$ with the same objects as $\mathcal{C}$ and with $(\pi_1\mathcal{C})(X, Y)$ the set of connected components of the category $\mathcal{C}(X, Y)$; we define $\mathcal{C}[\mathbb{A}^*]$ to be the category $\pi_1(\mathbf{Span}(\mathbb{A}))$. So objects of $\mathcal{C}[\mathbb{A}^*]$ are those of $\mathcal{C}$, while maps $A \to B$ are equivalence classes of $\mathbb{A}$-spans, where $(\boldsymbol{a}, f) \sim (\boldsymbol{b}, g)\colon A \to B$ just when they can be connected by a zigzag of $\mathbb{A}$-span maps.[3] There is an evident functor $J\colon \mathcal{C} \to \mathcal{C}[\mathbb{A}^*]$ which is the identity on objects and sends $f\colon A \to B$ to $[\mathbf{1}, f]\colon A \to B$. Note that, as in the preceding section, the category $\mathcal{C}[\mathbb{A}^\star]$ may not be locally small even if $\mathcal{C}$ is so.

[2] Nonetheless the structure of such bicategories appears to be independent interest. [33] introduces *crossed internal categories*, which can be viewed as monads in $\mathbf{Span}(\mathbb{A})$ where $\mathbb{A}$ is the concrete double category of split opfibrations over $\mathbf{Cat}$.

[3] For those $\mathbb{A}$ arising in practice, it is often the case that $V_1\colon \mathcal{A}_1 \to \mathcal{C}^{\mathbf{2}}$ creates pullbacks—for example when it is monadic. This ensures that each hom-category of $\mathbf{Span}(\mathbb{A})$ has pullbacks, from which it follows that that $(\boldsymbol{a}, f)\colon A \to B$ and $(\boldsymbol{b}, g)\colon A \to B$ are equal in $\mathcal{C}[\mathbb{A}^*]$ if and only if there exists a third $\mathbb{A}$-span $(\boldsymbol{c}, h)\colon A \to B$ and $\mathbb{A}$-span maps $(\boldsymbol{c}, h) \to (\boldsymbol{a}, f)$ and $(\boldsymbol{c}, h) \to (\boldsymbol{b}, g)$.

The above construction of $\mathcal{C}[\mathbb{A}^\star]$ and the proof of the following result are closely related to the 2-categorical constructions of Section 4.2 and Theorem 21 of [5].

Theorem 11. *Let $\mathcal{C}$ be a category with pullbacks and $\mathbb{A}$ a pullback-stable right-connected concrete double category over $\mathcal{C}$. There are 2-natural isomorphisms of categories*

$$\mathbf{CAT}(\mathcal{C}[\mathbb{A}^*], \mathcal{D}) \cong \mathbf{DBL}(\mathbb{A}, \mathbb{S}\mathbf{plEpi}(\mathcal{D}))$$

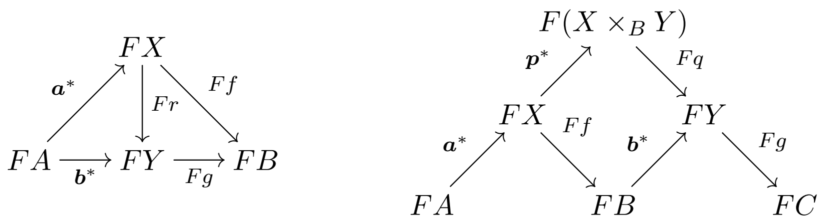
exhibiting $\mathcal{C}[\mathbb{A}^*]$ as $S \star \mathbb{A}$. These isomorphisms mediate between extensions of a functor $F\colon \mathcal{C} \to \mathcal{D}$ through $\mathcal{C}[\mathbb{A}^*]$ as on the left below, and liftings of F to a concrete double functor as on the right:

(3.8)

Proof. Arguing as in Theorem 10, it suffices to exhibit a natural bijection between extensions G and liftings H as in (3.8). Suppose first that we are given a lifting of F to a double functor H: such is specified by the assignment to each $\boldsymbol{a}\colon A \to B$ of a split epi $(Fa, \boldsymbol{a}^*)\colon FA \to FB$ in $\mathcal{D}$ subject to functoriality axioms.

Given this, we must define an extension $G\colon \mathcal{C}[\mathbb{A}^*] \to \mathcal{D}$ as on the left. Of course, we take $GX = FX$ on objects. On morphisms, given $[\boldsymbol{a}, f]\colon A \leftarrow X \to B$ in $\mathcal{C}[\mathbb{A}^*]$, we have an arrow $Ff \cdot \boldsymbol{a}^*\colon FA \to FX \to FB$ in $\mathcal{D}$; furthermore, for any $\mathbb{A}$-span map $r\colon (\boldsymbol{a}, f) \to (\boldsymbol{b}, g)$, the $\mathbb{A}$-square $(1, r)\colon \boldsymbol{a} \to \boldsymbol{b}$ is sent to a morphism $(1, Fr)\colon (Fa, \boldsymbol{a}^*) \to (Fb, \boldsymbol{b}^*)$ of split epis. Thus the left diagram in

(3.9)

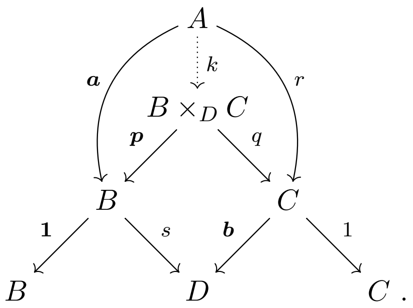


commutes, so that taking $G[\boldsymbol{a}, f] = Ff \cdot \boldsymbol{a}^*$ gives a well-defined assignation on morphisms. Given $f\colon A \to B \in \mathcal{C}$ we have $G[\mathbf{1}, f] = Ff \cdot \mathbf{1}^* = Ff$ so that G is indeed an extension of F, and the same argument shows that it preserves identities. As for binary composition, let $[\boldsymbol{a}, f]\colon A \to B$ and $[\boldsymbol{b}, g]\colon B \to C$ in $\mathcal{C}[\mathbb{A}^*]$ with composite $[\boldsymbol{a} \cdot \boldsymbol{p}, g \cdot q]$ as in (3.7). Now $G[\boldsymbol{b}, g] \cdot G[\boldsymbol{a}, f]$ and $G([\boldsymbol{b}, g] \cdot [\boldsymbol{a}, f])$ are the lower and upper composites on the right above, so it suffices to show that the inner square commutes—but as $(q, f)\colon \boldsymbol{p} \to \boldsymbol{b}$ in $\mathbb{A}$, its image under H is a map of split epis $(Fp, \boldsymbol{p}^*) \to (Fb, \boldsymbol{b}^*)$, whence the square commutes as required.

The passage from $H\colon \mathbb{A} \to \mathbb{S}\mathbf{plEpi}(\mathcal{D})$ to $G\colon \mathcal{C}[\mathbb{A}^*] \to \mathcal{D}$ is clearly natural in $\mathcal{D}$; it is also injective, as given $\boldsymbol{a}\colon A \to B$ we have $\boldsymbol{a}^* = G[\boldsymbol{a}, 1]\colon FB \to FA$. To complete the proof, it thus suffices to show that each extension $G\colon \mathcal{C}[\mathbb{A}^*] \to \mathcal{D}$ is induced by some $H\colon \mathbb{A} \to \mathbb{S}\mathbf{plEpi}(\mathcal{D})$. By naturality, it is enough to show that $1\colon \mathcal{C}[\mathbb{A}^*] \to \mathcal{C}[\mathbb{A}^*]$ is induced by some $H\colon \mathbb{A} \to \mathbb{S}\mathbf{plEpi}(\mathcal{C}[\mathbb{A}^*])$ lifting J.

We define H on vertical morphisms by $\boldsymbol{a} \mapsto (Ja, [\boldsymbol{a}, 1]) = ([\mathbf{1}, a], [\boldsymbol{a}, 1])$. So long as this is well-defined, the corresponding extension $\mathcal{C}[\mathbb{A}^*] \to \mathcal{C}[\mathbb{A}^*]$ will send $[\boldsymbol{a}, f] \colon A \to B$ to $Jf \cdot [\boldsymbol{a}, 1] = [1, \boldsymbol{f}] \cdot [\boldsymbol{a}, 1] = [\boldsymbol{a}, f]$ and so be the identity as required. It remains to show well-definedness of H. By $\mathbb{A}$'s right-connectedness, $a \colon (\boldsymbol{a}, a) \to (\mathbf{1}, 1)$ is an $\mathbb{A}$-span map, whence $[\boldsymbol{a}, 1] \cdot [\mathbf{1}, a] = [\boldsymbol{a}, a] = 1$ so that $([\mathbf{1}, a], [\boldsymbol{a}, 1])$ is indeed a split epi in $\mathcal{C}[\mathbb{A}^*]$. Vertical functoriality of H follows from the easy fact that $[\boldsymbol{a}, 1] \cdot [\boldsymbol{b}, 1] = [\boldsymbol{a} \cdot \boldsymbol{b}, 1]$, and so it remains to prove that H is well-defined on squares.

To this end, let $(r, s) \colon \boldsymbol{a} \to \boldsymbol{b}$ in $\mathbb{A}$; we must show that $([\mathbf{1}, r], [\mathbf{1}, s])$ is a map of split epis $([\mathbf{1}, a], [\boldsymbol{a}, 1]) \to ([\mathbf{1}, b], [\boldsymbol{b}, 1])$, so that $[\mathbf{1}, r] \cdot [\boldsymbol{a}, 1] = [\boldsymbol{b}, 1] \cdot [\mathbf{1}, s] \colon B \to C$ in $\mathcal{C}[\mathbb{A}^*]$. The left-hand side composes to $[\boldsymbol{a}, r] \colon B \to C$ whilst the right-hand is the composite $[\mathbf{1} \cdot \boldsymbol{p}, 1 \cdot q] = [\boldsymbol{p}, q] \colon B \to C$ in



There is a unique $k \colon A \to B \times_D C$ as displayed with $p \cdot k = a$ and $q \cdot k = r$; and since the cartesian $(q, s) \colon \boldsymbol{p} \to \boldsymbol{b}$ detects squares and $(r, s) \colon \boldsymbol{a} \to \boldsymbol{b}$ is an $\mathbb{A}$-square, it follows that $(k, 1) \colon \boldsymbol{a} \to \boldsymbol{p}$ is an $\mathbb{A}$-square. Thus $k \colon (\boldsymbol{a}, r) \to (\boldsymbol{p}, q)$ is a morphism of $\mathbb{A}$-spans, and so $[\mathbf{1}, r] \cdot [\boldsymbol{a}, 1] = [\boldsymbol{a}, r] = [\boldsymbol{p}, q] = [\boldsymbol{b}, 1] \cdot [\mathbf{1}, s]$ as required. ☐

In particular, if (L, R) is an AWFS on a category $\mathcal{C}$ with pullbacks, then by applying this theorem we may construct the category $\mathbf{Wk}_\ell(\mathsf{L}, \mathsf{R})$ of left weak maps as $\mathcal{C}[\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}^*]$. If $\mathcal{C}$ happens also to have an initial object, then our original construction of $\mathbf{Wk}_\ell(\mathsf{L}, \mathsf{R})$ as $\mathbf{Kl}(\mathsf{Q})$ also applies—and we conclude that:

Corollary 12. *If (L, R) is an* AWFS *on a category $\mathcal{C}$ with an initial object and pullbacks, then there is a unique isomorphism $\mathbf{Kl}(\mathsf{Q}) \cong \mathcal{C}[\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}^*]$ commuting with the maps from $\mathcal{C}$.*

Explicitly, this isomorphism identifies a map $f \colon QA \to B$ in the Kleisli category with the equivalence class of $(\boldsymbol{R}!_A, f) \colon A \leftarrow QA \to B$.

4. P-split epis

Consider once again the isomorphisms (3.1) defining the category of left weak maps of an AWFS. If the category $\mathcal{D}$ therein has binary coproducts, then the double category $\mathbb{S}\mathbf{plEpi}(\mathcal{D})$ is the double category of algebras for an AWFS $\mathsf{SE}(\mathcal{D})$ on $\mathcal{D}$; whence by Lemma 8 and Theorem 3, we may rewrite (3.1) as an isomorphism

$$\mathbf{CAT}(\mathbf{Wk}_\ell(\mathsf{L}, \mathsf{R}), \mathcal{D}) \cong \mathbf{AWFS}_{\text{lax}}(\,(\mathsf{L}, \mathsf{R}), \mathsf{SE}(\mathcal{D})\,)\ .$$

It is natural to try and see these isomorphisms as the action on homs of an adjunction between categories and AWFS. However, there is an obstruction to doing so: the Kleisli categories in the image of the apparent left adjoint $\mathbf{Wk}_\ell(-)$ will typically not admit binary coproducts, while the apparent right adjoint $\mathsf{SE}(-)$ can only be defined for those categories which do so.

The reason for this difficulty is that we should not be constructing an adjunction between AWFS and categories, but rather one between AWFS and *comonads*. The left adjoint will send an AWFS (L, R) to its cofibrant replacement comonad, while the right adjoint will send a comonad P to the following P-*split epi* AWFS.

4.1. **P-split epis.** Given a comonad P on $\mathcal{C}$, a P-*split epi* is a map $g\colon A \to B$ equipped with a section $p\colon B \rightsquigarrow A$ of g in the Kleisli category $\mathbf{Kl}(\mathsf{P})$: thus, a map $p\colon PB \to A$ of $\mathcal{C}$ with $gp = \epsilon_B\colon PB \to B$. The P-split epis are the vertical arrows of a concrete double category over $\mathcal{C}$, most efficiently described as a pullback

$$\begin{array}{ccc} \mathsf{P}\text{-}\mathbb{S}\mathbf{plEpi}(\mathcal{C}) & \longrightarrow & \mathbb{S}\mathbf{plEpi}(\mathbf{Kl}(\mathsf{P})) \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle U} \\ \mathbb{S}\mathbf{q}(\mathcal{C}) & \xrightarrow[\mathbb{S}\mathbf{q}(G)]{} & \mathbb{S}\mathbf{q}(\mathbf{Kl}(\mathsf{P})) \end{array} \tag{4.1}$$

along the cofree functor $G\colon \mathcal{C} \to \mathbf{Kl}(\mathsf{P})$. As in Example 7 we see that V will exhibit $\mathsf{P}\text{-}\mathbb{S}\mathbf{plEpi}(\mathcal{C})$ as the double category of R-algebras for an AWFS on $\mathcal{C}$ so long as $V_1\colon \mathsf{P}\text{-}\mathbf{SplEpi}(\mathcal{C}) \to \mathcal{C}^{\mathbf{2}}$ has a left adjoint. From our explicit description of the P-split epis, it is easy to see that this will be so whenever $\mathcal{C}$ admits binary coproducts, with the unit of the free P-split epi on f being given as on the left in

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & A+PB \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \langle f,\epsilon_B\rangle}\ \uparrow{\scriptstyle \iota_{PB}} \\ B & \xrightarrow[1]{} & B \end{array} \qquad \begin{array}{ccccc} A & & \xrightarrow{\quad 1 \quad} & & A \\ {\scriptstyle \iota_A}\downarrow & & & & \downarrow{\scriptstyle \iota_A} \\ A+PB & \xrightarrow[1+\Delta_B]{} & A+PPB & \xrightarrow[1+P\iota_{PB}]{} & A+P(A+PB) \end{array} . \tag{4.2}$$

The comonad L for this AWFS thus sends f to $\iota_A\colon A \to A+PB$ and has counit $(1, \langle f, \epsilon_B\rangle)\colon \iota_A \to f$, while its comultiplication can be calculated according to Remark 4 as having f-component as on the right above. In particular, the P-split epi AWFS has cofibrant replacement comonad P.

Though we shall not need the general L-coalgebras here, a straightforward calculation shows that, whenever $A \in \mathcal{C}$ and $b\colon B \to PB$ is a P-coalgebra, we obtain L-coalgebra structure on $\iota_A\colon A \to A+B$; and that when $\mathcal{C}$ is a lextensive category and P preserves pullbacks along coproduct injections, every L-coalgebra arises in this way.

4.2. **The awfs–comonad adjunction.** We are now ready to construct the adjunction between AWFS and comonads alluded to above. We write $\mathbf{AWFS}^{+}_{\mathrm{lax}}$ for the full sub-2-category of $\mathbf{AWFS}_{\mathrm{lax}}$ on those AWFS whose underlying categories admits finite coproducts, and $\mathbf{CMD}^{+}_{\mathrm{lax}}$ for the corresponding 2-category of comonads.

Theorem 13. *There is a 2-adjunction*

$$\mathbf{CMD}^{+}_{\text{lax}} \underset{(-)\text{-}\textit{split epi}}{\overset{\textit{cofibrant replacement}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{AWFS}^{+}_{\text{lax}} \tag{4.3}$$

with invertible counit.

Proof. The assignation sending a comonad $(\mathcal{C}, \mathsf{P})$ to the functor $G^{\mathsf{P}} \colon \mathcal{C} \to \mathbf{Kl}(\mathsf{P})$ is the action on objects of a 2-functor $\mathbf{CMD}_{\text{lax}} \to \mathbf{CAT}^{\mathbf{2}}$. It follows that the assignation sending $(\mathcal{C}, \mathsf{P})$ to the left-hand arrow of (4.1) is the action on objects of a 2-functor $\mathbf{CMD}_{\text{lax}} \to \mathbf{DBL}^{\mathbf{2}}$. By the preceding argument and Theorem 3, the restriction of this 2-functor to $\mathbf{CMD}^{+}_{\text{lax}}$ factors through $\mathbf{AWFS}^{+}_{\text{lax}}$; this defines the right adjoint 2-functor.

We now show that, for any $(\mathcal{C}, \mathsf{L}, \mathsf{R}) \in \mathbf{AWFS}^{+}_{\text{lax}}$, the cofibrant replacement comonad $(\mathcal{C}, \mathsf{Q})$ provides the value at $(\mathcal{C}, \mathsf{L}, \mathsf{R})$ of a left adjoint to this 2-functor. Indeed, to give a morphism $(\mathcal{C}, \mathsf{Q}) \to (\mathcal{D}, \mathsf{P})$ in $\mathbf{CMD}^{+}_{\text{lax}}$ is to give a square as on the left in

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle G^{\mathsf{Q}}}\downarrow & & \downarrow{\scriptstyle G^{\mathsf{P}}} \\ \mathbf{Kl}(\mathsf{Q}) & \xrightarrow[\bar{F}]{} & \mathbf{Kl}(\mathsf{P}) \end{array} \qquad \begin{array}{ccc} \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} & \longrightarrow & \mathbb{S}\mathbf{plEpi}(\mathbf{Kl}(\mathsf{P})) \\ \downarrow & & \downarrow \\ \mathbb{S}\mathbf{q}(\mathcal{C}) & \xrightarrow[\mathbb{S}\mathbf{q}(G^{\mathsf{P}}F)]{} & \mathbb{S}\mathbf{q}(\mathbf{Kl}(\mathsf{P})) \end{array} \qquad \begin{array}{ccc} \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} & \longrightarrow & \mathsf{P}\text{-}\mathbb{S}\mathbf{plEpi}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathbb{S}\mathbf{q}(\mathcal{C}) & \xrightarrow[\mathbb{S}\mathbf{q}(F)]{} & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{array} .$$

By Theorem 10, this is equally to give a square as in the centre; and since (4.1) is a pullback, this is equivalent to giving a square as on the right. But this is equally to give a lax AWFS morphism $(\mathcal{C}, \mathsf{L}, \mathsf{R}) \to (\mathcal{D}, \mathsf{L}_{\mathsf{P}}, \mathsf{R}_{\mathsf{P}})$ into the P-split epi AWFS on $\mathcal{C}$, as required. The naturality of these bijections in $(\mathcal{D}, \mathsf{P})$ is straightforward; to obtain the bijection on 2-cells, replace $(\mathcal{D}, \mathsf{P})$ by $(\mathcal{D}^{\mathbf{2}}, \mathsf{P}^{\mathbf{2}})$ above. Finally, the invertibility of the counit is the fact that the cofibrant replacement comonad of the P-split epi AWFS is P. ☐

Clearly, the 2-adjunction of the previous proposition is fibred over $\mathbf{CAT}$; on restricting to the fibre over a fixed category $\mathcal{C}$ with finite coproducts, we obtain the following corollary, telling us that the category of comonads on $\mathcal{C}$ with finite coproducts embeds coreflectively into the category of AWFS on $\mathcal{C}$. Note the reversal of the adjoints, caused by the fact that $\mathbf{AWFS}(\mathcal{C})$ and $\mathbf{CMD}(\mathcal{C})$ embed *contravariantly* into $\mathbf{AWFS}^{+}_{\text{lax}}$ and $\mathbf{CMD}^{+}_{\text{lax}}$.

Corollary 14. *For any category $\mathcal{C}$ with finite coproducts, there is an adjunction*

$$\mathbf{AWFS}(\mathcal{C}) \underset{\textit{cofibrant replacement}}{\overset{(-)\text{-}\textit{split epi}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{CMD}(\mathcal{C})$$

with invertible unit.

This restricted form of the adjunction—in the dual monad theoretic form of Section 4.4 below—was first constructed in [15, Theorem 34].

4.3. **A universal property of the split epi AWFS.** The following result—a corollary of Theorem 13 above—can be seen as providing a justification as to *why* split epimorphisms and cofibrant objects are so closely connected. In its statement, we write $\mathbf{AWFS}_{\mathrm{cocts}}$ for the 2-category obtained by restricting $\mathbf{AWFS}_{\mathrm{oplax}}$ to AWFS on cocomplete, locally small categories and to cocontinuous oplax morphisms between them.

Corollary 15. *The* AWFS *for split epis on* **Set** *is the free cocomplete* AWFS *on a cofibrant object; by which we mean that it is a birepresentation for the* 2*-functor*

$$\mathbf{Cof}(\text{–})\colon \mathbf{AWFS}_{\mathrm{cocts}} \to \mathbf{CAT}\ ,$$

sending an AWFS (L,R) *to its category* Q-**Coalg** *of algebraically cofibrant objects.*

Proof. We may restrict (4.3) to locally small, cocomplete categories and to functors with chosen left adjoint, yielding a 2-adjunction $\mathbf{CMD}^{\mathrm{coc}}_{\mathrm{radj}} \leftrightarrows \mathbf{AWFS}^{\mathrm{coc}}_{\mathrm{radj}}$. The doctrinal adjunction of Section 2.1, restricted to locally small, cocomplete categories, yields $(\mathbf{AWFS}^{\mathrm{coc}}_{\mathrm{radj}})^{\mathrm{coop}} \cong \mathbf{AWFS}^{\mathrm{coc}}_{\mathrm{ladj}}$; similarly $(\mathbf{CMD}^{\mathrm{coc}}_{\mathrm{radj}})^{\mathrm{coop}} \cong \mathbf{CMD}^{\mathrm{coc}}_{\mathrm{ladj}}$ and so we obtain a 2-adjunction $\mathbf{AWFS}^{\mathrm{coc}}_{\mathrm{ladj}} \leftrightarrows \mathbf{CMD}^{\mathrm{coc}}_{\mathrm{ladj}}$ with the left and right 2-adjoints now *reversed*. In particular, we have isomorphisms of categories

$$\mathbf{AWFS}^{\mathrm{coc}}_{\mathrm{ladj}}(\,\mathsf{SE}(\mathbf{Set}), (\mathcal{C},\mathsf{L},\mathsf{R})) \cong \mathbf{CMD}^{\mathrm{coc}}_{\mathrm{ladj}}(\,(\mathbf{Set},1), (\mathcal{C},\mathsf{Q})\,)$$

2-natural in $(\mathcal{C},\mathsf{L},\mathsf{R})$. Since a functor $\mathbf{Set} \to \mathcal{C}$ into a locally small category admits a right adjoint just when it preserves colimits, the above isomorphisms give rise to equivalences

$$\mathbf{AWFS}_{\mathrm{cocts}}(\,\mathsf{SE}(\mathbf{Set}), (\mathcal{C},\mathsf{L},\mathsf{R})) \simeq \mathbf{CMD}_{\mathrm{cocts}}(\,(\mathbf{Set},1), (\mathcal{C},\mathsf{Q})\,)$$

pseudonatural in $(\mathcal{C},\mathsf{L},\mathsf{R})$. Now by [31, Theorems 7 and 12], the category on the right is 2-naturally isomorphic to $\mathbf{CAT}_{\mathrm{cocts}}(\mathbf{Set}, \mathsf{Q}\text{-}\mathbf{Coalg})$—noting that Q-**Coalg** is cocomplete, since $\mathcal{C}$ is so and the forgetful functor creates colimits. Now as **Set** is the free cocomplete category on the object 1, the functor $\mathbf{CAT}_{\mathrm{cocts}}(\mathbf{Set}, \mathsf{Q}\text{-}\mathbf{Coalg}) \to \mathsf{Q}\text{-}\mathbf{Coalg}$ given by evaluation at 1 is an equivalence; combining with the previous equivalences yields the required pseudonatural equivalence $\mathbf{AWFS}_{\mathrm{cocts}}(\,\mathsf{SE}(\mathbf{Set}), (\mathcal{C},\mathsf{L},\mathsf{R})\,) \simeq \mathsf{Q}\text{-}\mathbf{Coalg}$. □

4.4. **T-split monos and sketches.** By dualising the preceding results, we obtain the AWFS for T-*split monos* associated to any monad T on a category $\mathcal{C}$ with binary products. Its coalgebras, the T-split monos, are maps $f\colon A \to B$ equipped with a Kleisli retraction $B \rightsquigarrow A$; and if $\mathcal{C}$ also has a terminal object, then its algebraically fibrant objects are the T-algebras. Dualising Theorem 13, we obtain an adjunction as on the left in:

$$\mathbf{MND}^{\times}_{\mathrm{oplax}} \underset{(\text{–})\text{-}\textit{split mono}}{\overset{(\mathcal{C},\mathsf{L},\mathsf{R})\mapsto(\mathcal{C},\mathsf{F})}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{AWFS}^{\times}_{\mathrm{oplax}} \qquad\qquad \mathbf{AWFS}(\mathcal{C}) \underset{(\text{–})\text{-}\textit{split mono}}{\overset{(\mathsf{L},\mathsf{R})\mapsto\mathsf{F}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{Mnd}(\mathcal{C})$$

with invertible counit; while restricting to a fixed category $\mathcal{C}$ with finite products, we obtain an adjunction as on the right, also with invertible counit.

We spell out this dual case primarily in order to highlight a connection between the AWFS for T-split monos and the *sketches* of [22]. Given a finitary monad T on a locally presentable category $\mathcal{C}$, [22] defines a T-*sketch* to be given by:

(i) A small family of 4-tuples $\mathcal{D} = (c_i, d_i, j_i \colon c_i \to d_i, k_i \colon d_i \to Tc_i)$ with each c_i and d_i finitely presentable and with $k_i j_i = \eta_{c_i}$ for each i;
(ii) An object X of $\mathcal{C}$ and a $\mathcal{D}$-indexed family of maps $\varphi_i \colon d_i \to X$;

and a (strict) *model* of this sketch in a T-algebra (A, a) to be a map $f \colon X \to A$ rendering commutative each square on the left in

$$\begin{array}{ccc} d_i & \xrightarrow{k_i} & Tc_i \\ {\scriptstyle \varphi_i}\downarrow & & \downarrow{\scriptstyle a\cdot T(f\varphi_i j_i)} \\ X & \xrightarrow[f]{} & A \end{array} \qquad\qquad \begin{array}{ccc} c_i & \xrightarrow{\psi_i} & X \\ {\scriptstyle \boldsymbol{j}_i}\downarrow & \nearrow{\scriptstyle \varphi_i} & \\ d_i & & \end{array} . \tag{4.4}$$

Now, it is immediate that the family $\mathcal{D}$ of 4-tuples in the definition of sketch is nothing other than a family of T-split monos $(j_i, k_i) \colon c_i \to d_i$. So we can see a T-sketch as being given by an object X and a family $\mathcal{D}$ of commuting triangles as to the right above wherein j_i bears T-split mono structure.

What about models for sketches? Note that a special case of the lifting property of any AWFS is that each map $c \to A$ into an algebraically fibrant object admits a canonical extension along any L-map $c \to d$. Since T is the fibrant replacement monad of the T-split mono AWFS, this means in particular that for any diagram as on the left in

$$\begin{array}{ccc} c & \xrightarrow{h} & \boldsymbol{A} \\ {\scriptstyle \boldsymbol{j}}\downarrow & \nearrow{\scriptstyle \bar h} & \\ d & & \end{array} \qquad\qquad \begin{array}{ccccc} c_i & \xrightarrow{\psi_i} & X & \xrightarrow{f} & \boldsymbol{A} \\ {\scriptstyle \boldsymbol{j}_i}\downarrow & \nearrow{\scriptstyle \varphi_i} & & & \\ d_i & & & & \end{array}$$

wherein $\boldsymbol{j} = (j, k)$ is a T-split mono and $\boldsymbol{A} = (A, a)$ is a T-algebra, there is a canonical filler $\bar h$; explicitly, we may calculate that $\bar h = a.Th.k$. Comparing with the definition of model given above, we see that a model for a sketch $(X, \mathcal{D})$ in a T-algebra $\boldsymbol{A} = (A, a)$ is a map $f \colon X \to A$ such that, for each triangle in $\mathcal{D}$ the composite triangle right above is a canonical lifting triangle.

This suggests the following general definition. Given an AWFS (L, R) on a category $\mathcal{C}$ with terminal object, an (L, R)-*sketch* is given by an object $X \in \mathcal{C}$ together with a family $\mathcal{D}$ of triangles as to the right of (4.4) wherein $\boldsymbol{j}_i$ is equipped with L-map structure. A *model* for a sketch $(X, \mathcal{D})$ in an algebraically fibrant object $\boldsymbol{A}$ is a morphism $f \colon X \to A$, composition with which sends chosen triangles in $\mathcal{D}$ to canonical (L, R)-lifting triangles. The sketches of [22] are then the specialisation of this definition to the T-split mono AWFS. A key result in [22] is one assuring the existence of initial models for sketches, and it not hard to see that these arguments may be carried out for sketches relative to any accessible AWFS on a locally presentable category.

5. Two dimensional monad theory

We conclude this paper by examining two applications of the theory of weak maps. The first is to the theory of 2-*monads.* If T is a 2-monad on a 2-category $\mathcal{C}$, then in addition to the usual Eilenberg–Moore 2-category $\mathsf{T}\text{-}\mathbf{Alg}_s$, one also has the 2-categories $\mathsf{T}\text{-}\mathbf{Alg}_l$ and $\mathsf{T}\text{-}\mathbf{Alg}_p$, whose objects are again the T-algebras, but whose maps are the *lax* or *pseudo* algebra morphisms $(f, \varphi)\colon (A, a) \to (B, b)$. A lax morphism involves a 1-cell $f\colon A \to B$ and a 2-cell $\varphi\colon b \cdot Tf \Rightarrow f \cdot a\colon TA \to B$ satisfying two coherence axioms [21, §3.5]; a pseudomorphism is the same but with φ invertible. Our objective in this section, as presaged in Examples 9 above, will be to exhibit $\mathsf{T}\text{-}\mathbf{Alg}_l$ and $\mathsf{T}\text{-}\mathbf{Alg}_p$ as the categories of left weak maps for suitable AWFS on $\mathsf{T}\text{-}\mathbf{Alg}_s$. Note we are being slightly loose here, since the theory of weak maps operates at the level of mere categories while $\mathsf{T}\text{-}\mathbf{Alg}_s$, $\mathsf{T}\text{-}\mathbf{Alg}_p$ and $\mathsf{T}\text{-}\mathbf{Alg}_l$ are in fact 2-categories. To capture the two-dimensional structure would require the theory of enrichment over the *monoidal* AWFS of [29]; as these are beyond our present scope, we will consider $\mathsf{T}\text{-}\mathbf{Alg}_s$, $\mathsf{T}\text{-}\mathbf{Alg}_p$ and $\mathsf{T}\text{-}\mathbf{Alg}_l$ as mere 1-categories and proceed accordingly.

Before continuing, let us note that the result we are aiming for allows us to reconstruct the main Theorem 3.13 of [4]; this states that for an accessible 2-monad T on a complete and cocomplete 2-category $\mathcal{C}$, the inclusion functors

$$I\colon \mathsf{T}\text{-}\mathbf{Alg}_s \to \mathsf{T}\text{-}\mathbf{Alg}_l \qquad \text{and} \qquad J\colon \mathsf{T}\text{-}\mathbf{Alg}_s \to \mathsf{T}\text{-}\mathbf{Alg}_p \tag{5.1}$$

have left adjoints.[4] In fact, this result is now a triviality: the inclusions (5.1) may be identified with the cofree functors into the categories of weak maps, so their left adjoints are given simply by the corresponding forgetful functors.

We suppose for the remainder of this section that T is an accessible monad on a complete and cocomplete 2-category $\mathcal{C}$. Since $\mathcal{C}$ is cocomplete, we may form the AWFS for lalis on the underlying category $\mathcal{C}_0$. Note that its underlying monad R, being induced as in Example 6 by left Kan extension and restriction, is cocontinuous. Moreover, the underlying ordinary monad T_0 is accessible since T is so, whence by [6, Proposition 14] the AWFS for lalis admits a projective lifting along $U\colon \mathsf{T}\text{-}\mathbf{Alg}_s \to \mathcal{C}_0$, which we term the AWFS for *U-lalis* on $\mathsf{T}\text{-}\mathbf{Alg}_s$. By starting instead from the AWFS for retract equivalences on $\mathcal{C}_0$ we obtain the lifted AWFS for *U-retract equivalences* on $\mathsf{T}\text{-}\mathbf{Alg}_s$; the underlying weak factorisation system of this latter AWFS was constructed in [24, §4.4].

As $\mathcal{C}$ admits an initial object, so too does $\mathsf{T}\text{-}\mathbf{Alg}_s$, and so we may form the category of left weak maps of any AWFS thereon. We will prove that, in the case of U-lalis or U-retract equivalences the inclusion $\mathsf{T}\text{-}\mathbf{Alg}_s \to \mathbf{Wk}_\ell(\mathsf{L}, \mathsf{R})$ may be identified with the inclusion of $\mathsf{T}\text{-}\mathbf{Alg}_s$ into $\mathsf{T}\text{-}\mathbf{Alg}_l$ or $\mathsf{T}\text{-}\mathbf{Alg}_p$; then the latter inclusions will have left adjoints since the former ones do.

Consider first the case of U-lalis. If $f\colon A \to B$ in $\mathsf{T}\text{-}\mathbf{Alg}_s$ bears U-lali structure—meaning that it comes equipped with a right adjoint section $p\colon UB \to UA$ of Uf in $\mathcal{C}$—then by the doctrinal adjunction of [20, Proposition 1.3], p bears a *unique* structure of lax T-algebra morphism $B \rightsquigarrow A$ making it into a right adjoint section of f in $\mathsf{T}\text{-}\mathbf{Alg}_l$. Using the unicity of the lax structure, it is easy to see that this assignation yields a pullback of double categories as in the left square

[4]In [4], $\mathsf{T}\text{-}\mathbf{Alg}_l$, $\mathsf{T}\text{-}\mathbf{Alg}_p$ and $\mathsf{T}\text{-}\mathbf{Alg}_s$ are considered as 2-categories, and I and J as 2-functors, which are shown to have left 2-adjoints; but as they explain, this two-dimensional aspect of the result is easily deduced from the one-dimensional one in the presence of cotensor products in $\mathcal{C}$.

below

$$\begin{array}{ccccc} U\text{-}\mathbb{L}\mathbf{ali}(\mathsf{T}\text{-}\mathbf{Alg}_s) & \longrightarrow & \mathbb{L}\mathbf{ali}(\mathsf{T}\text{-}\mathbf{Alg}_l) & \longrightarrow & \mathbb{S}\mathbf{plEpi}(\mathsf{T}\text{-}\mathbf{Alg}_l) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{S}\mathbf{q}(\mathsf{T}\text{-}\mathbf{Alg}_s) & \xrightarrow[\mathbb{S}\mathbf{q}(J)]{} & \mathbb{S}\mathbf{q}(\mathsf{T}\text{-}\mathbf{Alg}_l) & \xrightarrow[\mathbb{S}\mathbf{q}(1)]{} & \mathbb{S}\mathbf{q}(\mathsf{T}\text{-}\mathbf{Alg}_l)\ . \end{array}$$

By Theorem 10 the composite $K\colon U\text{-}\mathbb{L}\mathbf{ali}(\mathsf{T}\text{-}\mathbf{Alg}_s) \to \mathbb{S}\mathbf{plEpi}(\mathsf{T}\text{-}\mathbf{Alg}_l)$ induces an extension of $J\colon \mathsf{T}\text{-}\mathbf{Alg}_s \to \mathsf{T}\text{-}\mathbf{Alg}_l$ to a functor $\bar{J}\colon \mathbf{Wk}_\ell(\mathsf{L},\mathsf{R}) \to \mathsf{T}\text{-}\mathbf{Alg}_l$, defined as follows. For each $A \in \mathsf{T}\text{-}\mathbf{Alg}_s$, form the free U-lali $\boldsymbol{R}!_A\colon QA \to A$; applying K yields a splitting $q_A\colon A \rightsquigarrow QA$ for ϵ_A in $\mathsf{T}\text{-}\mathbf{Alg}_l$; and now $\bar{J}\colon \mathbf{Wk}_\ell(\mathsf{L},\mathsf{R}) \to \mathsf{T}\text{-}\mathbf{Alg}_l$ is the identity on objects, and on morphisms sends $f\colon QA \to B$ to $f \cdot q_A\colon A \rightsquigarrow B$. To complete the proof, it suffices to show that $\bar{J}$ is an isomorphism; of course, it is bijective on objects, and so it suffices to exhibit an inverse for each function $(-)\cdot q_A\colon \mathsf{T}\text{-}\mathbf{Alg}_s(QA,B) \to \mathsf{T}\text{-}\mathbf{Alg}_l(A,B)$.

So consider a lax morphism $f\colon A \rightsquigarrow B$. Since $\mathcal{C}$ is complete, [23, Theorem 3.2] assures us that the arrow f admits an oplax limit in $\mathsf{T}\text{-}\mathbf{Alg}_l$, as to the left in:

$$\begin{array}{ccc} & B/f & \\ {}^{u}\swarrow & \overset{\Leftarrow}{\lambda} & \searrow^{v} \\ A & \underset{f}{\rightsquigarrow} & B \end{array} \qquad\qquad \begin{array}{ccc} & A & \\ {}^{1}\swarrow & & \rightsquigarrow^{f} \\ A & \underset{f}{\rightsquigarrow} & B\ , \end{array}$$

whose projections are strict morphisms that *jointly detect strictness*—in the sense that any lax morphism $c\colon C \rightsquigarrow B/f$ with uc and vc strict, is itself strict.

Applying the universal property of B/f to the cone to the right above, we induce a unique lax morphism $r\colon A \rightsquigarrow B/f$ with $ur = 1$, $\lambda r = 1$ and $vr = f$. In fact, using the two-dimensional aspect of the universal property, we see that $u \dashv r$ is a lali in $\mathsf{T}\text{-}\mathbf{Alg}_l$, so that $\boldsymbol{u} = (u, Ur)\colon B/f \to A$ is a U-lali in $\mathsf{T}\text{-}\mathbf{Alg}_s$. The map $(!_{B/f}, 1_B)\colon !_A \to u$ in $\mathcal{C}^{\mathbf{2}}$ thus induces a map of U-lalis as on the left in:

$$\begin{array}{ccc} QA & \xrightarrow{k} & B/f \\ {\scriptstyle \boldsymbol{R}!_A}\downarrow & & \downarrow{\scriptstyle \boldsymbol{u}} \\ A & \xrightarrow{1} & A \end{array} \qquad\qquad \begin{array}{ccccc} QA & \xrightarrow{k} & B/f & \xrightarrow{v} & B\ , \\ {\scriptstyle q_A}\,\wr\!\uparrow & & \uparrow\!\wr\,{\scriptstyle r} & \nearrow_{f} & \\ A & \xrightarrow{1} & A & & \end{array}$$

applying K to which yields a morphism of split epis in $\mathsf{T}\text{-}\mathbf{Alg}_l$; in particular the diagram above right commutes, and so $vk\colon QA \to B$ is a strict map with $(vk)q_A = f$ as required. It remains to show unicity of vk: so given $s\colon QA \to B$ with $sq_A = f$, we must show that $s = vk$. Writing η_A for the unit of the adjunction $\epsilon_A \dashv q_A$ in $\mathsf{T}\text{-}\mathbf{Alg}_l$, we have $s\eta_A\colon s \Rightarrow sq_A\epsilon_A = f\epsilon_A$, so that we have a cone in $\mathsf{T}\text{-}\mathbf{Alg}_l$ as on the left in:

$$\begin{array}{ccc} & QA & \\ {}^{\epsilon_A}\swarrow & \overset{\Leftarrow}{s\eta_A} & \searrow^{s} \\ A & \underset{f}{\rightsquigarrow} & B \end{array} \qquad\qquad \begin{array}{ccc} QA & \xrightarrow{\bar{s}} & B/f \\ {\scriptstyle \boldsymbol{R}!_A}\downarrow & & \downarrow{\scriptstyle \boldsymbol{u}_f} \\ A & \xrightarrow{1} & A\ . \end{array}$$

As both projections are strict, there is a unique strict factorisation $\overline{s}\colon QA \to B/f$ with $u\overline{s} = p_A$ and $\lambda\overline{s} = s\eta_A$ and $v\overline{s} = s$. Using the universal property of B/f, it is easy to show that $(\overline{s}, 1)$ is a morphism of lalis $\boldsymbol{R}!_A \to \boldsymbol{u}$ as on the right above, whence by freeness of $\boldsymbol{R}!_A$, we have $\overline{s} = k$ and so $s = v\overline{s} = vk$ as required. Thus we have shown:

Theorem 16. *Let* T *be an accessible* 2*-monad on a complete and cocomplete* 2*-category. The category* $\mathsf{T}\text{-}\mathbf{Alg}_l$ *of algebras and lax morphisms is equally the category of left weak maps for the* AWFS *for* U*-lalis; in particular, the inclusion* $\mathsf{T}\text{-}\mathbf{Alg}_s \to \mathsf{T}\text{-}\mathbf{Alg}_l$ *has a left adjoint with counit a lali in* $\mathsf{T}\text{-}\mathbf{Alg}_l$*.*

Repeating this argument with U-retract equivalences in place of U-lalis allows us to identify the left weak maps for the lifted AWFS with the category $\mathsf{T}\text{-}\mathbf{Alg}_p$ of T-algebra pseudomorphisms, and so to deduce the existence of a left adjoint to $\mathsf{T}\text{-}\mathbf{Alg}_s \to \mathsf{T}\text{-}\mathbf{Alg}_p$. This pseudo case should be contrasted with [24, Theorem 4.12]: whereas our result shows that the cofibrant replacement comonad of the U-retract equivalence AWFS gives rise to the adjunction between strict and pseudo algebra maps, Theorem 4.12 of *ibid.* starts by assuming the adjunction between strict and pseudo maps, and deduces that the induced comonad provides a notion of cofibrant replacement for the U-retract equivalence AWFS.

6. DG-enriched monad theory

Our second application of the theory of weak maps will be to the description of *homotopy-coherent* morphisms between algebras for a dg-monad. First let us recall some basic definitions. Fixing a commutative ring R, we write $\mathbf{DG}$ for the category of unbounded chain complexes of R-modules

$$\cdots \xrightarrow{\partial} X_1 \xrightarrow{\partial} X_0 \xrightarrow{\partial} X_{-1} \xrightarrow{\partial} \cdots$$

This has a symmetric monoidal structure, whose unit I satisfies $I_0 = R$ and $I_k = 0$ for $k \neq 0$, whose binary tensor is defined by

$$(X \otimes Y)_n = \textstyle\sum_{p+q=n} X_p \otimes Y_q \quad \text{and} \quad \partial(x \otimes y) = \partial x \otimes y + (-1)^{\deg(x)} x \otimes \partial y \ ,$$

and whose symmetry $\sigma\colon X \otimes Y \to Y \otimes X$ satisfies $\sigma(x \otimes y) = (-1)^{\deg(x)\deg(y)} y \otimes x$. A *dg-category* [19] is a category enriched in $\mathbf{DG}$; it thus has R-modules of maps $\mathcal{C}(A,B)_n$ between any two objects, whose elements we write as $f\colon A \to_n B$ and call *graded maps of degree* n. Graded maps have a bilinear composition which adds degrees, and a differential ∂ such that $\partial(gf) = \partial g \cdot f + (-1)^{\deg(g)} g \cdot \partial f$ and $\partial(1_A) = 0$. Note that maps in the underlying ordinary category of $\mathcal{C}$ are graded maps of degree 0 with *zero* differential; we call such maps *chain maps* and write them as $f\colon A \to B$ with no subscript.

6.1. **Homotopy-coherent maps.** For the rest of this section, we suppose that T is an accessible dg-enriched monad on a cocomplete dg-category $\mathcal{C}$. We have, of course, the dg-category $\mathsf{T}\text{-}\mathbf{Alg}_s$ of T-algebras and *strict* maps; its object are T-algebras, and its graded maps $f\colon (A,a) \to_i (B,b)$ are maps $f\colon A \to_i B$ in $\mathcal{C}$ such that $b \cdot Tf = f \cdot a$. However, we may also define a dg-category $\mathsf{T}\text{-}\mathbf{Alg}_w$ of *homotopy-coherent maps*: its objects are again T-algebras, while its graded maps

$f\colon A \rightsquigarrow_i B$ are families $(f_n\colon T^n A \to_{n+i} B)_{n\in\mathbb{N}}$ of graded maps in $\mathcal{C}$ such that $f_n \cdot T^j \eta T^{n-j-1} = 0$ for all $0 \leqslant j \leqslant n-1$. The differential of f is the family

$$(\partial f)_n = (\partial f_n) - (-1)^i [b \cdot T f_{n\text{-}1} + f_{n\text{-}1} \sum_{j=1}^{n-1} (-1)^j T^{j\text{-}1} \mu_{T^{n\text{-}j\text{-}1}A} + (-1)^n f_{n\text{-}1} \cdot T^{n\text{-}1} a] \ .$$

The identity $A \rightsquigarrow_0 A$ has components $(1, 0, 0, \dots)$, while the composite of $f\colon A \rightsquigarrow_i B$ with $g\colon B \rightsquigarrow_k C$ is given by

$$(gf)_n = \textstyle\sum_{p+q=n} (-1)^{pi} g_p \cdot T^p f_q \ .$$

There is an evident forgetful dg-functor $\mathsf{T}\text{-}\mathbf{Alg}_w \to \mathcal{C}$ sending $f\colon (A,a) \rightsquigarrow_i (B,b)$ to $f_0\colon A \to_i B$, and an inclusion $J\colon \mathsf{T}\text{-}\mathbf{Alg}_s \to \mathsf{T}\text{-}\mathbf{Alg}_w$ which is the identity on objects and sends f to $(f, 0, 0, 0, \dots)$.

Example 17. Let $\mathcal{C}$ and $\mathcal{D}$ be dg-categories with $\mathcal{C}$ small and $\mathcal{D}$ cocomplete. We may identify the dg-functor category $[\mathcal{C}, \mathcal{D}]$ with the Eilenberg–Moore category $\mathsf{T}\text{-}\mathbf{Alg}_s$ for the dg-monad T on $[\mathrm{ob}\,\mathcal{C}, \mathcal{D}]$ induced by left Kan extension and restriction along the inclusion $\mathrm{ob}\,\mathcal{C} \to \mathcal{C}$. In this case, the dg-category $\mathsf{T}\text{-}\mathbf{Alg}_w$ has dg-functors as objects and hom-objects given by the complexes of *homotopy-coherent transformations* from F to G as defined in [32, §3.1], for example.

6.2. **Homological lalis and U-lalis.** Our objective is now to show that, under suitable assumptions on T, we may obtain the underlying ordinary category of $\mathsf{T}\text{-}\mathbf{Alg}_w$ as the category of weak maps associated to an AWFS on $\mathsf{T}\text{-}\mathbf{Alg}_s$. The AWFS in question will be constructed using the following analogue of the 2-categorical lalis of Example 2.

Definition 18. A *homological lali* in a dg-category $\mathcal{A}$ is a chain map $g\colon A \to B$ together with a section $p\colon B \to A$ and a graded map $\xi\colon A \to_1 A$ with $\partial\xi = 1 - pg$ and $g\xi = \xi p = \xi\xi = 0$. Homological lalis on $\mathcal{C}$ form a category $\mathbf{Lali}(\mathcal{A})$, wherein a morphism $(u,v)\colon (g,p,\xi) \to (g',p',\xi')$ is a commuting square of chain maps $(u,v)\colon g \to g'$ such that $up = p'v$ and $u\xi = \xi' u$. Homological lalis compose according to the formula

$$A \xrightarrow{(g',p',\xi')} B \xrightarrow{(g,p,\xi)} C \qquad \mapsto \qquad A \xrightarrow{(gg',p'p,\xi' + p'\xi g')} C$$

and so we obtain a double category $\mathbb{L}\mathbf{ali}(\mathcal{A}) \to \mathbb{S}\mathbf{q}(\mathcal{A}_0)$ of lalis which is concrete and right-connected over $\mathcal{A}_0$ via the double functor sending (g,p,ξ) to g.

Arguing as in Example 6, the homological lalis will comprise the right class of an AWFS in any sufficiently cocomplete dg-category $\mathcal{A}$; in this case, the colimits required are those for *mapping cylinders* [34, §1.5.5]. The underlying weak factorisation system of this AWFS is the (cofibration, trivial fibration) part of a model structure on $\mathcal{A}$, constructed in [8, Theorem 2.2], and there called the *relative model structure* for the trivial projective class.

In particular, the cocomplete dg-category $\mathcal{C}$ we are considering admits the AWFS for homological lalis; as the dg-monad T thereon is accessible, we may argue as in the preceding section to projectively lift this AWFS to one on the underlying category of $\mathsf{T}\text{-}\mathbf{Alg}_s$, which as before we call the AWFS for *U-lalis.* As $\mathcal{C}$ admits an initial object, so too does $\mathsf{T}\text{-}\mathbf{Alg}_s$, and so we may form the category $\mathbf{Wk}_\ell(\mathsf{L}, \mathsf{R})$

of left weak maps associated to the AWFS for U-lalis. In the remainder of this section, we will prove the following result; the side condition on preservation of codescent objects will be explained shortly.

Theorem 19. *Let* T *be an accessible dg-monad on a cocomplete dg-category. If* T *preserves codescent objects, then the underlying category of* $\mathsf{T}\text{-}\mathbf{Alg}_w$ *is equally the category of left weak maps for the* AWFS *for* U*-lalis; in particular, the inclusion* $\mathsf{T}\text{-}\mathbf{Alg}_s \to \mathsf{T}\text{-}\mathbf{Alg}_w$ *has a left adjoint with counit a lali in* $\mathsf{T}\text{-}\mathbf{Alg}_w$.

A key step in proving this is the following lemma, which is an analogue of the doctrinal adjunction [20] we used in the preceding section.

Lemma 20. *If* $(g, f_0, \epsilon_0)\colon (B, b) \to (A, a)$ *is a* U*-lali in* $\mathsf{T}\text{-}\mathbf{Alg}_s$*, then there is a unique lali* (g, f, ϵ) *in* $\mathsf{T}\text{-}\mathbf{Alg}_w$ *with* $Uf = f_0$ *and* $U\epsilon = \epsilon_0$ *and* $\epsilon_0 f_k = \epsilon_0\epsilon_k = 0$ *for all* k.

Proof. The zero components of f and ϵ are f_0 and ϵ_0, and for $n > 0$ we take

$$f_n = \epsilon_0 b \cdot Tf_{n-1} \qquad \text{and} \qquad \epsilon_n = -\epsilon_0 b \cdot Te_{n-1} \ .$$

A short calculation shows that (g, f, ϵ) is indeed a lali with $Uf = f_0$ and $U\epsilon = \epsilon_0$ and $\epsilon_0 f_k = \epsilon_0\epsilon_k = 0$ for all k. Suppose now that (f', ϵ') also satisfies these conditions; we show by induction on n that $f'_n = f_n$ and $\epsilon'_n = \epsilon_n$ for all n. The case $n = 0$ is clear. For the inductive step, assume the result for all $m < n$. It is easy to see that $\epsilon_p \cdot T^p f_q = 0$ for all p and q, and so

$$0 = (\epsilon' f)_n = \textstyle\sum_{p+q=n} \epsilon'_p \cdot T^p f_q = \epsilon'_n \cdot T^n f_0 \ .$$

We now have that $-f'_n = -f'_n \cdot T^n g \cdot T^n f_0 = (1 - f'g)_n \cdot T^n f_0 = (\mathbf{d}\epsilon')_n \cdot T^n f_0$; and so that

$$\begin{aligned} -f'_n = (\mathbf{d}\epsilon')_n \cdot T^n f_0 &= \mathbf{d}(\epsilon'_n) T^n f_0 + b \cdot T\epsilon_{n-1} \cdot T^n f_0 - \epsilon_{n-1} \textstyle\sum_{j=0}^{n-1} (-1)^j d_j T^n f_0 \\ &= -\epsilon_{n-1} \textstyle\sum_{j=0}^{n-2} (-1)^j T^{n-1} f_0 \cdot d_j + (-1)^n \epsilon_{n-1} \cdot T^{n-1} b \cdot T^n f_0 \\ &= (-1)^n \epsilon_{n-1} T^{n-1}(b \cdot Tf_0) = -f_n \ ; \end{aligned}$$

the calculation that $\epsilon'_n = \epsilon_n$ is identical in form. □

It follows easily from the existence and uniqueness established in this result that there is a lifting of the inclusion $J\colon \mathsf{T}\text{-}\mathbf{Alg}_s \to \mathsf{T}\text{-}\mathbf{Alg}_w$ to a double functor as on the left in:

$$\begin{array}{ccccc} U\text{-}\mathbb{L}\mathbf{ali}(\mathsf{T}\text{-}\mathbf{Alg}_s) & \longrightarrow & \mathbb{L}\mathbf{ali}(\mathsf{T}\text{-}\mathbf{Alg}_w) & \longrightarrow & \mathbb{S}\mathbf{plEpi}(\mathsf{T}\text{-}\mathbf{Alg}_w) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{S}\mathbf{q}(\mathsf{T}\text{-}\mathbf{Alg}_s) & \xrightarrow[\mathbb{S}\mathbf{q}(J)]{} & \mathbb{S}\mathbf{q}(\mathsf{T}\text{-}\mathbf{Alg}_w) & \xrightarrow[\mathbb{S}\mathbf{q}(1)]{} & \mathbb{S}\mathbf{q}(\mathsf{T}\text{-}\mathbf{Alg}_w) \ . \end{array}$$

So by Theorem 10, the composite $U\text{-}\mathbb{L}\mathbf{ali}(T\text{-}\mathbf{Alg}_s) \to \mathbb{S}\mathbf{plEpi}(T\text{-}\mathbf{Alg}_w)$ induces an extension of $J\colon \mathsf{T}\text{-}\mathbf{Alg}_s \to \mathsf{T}\text{-}\mathbf{Alg}_w$ to a functor

$$\bar{J}\colon \mathbf{Wk}_\ell(\mathsf{L}, \mathsf{R}) \to \mathsf{T}\text{-}\mathbf{Alg}_w \ . \tag{6.1}$$

To complete the proof of Theorem 19, it remains to show that this functor is an isomorphism. In the two-dimensional case, we did this using the fact that oplax limits of morphisms lift along the forgetful 2-functor $\mathsf{T}\text{-}\mathbf{Alg}_l \to \mathcal{C}$. The analogous limit in the dg-enriched case is the *mapping path space*, but unfortunately, it does not appear to be true that these limits lift along the forgetful dg-functor $\mathsf{T}\text{-}\mathbf{Alg}_w \to \mathcal{C}$. We are therefore forced to take a different approach: we will compute the free U-lali $\boldsymbol{R}!_A \colon QA \to A$ explicitly, and use this to show directly that (6.1) is an isomorphism. It turns out that this free U-lali is obtained precisely by the familiar *bar construction* of [7].

6.3. **Codescent objects and homological lalis.** Let $\mathbf{\Delta} \colon \Delta \to \mathbf{DG}$ be the functor sending $[n]$ to the standard homological n-simplex

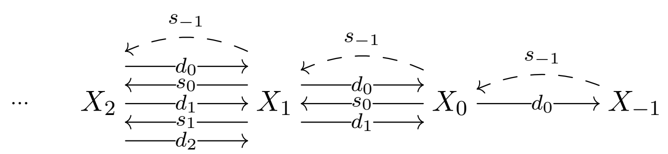
$$\mathbf{\Delta}[n]_k = \bigoplus_{f \colon [k] \rightarrowtail [n]} R \qquad \text{with} \qquad \partial(f) = \textstyle\sum_{j=0}^{n} (-1)^j f\delta_i \ .$$

By the *codescent object* of a simplicial diagram $X \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ in a dg-category, we mean the weighted colimit $|X| = \mathbf{\Delta} \star X$. The colimiting cocone comprises graded maps $\iota_n \colon X_n \to_n |X|$ such that $\iota_n s_j = 0$ for all $0 \leqslant j < n$ and such that

$$\partial(\iota_0) = 0 \qquad \text{and} \qquad \partial(\iota_n) = \iota_{n-1} \textstyle\sum_{j=0}^{n} (-1)^j d_j \text{ for } n > 0 \ ;$$

composition with these maps induces a bijection between maps $f \colon |X| \to_i B$ in $\mathcal{C}$, and families of maps $f_n \colon X_n \to_{n+i} B$ such that $f_n s_j = 0$ for all $0 \leqslant j < n$. If X is an *augmented* simplicial object—so equipped with a map $d_0 \colon X_0 \to X_{-1}$ satisfying $d_0 d_0 = d_0 d_1$—then there is an induced chain map $q \colon |X| \to X_{-1}$ characterised by $q\iota_0 = d_0$ and $q\iota_j = 0$ for $j > 0$.

There are essentially well-known conditions under which this induced q will be a homological lali. By a *contraction* [12, §III.5] on an augmented simplicial object X, we mean maps s_{-1} as in

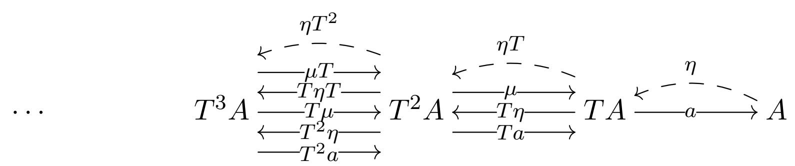


satisfying $d_0 s_{-1} = 1$ and $d_{i+1} s_{-1} = s_{-1} d_i$ and $s_{j+1} s_{-1} = s_{-1} s_j$. Such a contraction induces a structure of homological lali on the comparison $q \colon |X| \to X_{-1}$; indeed, q has the section $p = \iota_0 s_{-1} \colon X_{-1} \to X_0 \to |X|$, and we define $\xi \colon |X| \to_1 |X|$ to be the unique graded map with

$$\xi \iota_n = \iota_{n+1} s_{-1} \colon X_n \to X_{n+1} \to_{n+1} |X| \ .$$

Straightforward calculation now shows that (q, p, ξ) is a homological lali.

6.4. **Codescent objects and weak maps.** Using the preceding result, we may now give an explicit construction of the free U-lali $\boldsymbol{R}!_A \colon QA \to A$ for a dg-monad T which preserves codescent objects. Given $a \colon TA \to A$ a T-algebra, we consider in $\mathsf{T}\text{-}\mathbf{Alg}_s$ its *bar complex*, the augmented simplicial object $A_\bullet$ as in the solid part of



where each vertex except for the rightmost one bears its free algebra structure. We will continue to use s_j and d_j to denote the face and degeneracy maps; explicitly we have that:

$$s_j \colon T^nA \to T^{n+1}A = T^{j+1}\eta_{T_{n-j-1}A} \quad \text{for } -1 \leqslant j < n;$$

$$\text{and} \qquad d_j \colon T^{n+1}A \to T^n = \begin{cases} T^j\mu_{T^{n-j-1}A} & \text{for } 0 \leqslant j < n; \\ T^n a & \text{for } j = n. \end{cases}$$

note that we have $Ts_i = s_{i+1}$ and $Td_i = d_{i+1}$. Let QA be the codescent object of this bar complex and $q \colon QA \to A$ the comparison map to the augmentation. Note that the algebra structure $\bar{a} \colon TQA \to QA$ is uniquely determined by the equations

$$\bar{a} \cdot T\iota_n = \iota_{n+1}d_0 \colon T^{n+2}A \to QA\ .$$

Since T preserves codescent objects so does $U \colon \mathsf{T}\text{-}\mathbf{Alg}_s \to \mathcal{C}$, and so UQA is the codescent object of $UA_\bullet$ in $\mathcal{C}$. The unit maps η equip $UA_\bullet$ with a contraction, and so Uq is part of a lali (Uq, r, ξ) in $\mathcal{C}$; thus $(q, r, \xi) \colon QA \to A$ is a U-lali.

Proposition 21. *With the above assumptions and notation, (q, r, ξ) is the free U-lali $\boldsymbol{R}!_A \colon QA \to A$.*

Proof. Let $(g, f, \epsilon) \colon B \to A$ be a U-lali. We must show there is a unique T-algebra map $h \colon QA \to B$ comprising a map of U-lalis $(q, p, \xi) \to (g, f, \epsilon)$; this means that $gh = q$ and $f = hr$ and $\epsilon h = h\xi$. Now if we are to have $f = hr$ then $f = hr = h\iota_0 s_{-1} = h\iota_0\eta_A \colon A \to B$; since $h\iota_0$ is to be a map of T-algebras, this forces $h\iota_0 = b \cdot Tf \colon TA \to B$. Similarly, if we are to have $\epsilon h = h\xi$, then $\epsilon h\iota_n = h\xi\iota_n = h\iota_{n+1}s_{-1} = h\iota_{n+1}\eta_{T^{n+1}A} \colon T^{n+1}A \to B$; which since $h\iota_{n+1}$ is a T-algebra map forces $h\iota_{n+1} = b \cdot T(\epsilon h\iota_n) \colon T^{n+2}A \to B$. Since $h \colon QA \to B$ is determined by its precomposites with the ι_n's, this proves the uniqueness of h, and it remains only to check that these definitions do indeed yield a map of U-lalis. So let the algebra maps $h_n \colon T^{n+1}A \to_n B$ be defined by $h_0 = b \cdot Tf$ and $h_{n+1} = b \cdot T(\epsilon h_n)$; it follows easily that we have

$$h_0 s_{-1} = f \quad \text{and} \quad h_{n+1}s_{-1} = \epsilon h_n \quad \text{and} \quad h_n d_0 = b \cdot Th_n\ .$$

We first prove $h_n s_j = 0$ for all $0 \leqslant j < n$. We have that

$$h_n s_j = b \cdot T(\epsilon h_{n-1}) \cdot T(s_{j-1}) = b \cdot T(\epsilon h_{n-1}s_{j-1})\ .$$

If $j = 0$, then this is zero since $\epsilon h_0 s_{-1} = \epsilon f = 0$ and $\epsilon h_{n+1}s_{-1} = \epsilon\epsilon h_n = 0$. If $j > 0$, this is zero by induction on j. So by the universal property of QA, there is a unique algebra map $h \colon QA \to_0 B$ with $h\iota_n = h_n$. It remains to check that:

- $f = hr$ and $\epsilon h = h\xi$; which is forced by the method of definition.
- $gh = q$; which follows since we have $gh_0 = gb \cdot Tf = a \cdot Tg \cdot Tf = a = q\iota_0$, and for $n > 0$ that $gh_n = gb \cdot T(\epsilon h_{n-1}) = a \cdot T(g\epsilon h_{n-1}) = 0 = q\iota_n$.
- h is a chain map; which is to say that $\partial(h_n) = h\partial(\iota_n)$ for each n. But $\partial(h_0) = \partial(b \cdot Tf) = 0 = h\partial(\iota_0)$; and

$$\begin{aligned}\partial(h_1) &= \partial(b \cdot T(\epsilon h_0)) = b \cdot T((1 - fg)h_0 - \epsilon\partial(h_0)) = b \cdot T(h_0 - fa) \\ &= b \cdot Th_0 - b \cdot Tf \cdot Ta = h_0d_0 - h_0d_1 = h\partial(\iota_1)\ ;\end{aligned}$$

and for $n \geqslant 1$ we show that $\partial(h_n) = h\partial(\iota_n)$ by induction and the calculation

$$\begin{aligned}\partial(h_n) &= \partial(b \cdot T(\epsilon h_{n-1})) = b \cdot T((1-fg)h_{n-1} - \epsilon\partial(h_{n-1})) \\ &= b \cdot T(h_{n-1} - \epsilon h\partial(\iota_{n-1})) = b \cdot Th_{n-1} - b \cdot T(\epsilon h_{n-2} \textstyle\sum_{j=0}^{n-1}(-1)^j d_j) \\ &= h_{n-1}d_0 - h_{n-1}\textstyle\sum_{j=0}^{n-1}(-1)^j d_{j+1} = h\partial(\iota_n)\ . \end{aligned}$$

□

Given this result, we are now in a position to complete the proof of Theorem 19 by showing that the functor (6.1) is an isomorphism. Of course, it is bijective on objects, and on morphisms is defined as follows. For each $A \in \mathsf{T}\text{-}\mathbf{Alg}_s$, form the free U-lali $(q, r, \xi)\colon QA \to A$ as in the preceding result; now apply Lemma 20 to obtain a splitting $p\colon A \rightsquigarrow QA$ for q in $\mathsf{T}\text{-}\mathbf{Alg}_w$, which by direct calculation has components

$$p_n = \iota_n s_{-1} = \iota_n \eta_{T^nA}\colon T^nA \to T^{n+1} \to_n QA\ .$$

Now the action of (6.1) on morphisms sends $f\colon QA \to B$ to $f \cdot p\colon A \rightsquigarrow B$. But it is easy to see that this assignation is invertible; given a weak map $g\colon A \rightsquigarrow B$, the unique strict T-algebra map $\bar{f}\colon QA \to B$ inducing it is determined by the conditions

$$\bar{f}\iota_n = b \cdot Tf_n\colon T^{n+1}A \to B\ .$$

This completes the proof of Theorem 19.

Example 22. Let A be a unital dg-algebra—a monoid in **DG**, and let T be the monad $A \otimes (-)$ on **DG**. In this case, $QA \to A$ is the classical bar resolution of A [7, X, §6]. More generally, if $\mathcal{A}$ is a small dg-category, and T is the dg-monad on $[\mathrm{ob}\,\mathcal{A}, \mathbf{DG}]$ whose algebras are dg-modules $\mathcal{A} \to \mathbf{DG}$, then $QA \to A$ is the bar resolution $Y \circ A$ described in [18, Example 6.6].

It is worth also pointing out some of the examples which Theorem 19 does not encompass. The category of dg-algebras is itself monadic over **DG**, and as well as the usual strict morphisms of dg-algebras there are also the well-known weak ($=A_\infty$) morphisms. It is natural to attempt to re-find these by lifting the AWFS for homological lalis from **DG** to the category of dg-algebras. While this is certainly possible, it is does not fall under the scope of Theorem 19, since the monad for dg-algebras on **DG** is *not* a dg-monad. It is, however, a monad induced by a dg-operad, and in a sequel to this paper we will examine homotopy-coherent maps of algebras over dg-operads from the perspective of AWFS; we will see that they arise from a *dendroidal* [27] bar resolution which can also be understood in terms of the bar–cobar construction for operad algebras [25, Chapter 11].

References

[1] Athorne, T. The coalgebraic structure of cell complexes. *Theory and Applications of Categories 26* (2012), 304–330.
[2] Batanin, M., and Weber, M. Algebras of higher operads as enriched categories. *Applied Categorical Structures 19* (2011), 93–135.
[3] Beck, J. Distributive laws. In *Seminar on Triples and Categorical Homology Theory (Zürich, 1966/67)*, vol. 80 of *Lecture Notes in Mathematics*. Springer, 1969, pp. 119–140.
[4] Blackwell, R., Kelly, G. M., and Power, A. J. Two-dimensional monad theory. *Journal of Pure and Applied Algebra 59*, 1 (1989), 1–41.
[5] Bourke, J. Two-dimensional monadicity. *Advances in Mathematics 252* (2014), 708–747.

[6] Bourke, J., and Garner, R. Algebraic weak factorisation systems I: accessible AWFS. Preprint, available as arXiv:1412.6559, 2014.
[7] Cartan, H., and Eilenberg, S. *Homological algebra*. Princeton University Press, 1956.
[8] Christensen, J. D., and Hovey, M. Quillen model structures for relative homological algebra. *Mathematical Proceedings of the Cambridge Philosophical Society 133*, 2 (2002), 261–293.
[9] Gabriel, P., and Zisman, M. *Calculus of fractions and homotopy theory*, vol. 35 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 1967.
[10] Garner, R. Understanding the small object argument. *Applied Categorical Structures 17*, 3 (2009), 247–285.
[11] Garner, R. Homomorphisms of higher categories. *Advances in Mathematics 224*, 6 (2010), 2269–2311.
[12] Goerss, P. G., and Jardine, J. F. *Simplicial homotopy theory*, vol. 174 of *Progress in Mathematics*. Birkhäuser, 1999.
[13] Gordon, R., Power, J., and Street, R. Coherence for tricategories. *Memoirs of the American Mathematical Society 117* (1995).
[14] Grandis, M., and Tholen, W. Natural weak factorization systems. *Archivum Mathematicum 42*, 4 (2006), 397–408.
[15] Hébert, M. Weak reflections and weak factorization systems. *Applied Categorical Structures 19*, 1 (2011), 9–38.
[16] Joyal, A. Quasi-categories and Kan complexes. *Journal of Pure and Applied Algebra 175*, 1-3 (2002), 207–222.
[17] Joyal, A. The theory of quasi-categories and its applications. Quadern 45-2, CRM Barcelona, 2008.
[18] Keller, B. Deriving DG categories. *Annales Scientifiques de l'École Normale Supérieure (Ser. 4) 27*, 1 (1994), 63–102.
[19] Kelly, G. M. Chain maps inducing zero homology maps. *Proceedings of the Cambridge Philosophical Society 61* (1965), 847–854.
[20] Kelly, G. M. Doctrinal adjunction. In *Category Seminar (Sydney, 1972/1973)*, vol. 420 of *Lecture Notes in Mathematics*. Springer, 1974, pp. 257–280.
[21] Kelly, G. M., and Street, R. Review of the elements of 2-categories. In *Category Seminar (Sydney, 1972/1973)*, vol. 420 of *Lecture Notes in Mathematics*. Springer, 1974, pp. 75–103.
[22] Kinoshita, Y., Power, J., and Takeyama, M. Sketches. *Journal of Pure and Applied Algebra 143*, 1-3 (1999), 275–291.
[23] Lack, S. Limits for lax morphisms. *Applied Categorical Structures 13*, 3 (2005), 189–203.
[24] Lack, S. Homotopy-theoretic aspects of 2-monads. *Journal of Homotopy and Related Structures 7*, 2 (2007), 229–260.
[25] Loday, J.-L., and Vallette, B. *Algebraic operads*, vol. 346 of *Grundlehren der Mathematischen Wissenschaften*. Springer, 2012.
[26] Lurie, J. *Higher Topos Theory*. Princeton University Press, 2009.
[27] Moerdijk, I., and Weiss, I. Dendroidal sets. *Algebraic & Geometric Topology 7* (2007), 1441–1470.
[28] Riehl, E. Algebraic model structures. *New York Journal of Mathematics 17* (2011), 173–231.
[29] Riehl, E. Monoidal algebraic model structures. *Journal of Pure and Applied Algebra 217*, 6 (2013), 1069–1104.
[30] Rosolini, G. *Continuity and effectiveness in topoi*. PhD thesis, University of Oxford, 1986.
[31] Street, R. The formal theory of monads. *Journal of Pure and Applied Algebra 2*, 2 (1972), 149–168.
[32] Tamarkin, D. What do dg-categories form? *Compositio Mathematica 143*, 5 (2007), 1335–1358.
[33] Weber, M. Codescent for crossed internal categories. In preparation, 2014.

[34] Weibel, C. A. *An introduction to homological algebra*, vol. 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1994.

Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, Brno 60000, Czech Republic
E-mail address: bourkej@math.muni.cz

Department of Mathematics, Macquarie University, NSW 2109, Australia
E-mail address: richard.garner@mq.edu.au