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A characterisation of algebraic exactness

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ABSTRACT

An *algebraically exact* category is one that admits all of the limits and colimits which every variety of algebras possesses and every forgetful functor between varieties preserves, and which verifies the same interactions between these limits and colimits as hold in any variety. Such categories were studied by Adámek, Lawvere and Rosický: they characterised them as the categories with small limits and sifted colimits for which the functor taking sifted colimits is continuous. They conjectured that a complete and sifted-cocomplete category should be algebraically exact just when it is Barr-exact, finite limits commute with filtered colimits, regular epimorphisms are stable by small products, and filtered colimits distribute over small products. We prove this conjecture.

1. Introduction

In the series of papers [1,4,3] the notion of an *algebraically exact* category was introduced and studied. A category $\mathcal{C}$ is said to be algebraically exact if, firstly, it admits all of the operations $\mathcal{C}^{\mathcal{A}} \to \mathcal{C}$ of small arity which every variety of (finitary, many-sorted) algebras supports and every forgetful functor between varieties preserves, and secondly, it obeys all of the equations between such operations as are satisfied in every variety. Every variety admits small limits and sifted colimits, and every forgetful functor between varieties preserves them; recall from [2] that *sifted colimits* are those which commute with finite products in **Set**, most important amongst these being the filtered colimits, and the coequalisers of reflexive pairs (see [6] for a detailed exposition of these notions). It follows that any algebraically exact category also admits small limits and sifted colimits; and it turns out that these two kinds of operations in fact generate all of those required of an algebraically exact category. As regarding the equations that hold between these operations, we observe that in any variety, the following four exactness properties are verified:

(E1) Regular epimorphisms are stable under pullback, and equivalence relations are effective (i.e., the category is Barr-exact);
(E2) Finite limits commute with filtered colimits;
(E3) Regular epimorphisms are stable by small products;
(E4) Filtered colimits distribute over small products (in the sense of [6, Definition 3.19]).

It follows that these same conditions are verified in any algebraically exact category, and it was conjectured in [1] that, in fact, these four conditions completely characterise the algebraically exact categories amongst those categories with small limits and sifted colimits. The conjecture was proved in [4] for the case of cocomplete categories with a regular generator, and in [3] for the case of arbitrary cocomplete categories; the purpose of this article is to prove it in its full generality. We shall do so using techniques developed in [8], though the arguments are straightforward enough that we can reproduce them in full here, so making this article entirely self-contained.

In order to state the conjecture more precisely, we will make use of a different description of the algebraically exact categories. We recall from [2] the construction which to every locally small category $\mathcal{C}$ assigns its free completion $\mathcal{S}\mathrm{ind}(\mathcal{C})$

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under sifted colimits. As in [9, Theorem 5.35], we may obtain $\mathcal{S}\mathrm{ind}(\mathcal{C})$ as the closure of the representables in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ under sifted colimits, and now the restricted Yoneda embedding $W\colon \mathcal{C} \to \mathcal{S}\mathrm{ind}(\mathcal{C})$ provides the unit at $\mathcal{C}$ of a pseudomonad on **CAT** whose pseudoalgebras are the sifted-cocomplete categories. This pseudomonad is of the *Kock–Zöberlein* type—see [11] and the references therein—for which pseudoalgebra structure is left adjoint to unit: which is to say that a category $\mathcal{C}$ admits pseudoalgebra structure, and so sifted colimits, just when $W\colon \mathcal{C} \to \mathcal{S}\mathrm{ind}(\mathcal{C})$ admits a left adjoint.

It was shown in [1, Theorem 3.11] that if $\mathcal{C}$ is complete, then so too is $\mathcal{S}\mathrm{ind}(\mathcal{C})$; that if $F\colon \mathcal{C} \to \mathcal{D}$ is a continuous functor between complete categories, then so too is $\mathcal{S}\mathrm{ind}(F)$; and that the unit $\mathcal{C} \to \mathcal{S}\mathrm{ind}(\mathcal{C})$ and multiplication $\mathcal{S}\mathrm{ind}(\mathcal{S}\mathrm{ind}(\mathcal{C})) \to \mathcal{S}\mathrm{ind}(\mathcal{C})$ are always continuous functors. It follows that the pseudomonad $\mathcal{S}\mathrm{ind}$ restricts and corestricts to one on **CONTS**, the 2-category of complete categories and continuous functors; and it was shown in [1, Corollary 4.4] that the pseudoalgebras for this restricted pseudomonad are precisely the algebraically exact categories described above. Thus we are led, for the purposes of this paper, to adopt the following definition.

Definition 1.1. A complete category $\mathcal{C}$ is said to be *algebraically exact* when the restricted Yoneda embedding $\mathcal{C} \to \mathcal{S}\mathrm{ind}(\mathcal{C})$ admits a continuous left adjoint.

Our goal is to prove:

Theorem 1.2. *A complete and sifted-cocomplete category $\mathcal{C}$ is algebraically exact just when it satisfies conditions* (E1)–(E4).

Of course, any variety of algebras is algebraically exact; more generally, if $\mathcal{C}$ is an algebraically exact category, and T a monad thereon whose functor part preserves sifted colimits, then the category $\mathcal{C}^T$ of T-algebras will again be algebraically exact. Any presheaf topos is algebraically exact, and as we shall see, so too is any essential subtopos of a presheaf topos; as explained in [10], the toposes arising in this manner are those equivalent to categories of sheaves on some site $(\mathcal{C}, j)$ in which each object $X \in \mathcal{C}$ admits a smallest j-covering sieve. Combining the above two results, we see that for $(\mathcal{C}, j)$ a site of this form, and T a finitary monad on **Set**, the category of sheaves of T-algebras is an algebraically exact category. All of these examples are cocomplete, and so covered by the results of [3]; for an example which is not, we can as in [3, Examples 3.2.2] take an arbitrary subcategory $\mathcal{C}$ of an algebraically exact category $\mathcal{E}$ and form its closure under limits, filtered colimits and reflexive coequalisers. The subcategory $\bar{\mathcal{C}}$ so obtained will clearly satisfy (E1)–(E4), but need not be cocomplete; so whilst the results of [3] do not suffice to show $\bar{\mathcal{C}}$ to be algebraically exact, our theorem shows that this is, in fact, the case.

2. The result

We now give the proof of Theorem 1.2. As remarked above, any algebraically exact category does indeed satisfy (E1)–(E4); and so our task is to show that these conditions in turn imply algebraic exactness. The idea behind the proof is to show that any category $\mathcal{C}$ satisfying (E1)–(E4) admits a full structure-preserving embedding into some $\mathcal{E}$ which is an essential localisation of a presheaf topos; recall that an *essential localisation* of a category is a subcategory reflective via a small-limit-preserving reflector. Any such $\mathcal{E}$ will be algebraically exact, and now we may reflect this property along the full embedding, so concluding that $\mathcal{C}$ itself is algebraically exact. This argument does not quite work as it stands, for reasons of size. The $\mathcal{E}$ into which we would like to embed is a topos of sheaves on $\mathcal{C}$, but only when $\mathcal{C}$ is small may such a topos be constructed; in which situation, with $\mathcal{C}$ being small, and also small-complete, it is necessarily a preorder, which is far too restrictive. To overcome this problem, we will first prove a variant of Theorem 1.2, in which suitable bounds have been introduced on the size of the limits and colimits required, and then deduce the general result from this.

Our cardinality bounds will be governed by an infinite regular cardinal κ. Given any such κ, we define κ' to be the cardinal $(\Sigma_{\gamma<\kappa} 2^\gamma)^+$, and the pair (κ, κ') now has the property that whenever $\mu < \kappa$ and $\lambda < \kappa'$, we have $\lambda^\mu < \kappa'$: see [12, Proposition 2.3.5]. By a κ-limit we shall mean one indexed by a diagram of cardinality $< \kappa$, and we attach a corresponding meaning to the term κ'-colimit. We shall now describe a variant of the notion of algebraic exactness, which we term *κ-algebraic exactness*, that deals only with κ-limits and κ'-colimits.

There is a slight delicacy here as to the kinds of κ'-colimit we will consider; it turns out that there are two natural choices. For indeed, as a straightforward consequence of [4, Proposition 5.1], if $\mathcal{C}$ is complete, then the category $\mathcal{S}\mathrm{ind}(\mathcal{C})$, which we obtained as the closure of the representables in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ under sifted colimits, is equally well the closure of those representables under reflexive coequalisers and filtered colimits. Thus a complete category $\mathcal{C}$ admits sifted colimits just when it admits reflexive coequalisers and filtered colimits, and so the definition of an algebraically exact category could just as well be phrased in terms of these latter kinds of colimit. However, as was shown in [5], a category $\mathcal{C}$ which is *not* complete can admit all reflexive coequalisers and filtered colimits without admitting all sifted colimits. Thus, when we impose cardinality bounds, there are two distinct possibilities as to the kinds of κ'-colimit we consider: either the sifted κ'-colimits, or the reflexive coequalisers together with the filtered κ'-colimits. It turns out to be the *latter* choice which allows our proof to go through.

In order to define κ-algebraic exactness, we consider the 2-category κ-**CONTS** of κ-complete categories and κ-continuous functors between them; on this, we will describe a pseudomonad whose pseudoalgebras will be the κ-algebraically exact categories. Observe first that as well as the pseudomonad $\mathcal{S}\mathrm{ind}$ on **CAT** we also have the pseudomonad $\mathcal{P}$ which freely adds small colimits; its value at a category $\mathcal{C}$ is given by the closure of the representables in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ under all small colimits. Proposition 4.3 and Remark 6.6 of [7] prove that if $\mathcal{C}$ is κ-complete, then so is $\mathcal{P}\mathcal{C}$; that if $F\colon \mathcal{C} \to \mathcal{D}$ is a κ-continuous functor between such categories, then so is $\mathcal{P}F$; and that $\mathcal{P}$'s unit and multiplication are always κ-continuous. Thus we may restrict and corestrict $\mathcal{P}$ to a pseudomonad on κ-**CONTS**; and the pseudomonad of interest to us will be a submonad of this, defined as follows. For each $\mathcal{C}$ in κ-**CONTS**, we let $\mathcal{S}_{\kappa'}(\mathcal{C})$ denote the closure of $\mathcal{C}$ in $\mathcal{P}\mathcal{C}$ under κ-limits, reflexive coequalisers, and filtered κ'-colimits; this closure may be constructed by a transfinite iteration as in [9, Section 3.5]. Let $V\colon \mathcal{C} \to \mathcal{S}_{\kappa'}(\mathcal{C})$ denote the restricted Yoneda embedding; now [8, Proposition 3.1] ensures that this V provides the unit at $\mathcal{C}$ of a Kock–Zöberlein pseudomonad on κ-**CONTS**, and a κ-algebraically exact category will be a pseudoalgebra for this pseudomonad. In other words, we say that:

Definition 2.1. A κ-complete category $\mathcal{C}$ is *κ-algebraically exact* just when the embedding $V\colon \mathcal{C} \to \mathcal{S}_{\kappa'}(\mathcal{C})$ admits a κ-continuous left adjoint.

Observe that this implies that $\mathcal{C}$ has reflexive coequalisers and filtered κ'-colimits, but may not imply that it has all sifted κ'-colimits; cf. [5]. We shall now prove the following refinement of Theorem 1.2.

Theorem 2.2. *A category $\mathcal{C}$ with κ-limits, reflexive coequalisers and filtered κ'-colimits is κ-algebraically exact just when:*

(E1') *It is Barr-exact;*
(E2') *Finite limits commute with filtered κ'-colimits;*
(E3') *Regular epimorphisms are stable by κ-small products;*
(E4') *Filtered κ'-colimits distribute over κ-small products.*

Clearly, a complete and sifted-cocomplete $\mathcal{C}$ satisfies (E1')–(E4') for each regular κ if and only if it satisfies (E1)–(E4). On the other hand, we have:

Proposition 2.3. *A complete and sifted-cocomplete category $\mathcal{C}$ is algebraically exact if and only if it is κ-algebraically exact for each regular κ.*

By virtue of this Proposition and the comment preceding it, we may prove Theorem 1.2 by proving Theorem 2.2, and then taking the conjunction of all its instances as κ ranges across the small regular cardinals.

Proof of Proposition 2.3. For every κ, we observe that $\mathcal{S}\mathrm{ind}(\mathcal{C})$ is closed under κ-limits, reflexive coequalisers and filtered κ'-colimits in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$; whence $\mathcal{S}_{\kappa'}(\mathcal{C}) \subset \mathcal{S}\mathrm{ind}(\mathcal{C})$ with the inclusion preserving all κ-limits. Hence if $W\colon \mathcal{C} \to \mathcal{S}\mathrm{ind}(\mathcal{C})$ admits a continuous left adjoint, then by restriction each $V\colon \mathcal{C} \to \mathcal{S}_{\kappa'}(\mathcal{C})$ will admit a κ-continuous left adjoint.

Conversely, suppose that each $V\colon \mathcal{C} \to \mathcal{S}_{\kappa'}(\mathcal{C})$ admits a κ-continuous left adjoint. As observed above, since $\mathcal{C}$ is complete, it follows by [4, Proposition 5.1] that $\mathcal{S}\mathrm{ind}(\mathcal{C})$ is the closure of the representables in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ under reflexive coequalisers and filtered colimits. But it is easy to see that the collection of $\varphi \in \mathcal{S}\mathrm{ind}(\mathcal{C})$ which lie in some $\mathcal{S}_{\kappa'}(\mathcal{C})$ contains the representables and is closed under reflexive coequalisers and filtered colimits, and so must be all of $\mathcal{S}\mathrm{ind}(\mathcal{C})$; which is to say that $\mathcal{S}\mathrm{ind}(\mathcal{C}) = \bigcup_\kappa \mathcal{S}_{\kappa'}(\mathcal{C})$. Thus, since each $V\colon \mathcal{C} \to \mathcal{S}_{\kappa'}(\mathcal{C})$ admits a left adjoint, so too does $W\colon \mathcal{C} \to \mathcal{S}\mathrm{ind}(\mathcal{C})$, and it remains to show that this left adjoint is continuous. Given a small diagram $D\colon \mathcal{I} \to \mathcal{S}\mathrm{ind}(\mathcal{C})$, we may choose a regular cardinal κ such that $DI \in \mathcal{S}_{\kappa'}(\mathcal{C})$ for each $I \in \mathcal{I}$ and also $|\mathcal{I}| < \kappa$; now the diagram D factors as $D'\colon \mathcal{I} \to \mathcal{S}_{\kappa'}(\mathcal{C})$, and the left adjoint of $\mathcal{C} \to \mathcal{S}_{\kappa'}(\mathcal{C})$ preserves the limit of D': from which it follows that the left adjoint of W preserves that of D, as required. □

We now prove Theorem 2.2 for the case of a small $\mathcal{C}$. Given such a $\mathcal{C}$ satisfying the conditions of the theorem, we shall embed it into a κ-algebraically exact category via a functor preserving κ-limits, reflexive coequalisers and filtered κ'-colimits. It will then follow that $\mathcal{C}$ is κ-algebraically exact by virtue of the following result.

Proposition 2.4. *Let $J\colon \mathcal{C} \to \mathcal{E}$ be fully faithful; suppose moreover that $\mathcal{C}$ has, and that J preserves, κ-limits, reflexive coequalisers and filtered κ'-colimits, and that $\mathcal{E}$ is κ-algebraically exact. Then $\mathcal{C}$ is also κ-algebraically exact.*

Proof. Because $\mathcal{E}$ is κ-algebraically exact, the functor J admits a left Kan extension

$$\begin{tikzcd} \mathcal{C} \arrow[r, "J"] \arrow[d, "V"'] \arrow[dr, phantom, "\cong"] & \mathcal{E} \\ \mathcal{S}_{\kappa'}(\mathcal{C}) \arrow[ur, "\mathrm{Lan}_V J"'] & \end{tikzcd}$$

along V, which may be calculated as the composite

$$\mathcal{S}_{\kappa'}(\mathcal{C}) \xrightarrow{\mathcal{S}_{\kappa'}(J)} \mathcal{S}_{\kappa'}(\mathcal{E}) \xrightarrow{L} \mathcal{E}$$

with L the κ-continuous left adjoint of $V\colon \mathcal{E} \to \mathcal{S}_{\kappa'}(\mathcal{E})$. Now $\mathcal{S}_{\kappa'}(J)$ is an algebra morphism between free $\mathcal{S}_{\kappa'}$-algebras, and as such, preserves κ-limits, reflexive coequalisers and filtered κ'-colimits; whilst L preserves all colimits, being a left adjoint. It follows that $\mathrm{Lan}_V J$, like J, preserves κ-limits, reflexive coequalisers and filtered κ'-colimits; whence the collection of $\varphi \in \mathcal{S}_{\kappa'}(\mathcal{C})$ for which $\mathrm{Lan}_V J$ lands in the essential image of J contains the representables and is closed under κ-limits, reflexive coequalisers and filtered κ'-colimits, and so must be all of $\mathcal{S}_{\kappa'}(\mathcal{C})$. Hence $\mathrm{Lan}_V J$ factors as $\mathrm{Lan}_V J \cong JM$ for some functor $M\colon \mathcal{S}_{\kappa'}(\mathcal{C}) \to \mathcal{C}$. Since $\mathrm{Lan}_V J$ is κ-continuous and J is fully faithful, M is also κ-continuous. Moreover, we have natural isomorphisms of homsets

$$\mathcal{C}(MA, B) \cong \mathcal{E}(JMA, JB) \cong \mathcal{E}(L\mathcal{S}_{\kappa'}(J)A, JB) \cong \mathcal{S}_{\kappa'}(\mathcal{E})(\mathcal{S}_{\kappa'}(J)A, VJB) \cong \mathcal{S}_{\kappa'}(\mathcal{E})(\mathcal{S}_{\kappa'}(J)A, \mathcal{S}_{\kappa'}(J)VB) \cong \mathcal{S}_{\kappa'}(\mathcal{C})(A, VB)$$

so that M is left adjoint to $V\colon \mathcal{C} \to \mathcal{S}_{\kappa'}(\mathcal{C})$, and $\mathcal{C}$ is κ-algebraically exact. □

Given a small, κ-complete $\mathcal{C}$, admitting reflexive coequalisers and filtered κ'-colimits, and satisfying (E1')–(E4'), we now exhibit an embedding of the above form; as anticipated at the start of this section, it will in fact be an embedding into a topos. We consider the smallest topology on $\mathcal{C}$ for which all regular epimorphisms are covering, and for which the colimit injections into each filtered κ'-colimit are covering. (E1') and (E2') ensure that this topology is subcanonical and so we have a full embedding $J_\kappa\colon \mathcal{C} \to \mathbf{Sh}_\kappa(\mathcal{C})$.

Proposition 2.5. *The full embedding $J_\kappa\colon \mathcal{C} \to \mathbf{Sh}_\kappa(\mathcal{C})$ preserves κ-limits, reflexive coequalisers and filtered κ'-colimits.*

Proof. Clearly J_κ preserves all limits that exist, so in particular κ-limits. It also preserves regular epimorphisms, since the given topology contains the regular one, and we will show below that it preserves filtered κ'-colimits. It will then follow that it preserves reflexive coequalisers too, since in $\mathcal{C}$ and in $\mathbf{Sh}_\kappa(\mathcal{C})$, we may exploit (E1') and (E2') to construct such coequalisers from finite limits, countable filtered colimits and coequalisers of equivalence relations, all of which are preserved by J_κ; the argument is given in precisely the form we need it in [3, Theorem 2.6].

It remains to show that J_κ preserves filtered κ'-colimits. Observe that if $(p_k\colon Dk \to X \mid k \in \mathcal{K})$ is such a colimit in $\mathcal{C}$, then J_κ will preserve it just when every sheaf $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ sends it to a limit in $\mathbf{Set}$. So let F be a sheaf. Since the family $(p_k \mid k \in \mathcal{K})$ is covering, we may identify FX with the set of matching families for this covering. In other words, if

$$\begin{tikzcd} D_{jk} \arrow[r, "d_{jk}"] \arrow[d, "c_{jk}"'] & Dj \arrow[d, "p_j"] \\ Dk \arrow[r, "p_k"'] & X \end{tikzcd}$$

is a pullback for each $j, k \in \mathcal{K}$, then we may identify FX with the set

$$\{\vec{x} \in \Pi_k FDk \mid Fd_{jk}(x_j) = Fc_{jk}(x_k) \text{ for all } j, k \in \mathcal{K}\}. \tag{$*$}$$

Under this identification, the canonical comparison map $FX \to \lim FD$ is just the inclusion between these sets, seen as subobjects of $\Pi_k FDk$, and so injective; it remains to show that it is also surjective. Thus we must show that each $\vec{x} \in \lim FD$ lies in $(*)$, or in other words, that $Fd_{jk}(x_j) = Fc_{jk}(x_k)$ for each $\vec{x} \in \lim FD$ and each $j, k \in J$. To this end, we consider the category $\mathcal{K}'$ of cospans from j to k in $\mathcal{K}$; since $\mathcal{K}$ is filtered and κ'-small, it follows easily that $\mathcal{K}'$ is too. We define a functor $E\colon \mathcal{K}' \to \mathcal{C}$ by sending each cospan $f\colon j \to \ell \leftarrow k\colon g$ in $\mathcal{K}'$ to the apex of the pullback square

$$\begin{tikzcd} E(f, g) \arrow[r, "u_{f,g}"] \arrow[d, "v_{f,g}"'] & Dj \arrow[d, "Df"] \\ Dk \arrow[r, "Dg"'] & D\ell \end{tikzcd}$$

in $\mathcal{C}$. A simple calculation shows that $p_k.v_{f,g} = p_j.u_{f,g}$, so that we have induced maps $q_{f,g} := (u_{f,g}, v_{f,g})\colon E(f, g) \to D_{jk}$, constituting a cocone q under E with vertex D_{jk}. We claim that this cocone is colimiting; whereupon, by the preceding part of the argument, the comparison $FD_{jk} \to \lim FE$ induced by q will be monic, and so the family $(Fq_{f,g} \mid (f, g) \in \mathcal{K}')$ jointly monic. Thus in order to verify that $Fd_{jk}(x_j) = Fc_{jk}(x_k)$, and so complete the proof, it will be enough to observe that for each $f\colon j \to \ell \leftarrow k\colon g$ in $\mathcal{K}'$, we have:

$$Fq_{f,g}(Fd_{jk}(x_j)) = Fu_{f,g}(x_j) = Fu_{f,g}(FDf(x_\ell)) = Fv_{f,g}(FDg(x_\ell)) = Fv_{f,g}(x_k) = Fq_{f,g}(Fc_{jk}(x_k)).$$

It remains to verify that q is colimiting. For this, let $V\colon \mathcal{K}' \to \mathcal{K}$ denote the functor sending a j, k-cospan to its central object, and $\iota_1\colon \Delta j \to V \leftarrow \Delta k\colon \iota_2$ the evident natural transformations. Now we have a commutative cube

$$\begin{tikzcd} E \arrow[rr, "u"] \arrow[dr, "q"] \arrow[dd, "v"'] & & \Delta Dj \arrow[dr, equal] \arrow[dd, "D\iota_1" near start] & \\ & \Delta(D_{jk}) \arrow[rr, "\Delta d_{jk}"' near start] \arrow[dd, "\Delta c_{jk}" near end] & & \Delta(Dj) \arrow[dd, "\Delta p_j"] \\ \Delta(Dk) \arrow[rr, "D\iota_2" near start] \arrow[dr, equal] & & DV \arrow[dr, "pV"] & \\ & \Delta(Dk) \arrow[rr, "\Delta p_k"'] & & \Delta X \end{tikzcd}$$

in $[\mathcal{K}', \mathcal{C}]$; its front and rear faces are pullbacks, and by (E2') will remain so on applying the functor $\mathrm{colim}\colon [\mathcal{K}', \mathcal{C}] \to \mathcal{C}$. To show that q is colimiting is equally to show that it is inverted by colim; for which, by the previous sentence, it is enough to show that pV is likewise inverted. But $\mathcal{K}$'s filteredness implies easily that $V\colon \mathcal{K}' \to \mathcal{K}$ is a final functor, so that pV, like p, is a colimiting cocone, and so inverted by colim as required. □

We thus have a full structure-preserving embedding $\mathcal{C} \to \mathbf{Sh}_\kappa(\mathcal{C})$ and, in order to apply Proposition 2.4, the only thing left to verify is that $\mathbf{Sh}_\kappa(\mathcal{C})$ is in fact κ-algebraically exact. The key to doing so is the following proposition.

Proposition 2.6. *If $\mathcal{E}$ is reflective in a presheaf category via a κ-continuous reflector, then $\mathcal{E}$ is κ-algebraically exact.*

Proof. If $\mathcal{C}$ is small, then $\mathcal{PC} = [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, and now the restricted Yoneda embedding $\mathcal{PC} \to \mathcal{PPC}$ admits a continuous left adjoint $\mathcal{PPC} \to \mathcal{PC}$, this being the multiplication at $\mathcal{C}$ of the pseudomonad $\mathcal{P}$. Since $\mathcal{S}_{\kappa'}(\mathcal{PC})$ is closed in $\mathcal{PPC}$ under κ-limits, it follows by restriction that $\mathcal{PC} \to \mathcal{S}_{\kappa'}(\mathcal{PC})$ admits a κ-continuous left adjoint; and so every presheaf category is κ-algebraically exact. Now if $\mathcal{E}$ is reflective in the κ-algebraically exact $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ via a κ-continuous reflector, then it is a retract of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ in κ-**CONTS**, and so by a standard property of Kock–Zöberlein pseudomonads [11, Theorem 3.5], must itself be κ-algebraically exact. □

Thus it is enough to show that $\mathbf{Sh}_\kappa(\mathcal{C})$ is reflective in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ via a κ-continuous reflector. This will be a consequence of the following result, which may be found proven—though with "small" harmlessly replacing our "κ-small"—in [10, Theorem 4.2]; we shall not recall the details, since we shall not need them in what follows.

Proposition 2.7. *A left exact reflector $L\colon [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{E}$ preserves all κ-small limits if and only if the covering sieves for the corresponding topology are closed under κ-small intersections in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.*

We are therefore required to show that any κ-small intersection of covering sieves for the above-defined topology on $\mathcal{C}$ is again covering. Clearly it is sufficient to consider the case where the sieves participating in the intersection are generating ones for the topology. We can decompose any such intersection of sieves as an intersection

$$\bigcap_{i\in I} \mathcal{S}_i \cap \bigcap_{j\in J} \mathcal{T}_j$$

where each indexing set I and J is κ-small, each sieve $\mathcal{S}_i$ is generated by a regular epimorphism $e_i\colon A_i \twoheadrightarrow X$ and each sieve $\mathcal{T}_j$ is generated by a κ'-small filtered colimit cocone $((q_j)_x\colon D_j(x) \to X \mid x \in \mathcal{A}_j)$.

Now we can form the κ-small product $\Pi_i e_i\colon \Pi_i A_i \to \Pi_i X$; by condition (E3') this is a regular epimorphism in $\mathcal{C}$, and by regularity, so also is its pullback $e\colon A \to X$ along the diagonal $X \to \Pi_i X$. Clearly a map $Z \to X$ factors through e just when it factors through each e_i, and so the covering sieve $\mathcal{S}$ generated by e is the intersection $\bigcap_i \mathcal{S}_i$.

In a similar manner, we can form the filtered category $\Pi_j \mathcal{A}_j$; since $|J| < \kappa$, and each $\left|\mathcal{A}_j\right| < \kappa'$, we have also that $\left|\Pi_j \mathcal{A}_j\right| < \kappa'$. Now on considering the diagram $D\colon \Pi_j \mathcal{A}_j \to \mathcal{C}$ defined by $D(x_j \mid j \in J) = \Pi_j D_j(x_j)$, condition (E4') asserts that $\Pi_j X$ is a colimit for it; so that on pulling back along the diagonal $X \to \Pi_j X_j$, we conclude that X is a colimit for the diagram $D'\colon \Pi_j \mathcal{A}_j \to \mathcal{C}$ which sends $(x_j \mid j \in J)$ to the fibre product of the maps $(q_j)_{x_j}\colon D_j(x_j) \to X$. Now we see as before that the covering sieve $\mathcal{T}$ generated by this filtered κ'-colimit cocone is precisely $\bigcap_j \mathcal{T}_j$.

It follows that $\bigcap_i \mathcal{S}_i \cap \bigcap_j \mathcal{T}_j = \mathcal{S} \cap \mathcal{T}$ is a covering sieve, since covering sieves are always closed under finite intersections, and this completes the proof of:

Proposition 2.8. *If the small, κ-complete $\mathcal{C}$ with reflexive coequalisers and filtered κ'-colimits satisfies* (E1')–(E4')*, then it admits a full structure-preserving embedding into a κ-algebraically exact category, and so is itself κ-algebraically exact.*

It remains to prove Theorem 2.2 for categories of no matter what size. So let $\mathcal{C}$ be a category with κ-limits, reflexive coequalisers and filtered κ'-colimits, satisfying (E1')–(E4'). We call a full, replete subcategory *κ-closed* if it is closed in $\mathcal{C}$ under the limits and colimits just mentioned. Clearly, each small, κ-closed subcategory of $\mathcal{C}$ satisfies (E1')–(E4'), and so by the preceding proposition is κ-algebraically exact. We may now conclude that the same is true of $\mathcal{C}$ by way of the following result.

Proposition 2.9. *A κ-complete $\mathcal{C}$ admitting reflexive coequalisers and filtered κ'-colimits is κ-algebraically exact so long as all of its small κ-closed subcategories are.*

Proof. Suppose that each κ-closed subcategory of $\mathcal{C}$ is κ-algebraically exact; we must show that $\mathcal{C}$ is too, or in other words, that $V\colon \mathcal{C} \to \mathcal{S}_{\kappa'}(\mathcal{C})$ admits a κ-continuous left adjoint. To this end, consider the collection of $\varphi \in \mathcal{S}_{\kappa'}(\mathcal{C})$ for which there exists a small κ-closed $J\colon \mathcal{D} \hookrightarrow \mathcal{C}$ with φ lying in the essential image of the fully faithful $\mathcal{S}_{\kappa'}(J)\colon \mathcal{S}_{\kappa'}(\mathcal{D}) \to \mathcal{S}_{\kappa'}(\mathcal{C})$. It is easy to show that this collection contains the representables and is closed under κ-limits, reflexive coequalisers and filtered κ'-colimits, and so is all of $\mathcal{S}_{\kappa'}(\mathcal{C})$. It follows that $\mathcal{C} \to \mathcal{S}_{\kappa'}(\mathcal{C})$ admits a left adjoint, since each $\mathcal{D} \to \mathcal{S}_{\kappa'}(\mathcal{D})$ does by assumption.

To show that this left adjoint is moreover κ-continuous, consider a κ-small diagram $X\colon \mathcal{I} \to \mathcal{S}_{\kappa'}(\mathcal{C})$. For each $I \in \mathcal{I}$ we can find a small κ-closed $\mathcal{D}_I \subset \mathcal{C}$ with XI in the essential image of $\mathcal{S}_{\kappa'}(\mathcal{D}_I) \to \mathcal{S}_{\kappa'}(\mathcal{C})$; now taking $\mathcal{D}$ to be the closure of

$\bigcup_I \mathcal{D}_I$ in $\mathcal{C}$ under κ-limits, reflexive coequalisers and filtered κ'-colimits, we obtain another small κ-closed subcategory. The diagram X factors up-to-isomorphism through the fully faithful $\mathcal{S}_{\kappa'}(\mathcal{D}) \to \mathcal{S}_{\kappa'}(\mathcal{C})$ as $X' \colon \mathcal{I} \to \mathcal{S}_{\kappa'}(\mathcal{D})$, say; and now by assumption, the left adjoint of $\mathcal{D} \to \mathcal{S}_{\kappa'}(\mathcal{D})$ preserves the limit of X', whence the left adjoint of $\mathcal{C} \to \mathcal{S}_{\kappa'}(\mathcal{C})$ preserves that of X, as required. $\square$

This completes the proof of Theorem 2.2 for categories of any size; and now, as discussed previously, taking the conjunction of all instances of this theorem as κ ranges over the small regular cardinals completes the proof of Theorem 1.2.

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