



# Bousfield Localisation and Colocalisation of One-Dimensional Model Structures

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## Abstract

We give an account of Bousfield localisation and colocalisation for *one-dimensional* model categories—ones enriched over the model category of 0-types. A distinguishing feature of our treatment is that it builds localisations and colocalisations using only the constructions of *projective* and *injective* transfer of model structures along right and left adjoint functors, and without any reference to Smith’s theorem.

**Keywords** Quillen model structures · Bousfield (co)localisation

**Mathematics Subject Classification** Primary 55U35 · 18A40

## 1 Introduction

A (*Bousfield*) *localisation* of a model category  $\mathcal{E}$  is a model structure  $\mathcal{E}_\ell$  on the same underlying category with the same cofibrations, but a larger class of weak equivalences. If  $\mathcal{E}$  is left proper and combinatorial, one may construct a localisation from any set  $S$  of maps which one wishes to become weak equivalences in  $\mathcal{E}_\ell$ ; the fibrant objects of  $\mathcal{E}_\ell$  will be the *S-local* fibrant objects of  $\mathcal{E}$ —those which see each map in  $S$  as a weak equivalence—and the weak

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equivalences of  $\mathcal{E}_\ell$ , the *S-local equivalences*—those which every *S*-local fibrant object sees as a weak equivalence. The *S*-local equivalences and the original cofibrations determine the other classes of the  $\mathcal{E}_\ell$ -model structure; the hard part is exhibiting the needed factorisations, which is usually done using a subtle cardinality argument of Smith [4, Theorem 1.7].

This paper is the first step towards understanding localisations of combinatorial model categories in a way which avoids Smith’s theorem, and instead uses only the constructions of *projective* and *injective* liftings of model structures—that is, transfers along right and left adjoint functors. It is only a first step since, for reasons to be made clear soon, we only implement our idea here for the rather special class of *one-dimensional* model categories: those which are enriched over the cartesian model category of 0-types. While homotopically trivial, there are mathematically interesting examples of such model structures, and in this context, our approach yields the following complete characterisation:

**Theorem 26** *If  $\mathcal{E}$  is a left proper one-dimensional combinatorial model category, then the assignment  $\mathcal{E}_\ell \mapsto (\mathcal{E}_\ell)_{cf}$  yields an order-reversing bijection between combinatorial localisations of  $\mathcal{E}$  (ordered by inclusion of acyclic cofibrations) and full, replete, reflective, locally presentable subcategories of  $\mathcal{E}_{cf}$  (ordered by inclusion).*

Here,  $(-)_{cf}$  is the operation assigning to a model category its subcategory of cofibrant–fibrant objects. Since our approach relies only on injective and projective liftings, it dualises straightforwardly, giving the corresponding:

**Theorem 36** *If  $\mathcal{E}$  is a right proper one-dimensional combinatorial model category, then the assignment  $\mathcal{E}_r \mapsto (\mathcal{E}_r)_{cf}$  yields an order-reversing bijection between combinatorial colocalisations of  $\mathcal{E}$  (ordered by inclusion of acyclic fibrations) and full, replete, coreflective, locally presentable subcategories of  $\mathcal{E}_{cf}$  (ordered by inclusion).*

These results expand on the inquiry of [31], which characterises (co)localisations of *discrete* model categories: ones whose weak equivalences are the isomorphisms. However, it is our general approach to constructing (co)localisations, rather than the applications to the one-dimensional setting, which is the main conceptual contribution of this paper, and it therefore seems appropriate to now sketch this approach in the context of a general combinatorial model category  $\mathcal{E}$ .

As model structures are determined by their cofibrations and their fibrant objects, a localisation of  $\mathcal{E}$  can be determined by specifying its fibrant objects. So suppose given a class of fibrant objects in  $\mathcal{E}$ , which we call *local*, that we would like to form the fibrant objects of a localisation; for example, given a set *S* of maps in  $\mathcal{E}$ , we could take “local” to mean “*S*-local fibrant”. We will construct the localisation at issue with reference to an adjunction

$$\mathcal{L} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{G} \end{array} \mathcal{E} \quad (1.1)$$

between  $\mathcal{E}$  and a suitably-defined category of local objects  $\mathcal{L}$ . Naively, we might try taking  $\mathcal{L}$  to be the full subcategory of  $\mathcal{E}$  on the local objects; but since this subcategory is not typically complete nor cocomplete, its inclusion functor into  $\mathcal{E}$  will typically not have the required left adjoint. So instead, we take  $\mathcal{L}$ -objects to be  $\mathcal{E}$ -objects *endowed* with algebraic structure witnessing their locality, and take  $\mathcal{L}$ -maps to be  $\mathcal{E}$ -maps which *strictly* preserve this structure. This algebraicity of the definition of  $\mathcal{L}$  now ensures that it is a locally presentable category, and that the forgetful functor to  $\mathcal{E}$  has the desired left adjoint; this extends [26]’s construction of an adjunction with *algebraically fibrant* objects. Note that there can be many different ways of choosing the algebraic structure which witnesses locality, and not all of these are appropriate; indeed, choosing the correct definition of  $\mathcal{L}$  is the most subtle point in our argument.

Thereafter, the remainder of the argument is conceptually clear. We first *projectively* transfer the given model structure on  $\mathcal{E}$  along the right adjoint  $G: \mathcal{L} \rightarrow \mathcal{E}$ , and then *injectively* transfer back along  $F: \mathcal{E} \rightarrow \mathcal{L}$ . Local presentability ensures that these transfers exist so long as the requisite acyclicity conditions are satisfied (cf. Proposition 2 below). For the transfer to  $\mathcal{L}$ , we verify acyclicity using a path object argument, since every object of  $\mathcal{L}$  will be fibrant; for the transfer back to  $\mathcal{E}$ , acyclicity will be immediate so long as  $GF$  preserves weak equivalences—which might be verified, for example, using left properness of  $\mathcal{E}$ .

At this point, we have a new model structure  $\mathcal{E}'$  on the underlying category of  $\mathcal{E}$ , which has more weak equivalences and cofibrations, and makes every local object fibrant. However, it is not yet a localisation of  $\mathcal{E}$  since the cofibrations need not be the same. Thus, the final step is to note that, since  $\mathcal{C}_{\mathcal{E}} \subseteq \mathcal{C}_{\mathcal{E}'}$  and  $\mathcal{W}_{\mathcal{E}} \subseteq \mathcal{W}_{\mathcal{E}'}$ , we can use [13] to *mix* the model structures  $\mathcal{E}$  and  $\mathcal{E}'$ , obtaining a model structure  $\mathcal{E}_{\ell}$  whose cofibrations are those of  $\mathcal{E}$  and in which every local object is fibrant; under appropriate homotopical closure conditions on the class of local objects, the  $\mathcal{E}_{\ell}$ -fibrant objects will be *precisely* the local ones.

In this way, we may construct localisations using only the tools of projective and injective liftings, and of mixing of model structures. It turns out (cf. Proposition 5 below) that mixing of model structures may be reduced in turn to liftings, so that we have a construction of localisations from projective and injective liftings alone. Note that this approach does *not* avoid the cardinality arguments involved in Smith's theorem; rather, it pushes them elsewhere, namely into the construction of injective liftings of model structures as detailed in [25]. In particular, our approach gives no more of an explicit grasp on the classes of maps of a localisation than the usual one. However, we believe there are still good reasons for adopting it.

One advantage of our approach is that it dualises trivially to give a construction of Bousfield colocalisations, wherein one enlarges the class of weak equivalences while fixing the class of *fibrations*; this time, one starts from the *colocal* objects—those which should be the cofibrant objects of the colocalised model structure—and constructs the desired colocalisation with reference to an adjunction between  $\mathcal{E}$  and a category of “algebraically colocal cofibrant objects”.

Another positive consequence of our approach, and our original motivation for developing it, is that allows for an account of (co)localisation for the *algebraic* model structures of Riehl [28]. These are combinatorially rich presentations of model categories in which, among other things, (acyclic) fibrant replacement constitutes a monad on the category of arrows, and (acyclic) cofibrant replacement a comonad; they have been used to derive non-trivial homotopical results [2, 7, 12], and are of some importance in the *homotopy type theory* project [34]. However, there is no account of localisation for algebraic model structures as there seems to be no “algebraic” version of Smith's theorem. On the other hand, there *are* algebraic versions of injective and projective lifting [8, §4.5]; whence our interest. A potential application of this would be to the study of localisation for model structures which, while not cofibrantly generated in the classical sense, *are* cofibrantly generated in the algebraic sense; see the discussion in [3].

As noted above, the subtlest point in our approach lies in choosing the algebraic structure which constitutes the notion of “algebraically local object”. The key issue is whether one can construct the required path objects in  $\mathcal{L}$ , and this is sensitive both to the choice of  $\mathcal{L}$  and the nature of the model category  $\mathcal{E}$ ; see [12, 26] for some discussion of this point. This delicacy is somewhat orthogonal to the main thrust of our argument, and so in this paper, we sidestep it entirely by concentrating on the situation in which the *property* and the *structure* of locality necessarily coincide. This is the setting of one-dimensional model structures, and this is why we concentrate on this seemingly degenerate case.

In elementary terms, a model structure is one-dimensional when the liftings involved in its factorisations are *unique*. Such model categories were introduced and investigated in [27]; however, it was left open as to whether examples of such model structures arise in mathematical practice. A subsidiary objective of this paper is to show that, in fact, this is the case: for example, if  $A$  is a commutative ring, then there is a model structure on the category  $[\mathbf{Alg}_A^{\text{fp}}, \mathbf{Set}]$  of diagrams of finitely presented  $A$ -algebras whose fibrant objects are sheaves on the big Zariski topos of  $A$  (i.e., generalised algebraic spaces over  $\text{Spec } A$ ), and whose cofibrant–fibrant objects are sheaves on the topological space  $\text{Spec } A$ .

We conclude this introduction with a short overview of the contents of the paper. In Sect. 2 we recall the necessary model-categorical background on combinatoriality, lifting, and mixing of model structures. In Sect. 3, we introduce one-dimensional model structures and study their homotopical properties. Then in Sect. 4, we implement our general approach to localisation in the context of one-dimensional model structures, by providing a set of conditions which perfectly characterise the categories of fibrant objects in a localisation of a one-dimensional model structure. In Sect. 5, we explain how matters are simplified by the assumption of left properness, culminating in our first main result, Theorem 26; then in Sect. 6 we use this to recover the classical account of localisation at a given set of maps in a left proper one-dimensional model structure. In Sect. 7, we dualise our theory to the case of colocalisation for one-dimensional model structures, obtaining our second main Theorem 36; and finally, in Sect. 8, we illustrate our results with a range of examples of one-dimensional model structures.

## 2 Model-Categorical Background

Throughout the paper, we write  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  for a model structure with cofibrations  $\mathcal{C}$ , weak equivalences  $\mathcal{W}$  and fibrations  $\mathcal{F}$ , and write  $\mathcal{TC} = \mathcal{C} \cap \mathcal{W}$  and  $\mathcal{TF} = \mathcal{F} \cap \mathcal{W}$  for the acyclic cofibrations and fibrations. We assume our model categories to be locally small, complete and cocomplete, and endowed with functorial factorisations; these induce functorial fibrant and cofibrant replacements, which we write as  $\eta: 1 \Rightarrow R$  and  $\varepsilon: Q \Rightarrow 1$ . We write  $\text{RLP}(\mathcal{K})$  or  $\text{LLP}(\mathcal{K})$  for the class of maps with the right or left lifting property with respect to a class of maps  $\mathcal{K}$ , and write  $U^{-1}(\mathcal{K})$  for the inverse image of the class under a functor  $U$ .

**Definition 1** Suppose that  $\mathcal{E}$  is a category equipped with a model structure  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  and that  $U: \mathcal{D} \rightarrow \mathcal{E}$ .

- The *projectively lifted* model structure on  $\mathcal{D}$ , if it exists, is the one whose weak equivalences and fibrations are given by  $U^{-1}(\mathcal{W})$  and  $U^{-1}(\mathcal{F})$  respectively.
- The *injectively lifted* model structure on  $\mathcal{D}$ , if it exists, is that whose cofibrations and weak equivalences are given by  $U^{-1}(\mathcal{C})$  and  $U^{-1}(\mathcal{W})$  respectively.

The basic setting in which lifted model structures are guaranteed to exist is that of combinatorial model categories. Recall that a model category is called *combinatorial* if its underlying category is locally presentable [17], and its two weak factorisation systems  $(\mathcal{C}, \mathcal{TF})$  and  $(\mathcal{TC}, \mathcal{F})$  are cofibrantly generated.

**Proposition 2** *Let  $\mathcal{E}$  be a combinatorial model category, let  $\mathcal{D}$  be a locally presentable category, and let  $U: \mathcal{D} \rightarrow \mathcal{E}$ .*

- If  $U$  is a right adjoint, and the acyclicity condition  $\text{LLP}(U^{-1}(\mathcal{F})) \subset U^{-1}(\mathcal{W})$  holds, then the projective lifting along  $U$  exists and is combinatorial.*

(ii) *If  $U$  is a left adjoint, and the acyclicity condition  $\text{RLP}(U^{-1}(C)) \subset U^{-1}(\mathcal{W})$  holds, then the injective lifting along  $U$  exists and is combinatorial.*

**Proof** (i) follows from [19, Theorem 11.3.2] plus the fact that any set of maps in a locally presentable category permits the small object argument; the argument for (ii) is due to [25], but is given in the form we need in [3, Theorem 2.23]. □

Despite their surface similarity, the two parts of this result are sharply different from each other. In (i), we obtain explicit choices of generating (acyclic) cofibrations for  $\mathcal{D}$  by applying  $F$  to the corresponding generators for  $\mathcal{E}$ . In (ii), by contrast, it is typically impossible to write down explicit sets of generating (acyclic) cofibrations for  $\mathcal{D}$ ; one merely knows that they exist.

Note also the following result, which will be useful in the sequel. In its statement, an *accessible* functor is one preserving  $\kappa$ -filtered colimits for a regular cardinal  $\kappa$ .

**Proposition 3** *If  $\mathcal{E}$  is a combinatorial model category, then it admits a cofibrant replacement functor  $Q$  and fibrant replacement functor  $R$  which are accessible.*

**Proof** See [15, Proposition 2.3]. □

The use we make of this fact is encapsulated in the following standard result from the theory of locally presentable categories.

**Proposition 4** *If  $\mathcal{A} \subseteq \mathcal{E}$  is a full reflective (resp., coreflective) subcategory and  $\mathcal{E}$  is locally presentable, then  $\mathcal{A}$  is locally presentable if and only if the reflector  $R: \mathcal{E} \rightarrow \mathcal{E}$  (resp., coreflector  $Q: \mathcal{E} \rightarrow \mathcal{E}$ ) is accessible.*

**Proof** The “only if” direction follows on observing that, by [17, Satz 14.6], any adjunction between locally presentable categories induces both an accessible monad and an accessible comonad. In the “if” direction, note that, in either case, the subcategory  $\mathcal{A}$  is complete and cocomplete, and so by [1, Theorem 2.47] will be locally presentable so as long as it is an accessible category [24]. But  $\mathcal{A}$  is the universal subcategory of  $\mathcal{E}$  on which  $\eta: 1 \Rightarrow R$  (resp.,  $\varepsilon: Q \Rightarrow 1$ ) becomes invertible, and so by [24, Theorem 5.1.6] is accessible since  $R$  (resp.,  $Q$ ) is so. □

We now recall Cole’s result [13] on *mixing* model structures.

**Proposition 5** *Let  $(\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)$  and  $(\mathcal{C}_2, \mathcal{W}_2, \mathcal{F}_2)$  be combinatorial model structures on the same category  $\mathcal{E}$ . If  $\mathcal{F}_1 \subseteq \mathcal{F}_2$  and  $\mathcal{W}_1 \subseteq \mathcal{W}_2$ , then there is a combinatorial mixed model structure  $(\mathcal{C}_m, \mathcal{W}_m, \mathcal{F}_m)$  on  $\mathcal{E}$  with  $\mathcal{F}_m = \mathcal{F}_1$  and  $\mathcal{W}_m = \mathcal{W}_2$ .*

**Proof** Consider the combinatorial model structure  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  on  $\mathcal{E} \times \mathcal{E}$  which in its first component is given by  $(\mathcal{TC}_1, \text{all}, \mathcal{F}_1)$  and in its second by  $(\mathcal{C}_2, \mathcal{W}_2, \mathcal{F}_2)$ . The diagonal  $\Delta: \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E}$  is a right adjoint between locally presentable categories, and we have that  $\Delta^{-1}(\mathcal{F}) = \mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}_1$  and  $\Delta^{-1}(\mathcal{W}) = \text{all} \cap \mathcal{W}_2 = \mathcal{W}_2$ . Thus, since  $\text{LLP}(\Delta^{-1}(\mathcal{F})) = \mathcal{TC}_1 \subseteq \mathcal{W}_1 \subseteq \mathcal{W}_2 = \Delta^{-1}(\mathcal{W})$ , the projectively lifted model structure  $(\mathcal{C}_m, \mathcal{W}_m, \mathcal{F}_m)$  exists, and has  $\mathcal{W}_m = \mathcal{W}_2$  and  $\mathcal{F}_m = \mathcal{F}_1$ . □

The proof we give is less explicit than Cole’s; he constructs the required factorisations directly rather than appealing to a lifting result. This allows him to avoid the assumption of combinatoriality of the two starting model structures, but means that he does not derive it for the mixed model structure either.

It is in fact easy to derive explicit generating sets for the mixed model structure from ones for the two given model structures. Since  $\mathcal{F}_m = \mathcal{F}_1$ , a generating set of mixed acyclic cofibrations is given by any generating set for  $\mathcal{TC}_1$ ; and since  $\mathcal{TF}_m = \mathcal{F}_1 \cap \mathcal{W}_2 = \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{W}_2 = \mathcal{F}_1 \cap \mathcal{TF}_2$ , a generating set of mixed cofibrations is given by the union of any generating set for  $\mathcal{TC}_1$  and any generating set for  $\mathcal{C}_2$ .

Cole's construction of the mixed model structure dualises without difficulty. Our proof also dualises by using injective rather than projective liftings. Once again, we must add the assumption of combinatoriality of the input model structures, but gain it on the output side. This time we cannot, in general, find explicit generating sets of maps for the mixed model structure.

**Proposition 6** *Let  $(\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)$  and  $(\mathcal{C}_2, \mathcal{W}_2, \mathcal{F}_2)$  be combinatorial model structures on the same category  $\mathcal{E}$ . If  $\mathcal{C}_1 \subseteq \mathcal{C}_2$  and  $\mathcal{W}_1 \subseteq \mathcal{W}_2$ , then there is a combinatorial mixed model structure  $(\mathcal{C}_m, \mathcal{W}_m, \mathcal{F}_m)$  on  $\mathcal{E}$  with  $\mathcal{C}_m = \mathcal{C}_1$  and  $\mathcal{W}_m = \mathcal{W}_2$ .*

### 3 One-Dimensional Model Structures

**Definition 7** The *model category of 0-types* is the category of sets endowed with the cartesian monoidal model structure (all, iso, all). A model category  $\mathcal{E}$  is called *one-dimensional* if it is enriched over the model category of 0-types.

The following result characterises the underlying weak factorisation systems of one-dimensional model structures; for a yet more comprehensive list of characterisations, see [30, Proposition 2.3].

**Proposition 8** *The following are equivalent for a weak factorisation system  $(\mathcal{L}, \mathcal{R})$  on a finitely complete and cocomplete category:*

- (i) *Every  $\mathcal{L}$ -map has the unique lifting property against each  $\mathcal{R}$ -map;*
- (ii) *If  $f: A \rightarrow B$  is in  $\mathcal{L}$ , then so is the codiagonal  $\nabla: B +_A B \rightarrow B$ ;*
- (iii) *If  $f: A \rightarrow B$  is in  $\mathcal{R}$ , then so is the diagonal  $\Delta: A \rightarrow A \times_B A$ ;*
- (iv) *If  $f: A \rightarrow B$  and  $g: B \rightarrow C$  and  $f \in \mathcal{L}$ , then  $g \in \mathcal{L}$  iff  $gf \in \mathcal{L}$ ;*
- (v) *If  $f: A \rightarrow B$  and  $g: B \rightarrow C$  and  $g \in \mathcal{R}$ , then  $f \in \mathcal{R}$  iff  $gf \in \mathcal{R}$ .*

**Proof** (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iii) by [9, § 4.5], while (i)  $\Leftrightarrow$  (iv)  $\Leftrightarrow$  (v) by [29, Satz 3].  $\square$

We call a weak factorisation system satisfying these conditions *orthogonal*.

**Proposition 9** *The following are equivalent for a locally small model category  $\mathcal{E}$ :*

- (i)  *$\mathcal{E}$  is one-dimensional;*
- (ii) *The weak factorisation systems  $(\mathcal{TC}, \mathcal{F})$  and  $(\mathcal{C}, \mathcal{TF})$  of  $\mathcal{E}$  are orthogonal.*

**Proof** The model structure for 0-types on **Set** has no generating acyclic cofibrations, and generating cofibrations  $\{0 \rightarrow 1, 2 \rightarrow 1\}$ . For any map  $f: A \rightarrow B$  in  $\mathcal{E}$ , its pushout tensor with  $0 \rightarrow 1$  in **Set** is  $f$ , while its pushout tensor with  $2 \rightarrow 1$  is  $\nabla: B +_A B \rightarrow B$ . So  $\mathcal{E}$  is enriched over the model structure for 0-types if and only if both  $\mathcal{C}$  and  $\mathcal{TC}$  satisfy the closure condition in Proposition 8(ii).  $\square$

By the standard properties of orthogonal factorisation systems [16], factorisations of maps in a one-dimensional model category  $\mathcal{E}$  are unique to within unique isomorphism. We also have the following good behaviour of the full subcategories  $i: \mathcal{E}_f \hookrightarrow \mathcal{E}$  and  $j: \mathcal{E}_c \hookrightarrow \mathcal{E}$  of fibrant and cofibrant objects:

**Proposition 10** *For any one-dimensional model category  $\mathcal{E}$ , there are adjunctions*

$$\mathcal{E}_f \begin{array}{c} \xleftarrow{R} \\ \perp \\ \xrightarrow{i} \end{array} \mathcal{E} \qquad \mathcal{E}_c \begin{array}{c} \xleftarrow{Q} \\ \top \\ \xrightarrow{j} \end{array} \mathcal{E} .$$

*In particular, both  $\mathcal{E}_c$  and  $\mathcal{E}_f$  are complete and cocomplete; if  $\mathcal{E}$  is combinatorial, then they are moreover locally presentable.*

**Proof** Each  $Y \in \mathcal{E}_f$  lifts uniquely against each acyclic cofibration  $\eta_X : X \rightarrow RX$ , whence  $\mathcal{E}(X, Y) \cong \mathcal{E}_f(RX, Y)$ , naturally in  $X$  and  $Y$ . Thus  $\mathcal{E}_f$  is reflective in  $\mathcal{E}$ , and so complete and cocomplete since  $\mathcal{E}$  is; we argue dually for  $\mathcal{E}_c$ . Finally, if  $\mathcal{E}$  is combinatorial, then  $\mathcal{E}_c$  and  $\mathcal{E}_f$  are locally presentable by Propositions 3 and 4.  $\square$

Moreover, we have the following homotopical properties:

**Proposition 11** *The following are true in a one-dimensional model category:*

- (i) *Every map between (co)fibrant objects is a (co)fibration.*
- (ii)  *$Q$  and  $R$  preserve and reflect weak equivalences.*
- (iii)  *$R$  preserves cofibrations and inverts acyclic cofibrations.*
- (iv)  *$Q$  preserves fibrations and inverts acyclic fibrations.*
- (v) *Any weak equivalence between fibrant-cofibrant objects is an isomorphism.*
- (vi) *There is a natural isomorphism  $QR \cong RQ$ .*
- (vii) *A map is a weak equivalence if and only if it is inverted by  $QR \cong RQ$ .*

**Proof** For (i), apply Proposition 8(iv) and (v) to composites  $0 \rightarrow A \rightarrow B$  and  $A \rightarrow B \rightarrow 1$ . (ii) is standard in any model category. For (iii), consider the square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & RA \\ f \downarrow & & \downarrow Rf \\ B & \xrightarrow{\eta_B} & RB . \end{array} \tag{3.1}$$

Since  $\eta_A$  and  $\eta_B$  are (acyclic) cofibrations, if  $f$  is a cofibration, then so is  $Rf$  by Proposition 8(iv). If  $f$  is moreover acyclic, then  $Rf$  is both an acyclic cofibration and a fibration, whence invertible. Now (iv) is dual to (iii). For (v), note that any weak equivalence between cofibrant-fibrant objects is also a cofibration and a fibration, whence invertible. For (vi), note that by (iii),  $QRX$  is fibrant and  $Q\eta_X : QX \rightarrow QRX$  is an acyclic cofibration; so by the uniqueness of the  $(\mathcal{TC}, \mathcal{F})$ -factorisation of  $QX \rightarrow 1$ , we must have  $QRX \cong RQX$ . Finally, (vii) follows from (ii) and (v) as  $QR \cong RQ$  preserves and reflects weak equivalences.  $\square$

### 4 Localities for One-Dimensional Model Structures

We now begin to investigate the process of localisation for one-dimensional combinatorial model categories. First we fix our terminology.

**Definition 12** *A combinatorial localisation of a combinatorial one-dimensional model category  $\mathcal{E}$  is a combinatorial one-dimensional model category  $\mathcal{E}_\ell$  with the same underlying category, the same cofibrations, and at least as many acyclic cofibrations.*



As in the introduction, a localisation  $\mathcal{E}_\ell$  of  $\mathcal{E}$  is completely determined by its subcategory  $(\mathcal{E}_\ell)_f$  of fibrant objects. Our objective in this section is to show that, in the one-dimensional combinatorial setting, the subcategories so arising are captured perfectly by the following notion of *homotopical locality*.

**Definition 13** A *locality* for a combinatorial one-dimensional model category  $\mathcal{E}$  is a full subcategory  $\mathcal{E}_{\ell f} \subseteq \mathcal{E}_f$ , whose objects we call *local*, such that:

- (i)  $\mathcal{E}_{\ell f}$  is locally presentable and reflective in  $\mathcal{E}$  via a reflector  $\nu: 1 \rightarrow R_\ell$ ;
- (ii) If  $X, Y \in \mathcal{E}_f$  are weakly equivalent, then  $X$  is local just when  $Y$  is.

We call a locality *homotopical* if, in addition:

- (iii)  $R_\ell$  preserves weak equivalences.

Note that  $\mathcal{E}_f$  is itself a locality—indeed, the maximal one—and is homotopical by Proposition 11(ii).

**Remark 14** The data for a homotopical locality resemble the input data for the Bousfield–Friedlander approach to localisation [10, Theorem A.7]. Their setting also involves a right properness axiom which ensures that the necessary factorisations can be constructed in an elementary fashion; however, as noted in [32], this axiom can be dropped in the combinatorial setting, at the cost of losing an explicit grasp on the factorisations. The axioms for a homotopical locality above are a one-dimensional version of this more general form of the Bousfield–Friedlander axioms.

The easier direction is that any localisation gives rise to a homotopical locality:

**Proposition 15** *Let  $\mathcal{E}$  be a combinatorial one-dimensional model category, and let  $\mathcal{E}_\ell$  be a combinatorial localisation of  $\mathcal{E}$ . The subcategory  $(\mathcal{E}_\ell)_f$  of  $\mathcal{E}_\ell$ -fibrant objects is a homotopical locality for  $\mathcal{E}$ .*

**Proof**  $(\mathcal{E}_\ell)_f \subseteq \mathcal{E}_f$  since  $\mathcal{E}_\ell$  has more acyclic cofibrations than  $\mathcal{E}$ ; it is of course full, replete and reflective in  $\mathcal{E}$ , and is locally presentable by Proposition 10. We next verify (iii). If  $f$  is an  $\mathcal{E}$ -weak equivalence, then it is an  $\mathcal{E}_\ell$ -weak equivalence, whence by Proposition 11(vii) inverted by  $QR_\ell$ . Since  $Q$  is also the cofibrant replacement for  $\mathcal{E}$ , we conclude by Proposition 11(ii) that  $R_\ell(f)$  is an  $\mathcal{E}$ -weak equivalence. Finally, for (ii), let  $f: X \rightarrow Y$  be a weak equivalence in  $\mathcal{E}_f$  and consider the square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \nu_X \downarrow & & \downarrow \nu_Y \\ R_\ell X & \xrightarrow{R_\ell f} & R_\ell Y \end{array}$$

We must show  $\nu_X$  is invertible just when  $\nu_Y$  is. But both are (acyclic) cofibrations in  $\mathcal{E}_\ell$ , whence cofibrations in  $\mathcal{E}$ , between  $\mathcal{E}$ -fibrant objects; so they are invertible just when they are  $\mathcal{E}$ -weak equivalences. But  $R_\ell$  preserves  $\mathcal{E}$ -weak equivalences, so both horizontal maps are  $\mathcal{E}$ -weak equivalences, whence  $\nu_X$  is an  $\mathcal{E}$ -weak equivalence if and only if  $\nu_Y$  is.  $\square$

To show conversely that every homotopical locality  $\mathcal{E}_{\ell f}$  arises from a localisation, we carry out the procedure outlined in the introduction: lifting the model structure from  $\mathcal{E}$  to  $\mathcal{E}_{\ell f}$  and back again, and mixing the result with the original model structure. Before doing so, we establish some necessary properties of localities.



**Lemma 16** *If  $\mathcal{E}_{\ell f}$  is a locality for  $\mathcal{E}$ , then:*

- (i)  $Q$  preserves and reflects locality of fibrant objects;
- (ii) Each  $\nu_X: X \rightarrow R_\ell X$  is a cofibration;
- (iii)  $R_\ell$  preserves cofibrations.

**Proof** For (i), apply property (ii) of a locality to  $\varepsilon_X: QX \rightarrow X$ . For (ii), we factor  $\nu_X$  as a cofibration  $f: X \rightarrow P$  followed by an acyclic fibration  $g: P \rightarrow R_\ell X$ . Now  $P$  is fibrant since  $R_\ell X$  is, and  $R_\ell X$  is local; so applying property (ii) of a locality to  $g$ , we conclude that  $P$  is local. We can thus extend  $f: X \rightarrow P$  to a map  $h: R_\ell X \rightarrow P$  with  $h\nu_X = f$ ; now  $\nu_X = gf = gh\nu_X$ , whence  $gh = 1$ , and so

$$\begin{array}{ccccc}
 X & \xlongequal{\quad} & X & \xlongequal{\quad} & X \\
 \nu_X \downarrow & & f \downarrow & & \nu_X \downarrow \\
 R_\ell X & \xrightarrow{h} & P & \xrightarrow{g} & R_\ell X
 \end{array}$$

exhibits  $\nu_X$  as a retract of the cofibration  $f$  and so a cofibration. Finally, for (iii), apply (ii) and Proposition 8(iv) to the naturality square for  $R_\ell$  at a cofibration. □

We now show that the model structure on  $\mathcal{E}$  lifts along the inclusion  $\mathcal{E}_{\ell f} \rightarrow \mathcal{E}$ .

**Proposition 17** *If  $\mathcal{E}_{\ell f}$  is a locality for the one-dimensional combinatorial  $\mathcal{E}$ , then the model structure on  $\mathcal{E}$  restricts to one on  $\mathcal{E}_{\ell f}$ , with classes as follows:*

- Cofibrations = LLP(*maps inverted by  $Q$* );
- Acyclic cofibrations = *isomorphisms*;
- Fibrations = *all maps*;
- Acyclic fibrations = *weak equivalences = maps inverted by  $Q$* .

*The restricted model structure is one-dimensional and projectively lifts that on  $\mathcal{E}$ .*

**Proof** On taking either factorisation of a map between objects of  $\mathcal{E}_{\ell f}$ , the interposing object clearly lies in  $\mathcal{E}_f$ , and is moreover local by Lemma 16(i). So the model structure restricts; in particular, it is a projective lifting along  $\mathcal{E}_{\ell f} \rightarrow \mathcal{E}$  and so also one-dimensional. Since  $\mathcal{E}_{\ell f} \subseteq \mathcal{E}_f$ , Proposition 11(i) gives the characterisation of the restricted fibrations and acyclic cofibrations. The restricted weak equivalences therefore equal the restricted acyclic fibrations; and as every local object is fibrant, these are by Proposition 11(ii), (iii) and (v), exactly the maps inverted by  $Q$ . □

And now we lift back in the other direction:

**Proposition 18** *Let  $\mathcal{E}_{\ell f}$  be a homotopical locality for  $\mathcal{E}$ . The restricted model structure on  $\mathcal{E}_{\ell f}$  lifts injectively along  $R_\ell: \mathcal{E} \rightarrow \mathcal{E}_{\ell f}$ ; this new model structure  $\mathcal{E}'$  is one-dimensional, with acyclic cofibrations the maps inverted by  $R_\ell$ , and with fibrant objects the local objects. The identity functor  $\mathcal{E} \rightarrow \mathcal{E}'$  is left Quillen.*

**Proof** By assumption,  $\mathcal{E}_{\ell f}$  is locally presentable and  $R_\ell$  is a left adjoint. Now as  $R_\ell$  preserves weak equivalences by assumption and cofibrations by Lemma 16(iii), we have  $\mathcal{W} \subseteq R_\ell^{-1}(\mathcal{W})$  and  $\mathcal{C} \subseteq R_\ell^{-1}(\mathcal{C})$ , whence  $\text{RLP}(R_\ell^{-1}(\mathcal{C})) \subseteq \text{RLP}(\mathcal{C}) = \mathcal{TF} \subseteq \mathcal{W} \subseteq R_\ell^{-1}(\mathcal{W})$ . Thus by Proposition 2, the injectively lifted model structure  $\mathcal{E}'$  exists, and is one-dimensional by Proposition 8(iv); since  $\mathcal{E}'$  has more cofibrations and acyclic cofibrations than  $\mathcal{E}$ , the identity  $\mathcal{E} \rightarrow \mathcal{E}'$  is left Quillen. The characterisation of the acyclic cofibrations follows since the  $\mathcal{E}_{\ell f}$ -acyclic cofibrations are the isomorphisms. Finally, an object is  $\mathcal{E}'$ -fibrant if and only if it is orthogonal to all maps inverted by the reflector  $R_\ell: \mathcal{E} \rightarrow \mathcal{E}_{\ell f}$ , if and only if it is in  $\mathcal{E}_{\ell f}$ . □

Finally, we mix this new model structure with our original one:

**Proposition 19** *If  $\mathcal{E}_{\ell f}$  is a homotopical locality for  $\mathcal{E}$ , then there exists a combinatorial localisation  $\mathcal{E}_{\ell}$  of  $\mathcal{E}$  such that  $(\mathcal{E}_{\ell})_f = \mathcal{E}_{\ell f}$ .*

**Proof** The model structure  $\mathcal{E}'$  of the last result has more cofibrations and weak equivalences than  $\mathcal{E}$ ; so we can mix with  $\mathcal{E}$  to obtain a model structure  $\mathcal{E}_{\ell}$  with the cofibrations of  $\mathcal{E}$  and the weak equivalences of  $\mathcal{E}'$ . It remains to show  $(\mathcal{E}_{\ell})_f = \mathcal{E}_{\ell f}$ . As the  $\mathcal{E}'$ -fibrant objects are the local ones, and as  $\mathcal{TC}_{\ell} \subseteq \mathcal{TC}'$ , each local object is  $\mathcal{E}_{\ell}$ -fibrant. Conversely, if  $X$  is  $\mathcal{E}_{\ell}$ -fibrant, then since  $v_X: X \rightarrow R_{\ell}X$  is in  $\mathcal{TC}_{\ell}$ —being both an  $\mathcal{E}$ -cofibration and inverted by  $R_{\ell}$ —the identity  $X \rightarrow X$  extends to a retraction  $p: RX \rightarrow X$  for  $v_X$ . Now since  $v_X p v_X = v_X$  also  $v_X p = 1$  and so  $X$  is local as an isomorph in  $\mathcal{E}_f$  of the local  $RX$ .  $\square$

Putting together the preceding results, we obtain:

**Theorem 20** *Let  $\mathcal{E}$  be a one-dimensional combinatorial model category. The assignation  $\mathcal{E}_{\ell} \mapsto (\mathcal{E}_{\ell})_f$  yields an order-reversing bijection between combinatorial localisations of  $\mathcal{E}$  (ordered by inclusion of their acyclic cofibrations) and homotopical localities for  $\mathcal{E}$  (ordered by inclusion of their subcategories of local objects).*

**Proof** The assignation is well-defined by Proposition 15, clearly order-reversing, and surjective by Proposition 19. Moreover, two localisations of  $\mathcal{E}$  which induce the same localities must have isomorphic cofibrant–fibrant replacement functors, whence by Proposition 11(vii) the same weak equivalences; and so must coincide.  $\square$

## 5 Left Properness

Localisation of model structures is often carried out under the assumption of left properness; recall that a model structure is called *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence. We now explain the significance of this condition in the one-dimensional context, by proving:

**Proposition 21** *If  $\mathcal{E}$  is one-dimensional, combinatorial and left proper, then any locality for  $\mathcal{E}$  is homotopical.*

Before proving this, we establish some preparatory lemmas. In stating the first, note that, as a special case of Proposition 17, any one-dimensional model structure on a category  $\mathcal{E}$  restricts to a one-dimensional model structure on  $\mathcal{E}_f$ .

**Lemma 22** *A one-dimensional model category  $\mathcal{E}$  is left proper if and only if the restricted model structure on  $\mathcal{E}_f$  is left proper.*

**Proof**  $R: \mathcal{E} \rightarrow \mathcal{E}_f$  preserves pushouts, weak equivalences and cofibrations; so if  $\mathcal{E}_f$  is left proper, then the cobase change in  $\mathcal{E}$  of a weak equivalence along a cofibration is sent by  $R$  to another such cobase change in  $\mathcal{E}_f$  and so, by left properness, to a weak equivalence. As  $R$  reflects weak equivalences, this shows  $\mathcal{E}$  is left proper.

Conversely, if  $\mathcal{E}$  is left proper, then the cobase change in  $\mathcal{E}_f$  of a weak equivalence along a cofibration may be calculated by forming the cobase change in  $\mathcal{E}$ —which yields a weak equivalence by left properness—and then applying  $R$ —which yields a weak equivalence since  $R$  preserves such. This shows  $\mathcal{E}_f$  is left proper.  $\square$

**Remark 23** It follows that any localisation of the left proper one-dimensional  $\mathcal{E}$  is left proper. For indeed, by copying the “only if” direction of the preceding proof, we see that the restriction of the model structure  $\mathcal{E}$  to  $(\mathcal{E}_\ell)_f$  is left proper. But this model structure is equally the restriction of the model structure  $\mathcal{E}_\ell$  to  $(\mathcal{E}_\ell)_f$  and so by the “if” direction of the preceding result,  $\mathcal{E}_\ell$  is also left proper.

**Lemma 24** If  $\mathcal{E}_{\ell f}$  is a locality for the left proper  $\mathcal{E}$ , then for any weak equivalence  $f : X \rightarrow Y$  between fibrant objects, the following is a pushout in  $\mathcal{E}_f$ :

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \downarrow v_X & & \downarrow v_Y \\
 R_\ell X & \xrightarrow{R_\ell f} & R_\ell Y
 \end{array} \tag{5.1}$$

**Proof** Let us form the pushout

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \downarrow v_X & & \downarrow p \\
 R_\ell X & \xrightarrow{q} & P
 \end{array}$$

in the left proper  $\mathcal{E}_f$ . Since  $f$  is a weak equivalence and  $v_X$  is a cofibration,  $q$  is also a weak equivalence in  $\mathcal{E}_f$ ; thus since  $R_\ell X$  is local,  $P$  is too. Moreover, since  $v_X$  has the left lifting property against any local object, so does its pushout  $p$ ; whence  $p : Y \rightarrow P$  is a reflection of  $Y$  into  $\mathcal{E}_{\ell f}$ . As  $v_Y : Y \rightarrow R_\ell Y$  is another such reflection, the unique induced map  $P \rightarrow R_\ell Y$  is thus invertible.  $\square$

We can now give:

**Proof of Proposition 21** Let  $\mathcal{E}_{\ell f}$  be a locality. If  $g : X \rightarrow Y$  is a weak equivalence in  $\mathcal{E}$ , then  $Rg$  is one in  $\mathcal{E}_f$ , and so taking  $f = Rg$  in (5.1) shows that  $R_\ell Rg$ , as a pushout of a weak equivalence along a cofibration in the left proper  $\mathcal{E}_f$ , is also a weak equivalence. Now  $\eta_X : X \rightarrow R_\ell X$  and  $\eta_Y : X \rightarrow R_\ell Y$  have the unique left lifting property against every (fibrant and so every) local object, and as such are inverted by  $R_\ell$ ; whence

$$R_\ell g = R_\ell X \xrightarrow{R_\ell \eta_X} R_\ell R_\ell X \xrightarrow{R_\ell Rg} R_\ell R_\ell Y \xrightarrow{(R_\ell \eta_Y)^{-1}} R_\ell Y$$

is a composite of weak equivalences and so a weak equivalence.  $\square$

We have thus shown that, in the left proper context, we can drop the modifier “homotopical” from the statement of Theorem 20: that is, localisations of the left proper  $\mathcal{E}$  correspond to localities on  $\mathcal{E}$ . The value of this is that localities are rather easy to construct, by virtue of:

**Proposition 25** Let  $\mathcal{E}$  be a combinatorial one-dimensional model category. The assignation  $\mathcal{E}_{\ell f} \mapsto \mathcal{E}_{\ell f} \cap \mathcal{E}_c$  yields an order-preserving bijection between localities for  $\mathcal{E}$  and full, replete, reflective, locally presentable subcategories of  $\mathcal{E}_{cf}$  (where in each case the order is given by inclusion of subcategories).

**Proof** If  $\mathcal{E}_{\ell f}$  is a locality, then by Lemma 16(iii) its reflector  $R_\ell$  maps  $\mathcal{E}_{cf}$  into  $\mathcal{E}_{\ell f} \cap \mathcal{E}_c$ , so that  $\mathcal{E}_{\ell f} \cap \mathcal{E}_c$  is reflective in  $\mathcal{E}_{cf}$  via  $R_\ell$ . Moreover, since  $\mathcal{E}_{\ell f}$  is reflective in  $\mathcal{E}_f$  via  $R_\ell$ , the functor  $R_\ell : \mathcal{E}_f \rightarrow \mathcal{E}_f$  is accessible by Proposition 4. Since  $\mathcal{E}_{cf}$  is coreflective in  $\mathcal{E}_f$  and

hence closed under colimits,  $R_\ell : \mathcal{E}_{cf} \rightarrow \mathcal{E}_{cf}$  is also accessible and hence  $\mathcal{E}_{\ell f} \cap \mathcal{E}_c$  is locally presentable by Proposition 4 again.

This shows that  $\mathcal{E}_{\ell f} \mapsto \mathcal{E}_{\ell f} \cap \mathcal{E}_c$  is well-defined, and it is injective since for any locality, the objects in  $\mathcal{E}_{\ell f}$  are, by Lemma 16(i), those  $X \in \mathcal{E}_f$  for which  $QX \in \mathcal{E}_{\ell f} \cap \mathcal{E}_c$ . To show surjectivity, let  $\mathcal{E}_{\ell cf} \subseteq \mathcal{E}_{cf}$  be reflective and locally presentable. Form the pullback

$$\begin{array}{ccc} \mathcal{E}_{\ell f} & \xrightarrow{\quad} & \mathcal{E}_f \\ \downarrow Q' & \lrcorner & \downarrow Q \\ \mathcal{E}_{\ell cf} & \xrightarrow{\quad i \quad} & \mathcal{E}_{cf} \end{array};$$

so objects of  $\mathcal{E}_{\ell f}$  are objects  $X \in \mathcal{E}_f$  with  $QX \in \mathcal{E}_{\ell cf}$ . Since  $QX \cong X$  whenever  $X \in \mathcal{E}_c$ , we have  $\mathcal{E}_{\ell f} \cap \mathcal{E}_c = \mathcal{E}_{\ell cf}$ , and so will be done so long as  $\mathcal{E}_{\ell f}$  is a locality.

To check condition (i) of Definition 13, note that  $\mathcal{E}_{\ell cf}, \mathcal{E}_{cf}$  and  $\mathcal{E}_f$  are locally presentable,  $i$  and  $Q$  are right adjoints, and  $i$  is an isofibration; so by [6, Theorem 2.18],  $\mathcal{E}_{\ell f}$  is also locally presentable, and  $i'$  is also a right adjoint. So  $\mathcal{E}_{\ell f}$  is reflective in  $\mathcal{E}_f$ , and  $\mathcal{E}_f$  is reflective in  $\mathcal{E}$ , whence  $\mathcal{E}_{\ell f}$  is reflective in  $\mathcal{E}$ . To check Definition 13(ii), note that if  $f : X \rightarrow Y$  is a weak equivalence in  $\mathcal{E}_f$ , then  $Qf : QX \rightarrow QY$  is invertible in  $\mathcal{E}_{cf}$ ; since  $\mathcal{E}_{\ell cf}$  is replete in  $\mathcal{E}_{cf}$ , we thus have that  $X \in \mathcal{E}_{\ell f}$  iff  $QX \in \mathcal{E}_{\ell cf}$  iff  $QY \in \mathcal{E}_{\ell cf}$  iff  $Y \in \mathcal{E}_{\ell f}$ .  $\square$

Combining this with Theorem 20 and Proposition 21, we therefore obtain:

**Theorem 26** *If  $\mathcal{E}$  is a left proper one-dimensional combinatorial model category, then the assignment  $\mathcal{E}_\ell \mapsto (\mathcal{E}_\ell)_{cf}$  yields an order-reversing bijection between combinatorial localisations of  $\mathcal{E}$  (ordered by inclusion of acyclic cofibrations) and full, replete, reflective, locally presentable subcategories of  $\mathcal{E}_{cf}$  (ordered by inclusion).*

## 6 Localisation at a Set of Maps

In practice, one often constructs localisations of a left proper model category starting from a set of maps which one wishes to make into weak equivalences. We now use the theory of the preceding section to reproduce this construction in the one-dimensional context. First we recall the basic definitions:

**Definition 27** If  $\mathcal{E}$  is a model category enriched over the monoidal model category  $\mathcal{V}$ , then the *derived hom* of  $\mathcal{E}$  is the functor

$$\mathcal{E}_h : \mathcal{E}^{\text{op}} \times \mathcal{E} \xrightarrow{\mathcal{E}(Q-, R-)} \mathcal{V} \xrightarrow{\text{Ho}} \text{Ho } \mathcal{V}.$$

For enrichment over the model structure for 0-types on **Set**, the functor  $\text{Ho} : \mathbf{Set} \rightarrow \text{Ho } \mathbf{Set}$  is the identity, so that for a one-dimensional model category  $\mathcal{E}$  we have  $\mathcal{E}_h(A, B) = \mathcal{E}(QA, RB)$ .

**Definition 28** If  $\mathcal{E}$  is a model  $\mathcal{V}$ -category,  $X \in \mathcal{E}$  and  $f \in \mathcal{E}(A, B)$ , then we write  $f \perp_h X$  if  $\mathcal{E}_h(f, X) : \mathcal{E}_h(B, X) \rightarrow \mathcal{E}_h(A, X)$  is invertible. Given a class of maps  $S$  in  $\mathcal{E}$ , we now say that:

- An object  $X \in \mathcal{E}$  is *S-local* if  $f \perp_h X$  for all  $f \in S$ ;
- A map  $f \in \mathcal{E}$  is an *S-local equivalence* if  $f \perp_h X$  for all  $S$ -local  $X \in \mathcal{E}$ .

**Remark 29** The derived hom  $\mathcal{E}_h$  has the property of sending weak equivalences in each variable to isomorphisms; in particular, we have

$$\mathcal{E}_h(A, B) \cong \mathcal{E}_h(A, QB) \cong \mathcal{E}_h(A, RB) \text{ and } \mathcal{E}_h(A, B) \cong \mathcal{E}_h(RA, B) \cong \mathcal{E}_h(QA, B).$$

It follows that  $Q$  and  $R$  preserve and reflect both  $S$ -local objects and  $S$ -local equivalences. As a consequence, in showing that a map  $f$  is an  $S$ -local equivalence, it suffices to check that  $f \perp_h X$  for each *cofibrant–fibrant*  $S$ -local  $X$ .

**Remark 30** If  $\mathcal{E}_\ell$  is any combinatorial localisation of  $\mathcal{E}$  whatsoever, then each  $\mathcal{E}$ -weak equivalence is an  $\mathcal{E}_\ell$ -weak equivalence. In particular, both  $\eta: 1 \rightarrow R$  and  $\varepsilon: Q \rightarrow 1$  are pointwise  $\mathcal{E}_\ell$ -weak equivalences so that, by the two-out-of-three property for  $\mathcal{E}_\ell$ -weak equivalences,  $Q$  and  $R$  preserve and reflect  $\mathcal{E}_\ell$ -equivalences.

**Theorem 31** *Let  $\mathcal{E}$  be combinatorial, left proper and one-dimensional. For any set of maps  $S$  of  $\mathcal{E}$ , there exists a combinatorial localisation  $\mathcal{E}_\ell$  of the model structure  $\mathcal{E}$  for which:*

- *The fibrant objects are the  $S$ -local  $\mathcal{E}$ -fibrant objects;*
- *The weak equivalences are the  $S$ -local equivalences.*

*Moreover, every combinatorial localisation of  $\mathcal{E}$  arises thus.*

**Proof** Given a set  $S$  of maps, let  $\mathcal{E}_{lcf} \subseteq \mathcal{E}_{cf}$  be the full subcategory of  $S$ -local fibrant–cofibrant objects. Note that  $X \in \mathcal{E}_{cf}$  is in  $\mathcal{E}_{lcf}$  just when  $\mathcal{E}_{cf}(QRf, X)$  is invertible for each  $f \in S$ . Thus, on taking  $S' = \{QRf : f \in S\}$  and

$$I = J = S' \cup \{\nabla_g : B +_A B \rightarrow B \mid g: A \rightarrow B \in S'\}$$

as generating (acyclic) cofibrations, the small object argument yields a combinatorial one-dimensional model structure on  $\mathcal{E}_{cf}$  with subcategory of fibrant objects  $\mathcal{E}_{lcf}$ . So by Proposition 10,  $\mathcal{E}_{lcf}$  is reflective in  $\mathcal{E}_{cf}$  and locally presentable.

Now, applying Theorem 26 to  $\mathcal{E}_{lcf}$  yields a localisation  $\mathcal{E}_\ell$  of  $\mathcal{E}$  with  $(\mathcal{E}_\ell)_{cf} = \mathcal{E}_{lcf}$ . As  $Q$  preserves and reflects both the  $\mathcal{E}_\ell$ -fibrancy and the  $S$ -locality of  $\mathcal{E}$ -fibrant objects, the  $\mathcal{E}_\ell$ -fibrant objects are the  $S$ -local  $\mathcal{E}$ -fibrant ones. Moreover, the  $\mathcal{E}_\ell$ -weak equivalences in  $\mathcal{E}_{cf}$  are the maps inverted by the reflector into  $\mathcal{E}_{lcf}$ , which are those  $f$  such that  $\mathcal{E}_{cf}(f, X) \cong \mathcal{E}_h(f, X)$  is invertible for all  $X \in \mathcal{E}_{lcf}$ —which, by Remark 29, are exactly the  $S$ -local equivalences in  $\mathcal{E}_{cf}$ . Since, by Remark 29 and Remark 30,  $QR$  preserves and reflects both  $S$ -local equivalences and  $\mathcal{E}_\ell$ -weak equivalences, it follows that the  $\mathcal{E}_\ell$ -weak equivalences are the  $S$ -local equivalences.

Finally, if  $\mathcal{E}_\ell$  is any localisation of  $\mathcal{E}$ , then by Theorem 26,  $(\mathcal{E}_\ell)_{cf}$  is locally presentable and reflective in  $\mathcal{E}_{cf}$ . Thus, by [1, Theorem 1.39], there is a set  $S$  of maps in  $\mathcal{E}_{cf}$  so that  $(\mathcal{E}_\ell)_{cf}$  comprises those  $X \in \mathcal{E}_{cf}$  for which  $\mathcal{E}_{cf}(-, X) \cong \mathcal{E}_h(-, X)$  inverts each  $g \in S$ —in other words, the  $S$ -local cofibrant–fibrant objects. As a model structure is determined by its cofibrations and cofibrant–fibrant objects,  $\mathcal{E}_\ell$  is thus the localisation of  $\mathcal{E}$  at  $S$ . □

## 7 Colocalisation

As noted in the introduction, an advantage of our approach is that everything we have done adapts without fuss from the case of localisations to *colocalisations*.

**Definition 32** *A combinatorial colocalisation of a combinatorial one-dimensional model category  $\mathcal{E}$  is a combinatorial one-dimensional model category  $\mathcal{E}_r$  with the same underlying category, the same fibrations, and at least as many acyclic fibrations.*

The arguments of Sect. 4 dualise immediately to show that colocalisations correspond to homotopical *colocalities*:

**Definition 33** A *colocality* for a combinatorial one-dimensional model category  $\mathcal{E}$  is a full subcategory  $\mathcal{E}_{rc} \subseteq \mathcal{E}_c$ , whose objects we call *colocal*, such that:

- (i)  $\mathcal{E}_{rc}$  is locally presentable and coreflective in  $\mathcal{E}$  via a coreflector  $\xi: Q_r \rightarrow 1$ ;
- (ii) If  $X, Y \in \mathcal{E}_c$  are weakly equivalent, then  $X$  is colocal just when  $Y$  is.

We call a colocality *homotopical* if, in addition:

- (iii)  $Q_r$  preserves weak equivalences.

**Theorem 34** Let  $\mathcal{E}$  be a one-dimensional combinatorial model category. The assignment  $\mathcal{E}_r \mapsto (\mathcal{E}_r)_c$  yields an order-reversing bijection between combinatorial colocalisations of  $\mathcal{E}$  (ordered by inclusion of their fibrations) and homotopical colocalities for  $\mathcal{E}$  (ordered by inclusion of their subcategories of colocal objects).

Now the arguments of Sect. 5 dualise to show that every colocality for the right proper one-dimensional combinatorial  $\mathcal{E}$  is homotopical. The analogue of Proposition 25, however, requires a proof which is not exactly dual, and which we therefore give in more detail:

**Proposition 35** Let  $\mathcal{E}$  be a combinatorial one-dimensional model category. The assignment  $\mathcal{E}_{rc} \mapsto \mathcal{E}_{rc} \cap \mathcal{E}_f$  yields an order-preserving bijection between colocalities for  $\mathcal{E}$  and full, replete, coreflective, locally presentable subcategories of  $\mathcal{E}_{cf}$  (where in each case the order is given by inclusion of subcategories).

**Proof** The coreflectivity of  $\mathcal{E}_{rc} \cap \mathcal{E}_f$  in  $\mathcal{E}_{cf}$  is dual to before. For its local presentability, as  $\mathcal{E}_{rc}$  is coreflective in  $\mathcal{E}_c$  via  $Q_r$ , Proposition 4 implies that  $Q_r: \mathcal{E}_c \rightarrow \mathcal{E}_c$  preserves  $\lambda$ -filtered colimits for some  $\lambda$ ; and as  $\mathcal{E}_{cf}$  is reflective in  $\mathcal{E}_c$ , it is by Proposition 4 closed in  $\mathcal{E}_c$  under  $\kappa$ -filtered colimits for some  $\kappa \geq \lambda$ . So  $Q_r: \mathcal{E}_{cf} \rightarrow \mathcal{E}_{cf}$  preserves  $\kappa$ -filtered colimits, and so  $\mathcal{E}_{rc} \cap \mathcal{E}_f$  is locally presentable by Proposition 4. Thus  $\mathcal{E}_{rc} \mapsto \mathcal{E}_{rc} \cap \mathcal{E}_f$  is well-defined, and it is injective as before; for surjectivity, given  $\mathcal{E}_{rcf} \subseteq \mathcal{E}_{cf}$  coreflective and locally presentable, we form the pullback

$$\begin{array}{ccc}
 \mathcal{E}_{rc} & \xrightarrow{\quad} & \mathcal{E}_c \\
 \downarrow R' & \lrcorner & \downarrow R \\
 & j' & \\
 \mathcal{E}_{rcf} & \xrightarrow{\quad} & \mathcal{E}_{cf}
 \end{array} ;$$

now the previous argument will carry over, *mutatis mutandis*, so long as  $\mathcal{E}_{rc}$  is locally presentable and  $j'$  has a right adjoint. Since  $R$  and  $j$  are left adjoint functors between locally presentable categories, this follows like before but now appealing to Theorem 3.15, rather than Theorem 2.18, of [6]. □

Putting these results together, we now obtain:

**Theorem 36** If  $\mathcal{E}$  is a right proper one-dimensional combinatorial model category, then the assignment  $\mathcal{E}_r \mapsto (\mathcal{E}_r)_{cf}$  yields an order-reversing bijection between combinatorial colocalisations of  $\mathcal{E}$  (ordered by inclusion of acyclic fibrations) and full, replete, coreflective, locally presentable subcategories of  $\mathcal{E}_{cf}$  (ordered by inclusion).

Analogously to Sect. 6, one often constructs colocalisations of a right proper model category from a set of objects which generate the colocal ones under homotopy colimits. We now rederive this result in the one-dimensional setting.

**Definition 37** Given a model  $\mathcal{V}$ -category  $\mathcal{E}$ , an object  $X \in \mathcal{E}$  and a map  $f \in \mathcal{E}(A, B)$ , we write  $X \perp_h f$  if  $\mathcal{E}_h(X, f): \mathcal{E}_h(X, A) \rightarrow \mathcal{E}_h(X, B)$  is invertible. Given a class of objects  $K$  in  $\mathcal{E}$ , we now say that:

- A map  $f \in \mathcal{E}$  is a  $K$ -colocal equivalence if  $X \perp_h f$  for all  $X \in K$ ;
- An object  $X \in \mathcal{E}$  is  $K$ -colocal if  $X \perp_h f$  for all  $K$ -colocal equivalences  $f \in \mathcal{E}$ .

**Theorem 38** Let  $\mathcal{E}$  be combinatorial, right proper and one-dimensional. For any set of objects  $K$  in  $\mathcal{E}$ , there exists a combinatorial colocalisation  $\mathcal{E}_r$  of the model structure  $\mathcal{E}$  for which:

- The cofibrant objects are the  $K$ -colocal  $\mathcal{E}$ -cofibrant objects;
- The weak equivalences are the  $K$ -colocal equivalences.

Moreover, every combinatorial colocalisation of  $\mathcal{E}$  arises thus.

**Proof** Given a set  $K$  of objects, let  $\mathcal{E}_{ref} \subseteq \mathcal{E}_{cf}$  be the full subcategory of  $K$ -colocal fibrant-cofibrant objects. Taking  $K' = \{QRX : X \in K\}$  and taking

$$I = J = \{0 \rightarrow Y : Y \in K'\} \cup \{\nabla : Y + Y \rightarrow Y \mid Y \in K'\}, \tag{7.1}$$

we obtain by the small object argument a combinatorial one-dimensional model structure on  $\mathcal{E}_{cf}$  with acyclic fibrations the  $K$ -colocal equivalences in  $\mathcal{E}_{cf}$ , and so, by the dual of Remark 29, with cofibrant objects the  $K$ -colocal objects in  $\mathcal{E}_{cf}$ . Thus, by Proposition 10,  $\mathcal{E}_{ref}$  is coreflective in  $\mathcal{E}_{cf}$  and locally presentable, and so applying Theorem 36 to  $\mathcal{E}_{ref}$  yields a colocalisation  $\mathcal{E}_r$  of  $\mathcal{E}$  with  $(\mathcal{E}_r)_{cf} = \mathcal{E}_{ref}$ .

The same argument as previously shows that the  $\mathcal{E}_r$ -cofibrant objects are the  $K$ -colocal  $\mathcal{E}$ -cofibrant ones. Moreover, the  $\mathcal{E}_r$ -weak equivalences in  $\mathcal{E}_{cf}$  are the maps inverted by the coreflector into  $\mathcal{E}_{ref}$ , which are those  $f$  such that  $\mathcal{E}_{cf}(X, f) \cong \mathcal{E}_h(X, f)$  is invertible for all  $X \in \mathcal{E}_{ref}$ . By the dual of Remark 29, these are exactly the  $K$ -colocal equivalences in  $\mathcal{E}_{cf}$ ; so arguing as before, the  $\mathcal{E}_r$ -weak equivalences are the  $K$ -colocal equivalences.

Finally, if  $\mathcal{E}_r$  is any colocalisation of  $\mathcal{E}$ , then by Theorem 36,  $(\mathcal{E}_r)_{cf}$  is locally presentable and coreflective in  $\mathcal{E}_{cf}$ . Since  $(\mathcal{E}_r)_{cf}$  is locally presentable, it has a small full subcategory  $\mathcal{A}$  whose colimit-closure in  $(\mathcal{E}_r)_{cf}$  is the whole category; thus, since  $(\mathcal{E}_r)_{cf}$  is closed in  $\mathcal{E}_{cf}$  under colimits, the colimit-closure of  $\mathcal{A}$  in  $\mathcal{E}_{cf}$  is  $(\mathcal{E}_r)_{cf}$ . Now let  $K = \text{ob } \mathcal{A}$ . The  $K$ -colocal objects in  $\mathcal{E}_{cf}$  comprise a coreflective subcategory, which is colimit-closed, and so includes every object in  $(\mathcal{E}_r)_{cf}$ . On the other hand, each  $K$ -colocal object is a retract of an  $I$ -cell complex with  $I$  as in (7.1), so constructible from objects in  $\mathcal{A}$  via colimits, and so in  $(\mathcal{E}_r)_{cf}$ . So the subcategory of  $K$ -colocal objects in  $\mathcal{E}_{cf}$  is precisely  $(\mathcal{E}_r)_{cf}$ . As a model structure is determined by its fibrations and cofibrant-fibrant objects,  $\mathcal{E}_r$  is thus the colocalisation of  $\mathcal{E}$  with respect to  $K$ . □

## 8 Examples

We conclude this paper by describing some examples of one-dimensional model categories obtained via Bousfield (co)localisation. While the one-dimensionality means that there is no real homotopy theory, we can at least find examples in which the fibrant, cofibrant or fibrant-cofibrant objects are mathematically interesting.

As a first step, we may apply Theorem 26 to see that combinatorial localisations of the *discrete* model structure on a locally presentable category  $\mathcal{E}$  correspond bijectively with full, replete, reflective, locally presentable subcategories of  $\mathcal{E}$ ; this recovers Theorem 4.3 of [31].<sup>1</sup>

<sup>1</sup> Or rather, its restriction to the combinatorial case; when starting from a discrete model structure, it is possible to construct (co)localisations under rather weaker assumptions than combinatoriality.



The localised model structure corresponding to the subcategory  $\mathcal{B}$  is obtained by lifting the discrete model structure on  $\mathcal{B}$  injectively along the reflector  $R : \mathcal{E} \rightarrow \mathcal{B}$ . This model structure is always left proper, since every object is cofibrant, but with an eye towards subsequent colocalisation, it will be useful to know when it is also *right* proper.

**Definition 39** A reflection  $V : \mathcal{B} \hookrightarrow \mathcal{E} : F$  is called *semi-left-exact* if the reflector  $F : \mathcal{E} \rightarrow \mathcal{B}$  preserves pullbacks along maps in the essential image of  $V$ .

This definition originates in Section 4 of [11]; the following result, describing the relation with right proper model structures, was first observed in [30].

**Lemma 40** A localisation of the discrete model structure on the locally presentable  $\mathcal{E}$  is right proper if and only if the reflection  $i : \mathcal{E}_{\ell f} \hookrightarrow \mathcal{E} : R_{\ell}$  is semi-left-exact.

**Proof** The acyclic fibrations of the localised model structure are the isomorphisms, whence the weak equivalences are the acyclic cofibrations; so right properness is the condition that  $\mathcal{TC}$ -maps are stable under pullback along  $\mathcal{F}$ -maps. Since the acyclic cofibrations are equally the maps inverted by  $R_{\ell}$ , its  $(\mathcal{TC}, \mathcal{F})$ -factorisation system is, in the terminology of [11], the *reflective factorisation system* corresponding to the subcategory  $\mathcal{E}_{\ell f}$ ; now Theorem 4.3 of *ibid.* proves that  $\mathcal{TC}$ -maps are stable under pullback along  $\mathcal{F}$ -maps just when the reflection is semi-left-exact. □

Putting this together with Theorem 36, we get:

**Proposition 41** Let  $\mathcal{A}, \mathcal{B}$  and  $\mathcal{E}$  be locally presentable. For any semi-left-exact reflection  $i : \mathcal{B} \hookrightarrow \mathcal{E} : R$  and coreflection  $j : \mathcal{A} \hookrightarrow \mathcal{B} : Q$  there is a one-dimensional model structure on  $\mathcal{E}$  with fibrant objects those in the essential image of  $i$ , with cofibrant objects those  $X \in \mathcal{E}$  such that  $RX$  is in the essential image of  $j$ , and with cofibrant–fibrant objects those in the essential image of  $ij$ .

Dually, we can construct a model structure on  $\mathcal{E}$  from a semi-right-exact coreflection  $j : \mathcal{B} \hookrightarrow \mathcal{E} : Q$  together with a reflection  $i : \mathcal{A} \hookrightarrow \mathcal{B} : R$  by first colocalising and then localising.

With these results in hand, we are now ready to give some examples. It is readily checked that all of the categories we deal with are locally presentable, and so we will make no mention of this in what follows.

**Example 42** Let  $A$  be a commutative ring, and let  $\mathbf{Zar}(A)$  denote the big Zariski topos of  $A$ . That is,  $\mathbf{Zar}(A)$  the category of sheaves on the dual of the category  $\mathbf{Alg}_A^{\text{fp}}$  of finitely presentable  $A$ -algebras, with the topology defined by surjective families of Zariski open inclusions. Sheafification gives a (semi-)left-exact reflection

$$\mathbf{Zar}(A) \xleftarrow{\perp} [\mathbf{Alg}_A^{\text{fp}}, \mathbf{Set}] \xrightarrow{\top} \tag{8.1}$$

Now let  $\mathbf{zar}(A)$  denote the small Zariski topos of  $A$ : the category of sheaves on the dual of the subcategory  $\mathbf{Loc}_A \subseteq \mathbf{Alg}_A^{\text{fp}}$  on the basic Zariski opens of  $A$  (i.e., the localisations of  $A$  at a single element) under the restricted Zariski topology. The inclusion  $j : \mathbf{Loc}_A \rightarrow \mathbf{Alg}_A^{\text{fp}}$  is fully faithful, left exact, and preserves and reflects covers; so by [21, Example C2.3.23] there is a coreflection

$$\mathbf{zar}(A) \xleftarrow{\top} \mathbf{Zar}(A) \xrightarrow{\perp} \tag{8.2}$$

with right adjoint given by restriction along  $j$  and left adjoint by left Kan extension followed by sheafification. More concretely, the left adjoint sends  $X \in \mathbf{zar}(A)$  to the functor of points of the  $A$ -scheme  $p: \Lambda_X \rightarrow \text{Spec } A$  obtained by glueing Zariski open subschemes of  $\text{Spec } A$  in the manner specified by  $X$ . As such, we can see the image of this left adjoint as comprising the “local homeomorphisms” over  $\text{Spec } A$ .

Applying Proposition 41 to (8.1) and (8.2), we thus have a model structure on  $[\mathbf{Alg}_A^{\text{fp}}, \mathbf{Set}]$  with fibrant objects the big Zariski sheaves and with as cofibrant–fibrant objects, the small Zariski sheaves seen as local homeomorphisms over  $\text{Spec } A$ . The general fibrant objects are those functors of points  $\mathbf{Alg}_A^{\text{fp}} \rightarrow \mathbf{Set}$  whose sheafification lands in  $\mathbf{zar}(A) \subseteq \mathbf{Zar}(A)$ .

**Example 43** Let  $k$  be an algebraically closed field, and let  $\mathbf{LocArt}_k \subseteq \mathbf{Alg}_k^{\text{fp}}$  denote the full subcategory on the local Artinian  $k$ -algebras. The topology on the dual of  $\mathbf{LocArt}_k$  induced from the Zariski topology is easily seen to be *discrete*, so that the category of sheaves thereon is equally the category of presheaves; now, as in the preceding example, we induce a coreflection

$$[\mathbf{LocArt}_k, \mathbf{Set}] \xrightleftharpoons[\top]{} \mathbf{Zar}(k) \tag{8.3}$$

whose right adjoint has a further right adjoint given by right Kan extension along the inclusion  $\mathbf{LocArt}_k \subseteq \mathbf{Alg}_k^{\text{fp}}$ . It follows that this coreflection is semi-right-exact.

The linear duals of local Artinian  $k$ -algebras are the cocommutative  $k$ -coalgebras which are finite-dimensional and *irreducible*: that is, contain a unique grouplike element. By [33, Corollary 8.0.7], any cocommutative  $k$ -coalgebra is the direct sum of irreducible ones, and by [33, Theorem 2.2.1], any irreducible  $k$ -coalgebra is the union of its (irreducible) finite-dimensional subcoalgebras. It follows that the linear duals of local Artinian  $k$ -algebras are dense in the (cocomplete) category  $k\text{-Cocomm}$  of cocommutative coalgebras, and so we have a reflection

$$k\text{-Cocomm} \xrightleftharpoons[\perp]{} [\mathbf{LocArt}_k, \mathbf{Set}]. \tag{8.4}$$

Applying the dual of Proposition 41 to (8.3) and (8.4), we thus have a model structure on the big Zariski topos of  $k$  whose cofibrant objects are the colimits in  $\mathbf{Zar}(k)$  of the spectra of local Artinian  $k$ -algebras, and whose cofibrant–fibrant objects are cocommutative  $k$ -algebras; the inclusion into  $\mathbf{Zar}(k)$  identifies these with the filtered colimits of the spectra of Artinian  $k$ -algebras. The general fibrant objects are Zariski sheaves  $X$  satisfying a form of “infinitesimal linearity” [22] which is satisfied, for example, by any scheme over  $\mathbf{Spec}(k)$ . Among other things, this infinitesimal linearity ensures the set of tangent vectors  $T_e(X)$  to a  $k$ -valued point  $e: \mathbf{Spec}(k) \rightarrow X$ —that is, the set of extensions of  $e$  through the map  $\mathbf{Spec}(k) \rightarrow \mathbf{Spec}(k[\varepsilon]/\varepsilon^2)$ —has the structure of a  $k$ -vector space, which is moreover a Lie algebra if  $e$  is the neutral element for a group structure on  $X$ .

**Example 44** Let  $X$  be a connected, locally connected and semi-locally simply connected topological space. As for any space, we have the left exact reflection

$$\mathbf{Sh}(X) \xrightleftharpoons[\perp]{} [\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$$

of presheaves into sheaves. Now let  $U$  be a universal covering space for  $X$ , seen as an object in  $\mathbf{Sh}(X)$ , let  $\pi_1(X) = \mathbf{Sh}(X)(U, U)$  be the fundamental group, and let  $j: \pi_1(X) \rightarrow \mathbf{Sh}(X)$  be the inclusion of the full subcategory on  $U$ . By standard properties of covering spaces, the cocontinuous extension  $j_!: \pi_1(X)\text{-Set} \rightarrow \mathbf{Sh}(X)$  of  $j$  is fully faithful and has as essential image the covering spaces over  $X$ . In particular, we have a coreflection

$$\pi_1(X)\text{-Set} \xrightleftharpoons[\top]{} \mathbf{Sh}(X)$$

with right adjoint sending a sheaf  $S$  to the set  $\mathbf{Sh}(X)(U, S)$  with  $\pi_1(X)$ -action induced from  $U$ . So applying Proposition 41, we have a model structure on  $[\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$  whose fibrant objects are sheaves on  $X$ , and whose cofibrant–fibrant objects are  $\pi_1(X)$ -sets, identified with the corresponding covering spaces. General cofibrant objects are presheaves whose sheaf of local sections is a covering space.

**Example 45** The preceding example arose by colocalising the model structure for sheaves on  $[\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$  at the single object  $U$  given by the universal covering space; however, if  $X$  is not locally semi-locally simply connected, then  $U$  need not exist. However, we can instead take the colocalisation at the set  $K$  of all (isomorphism-class representatives) of *finite* covering spaces; we then obtain a model structure on  $[\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$  with sheaves as fibrant objects, and cofibrant–fibrant objects the continuous  $G$ -sets for  $G$  the profinite completion of  $\pi_1(X)$ .

**Example 46** Generalising Example 44 in a different direction, we can construct a model structure on the category  $[\mathcal{O}(X)^{\text{op}}, \mathbf{Vect}_k]$  of presheaves of  $k$ -vector spaces on the connected, locally connected and semi-locally simply connected  $X$  whose fibrant objects are the sheaves of  $k$ -vector spaces and whose category of cofibrant–fibrant objects is the category of  $k$ -linear representations of  $\pi_1(X)$ , with these being identified in  $[\mathcal{O}(X)^{\text{op}}, \mathbf{Vect}_k]$  with the corresponding local systems.

**Example 47** Let  $X$  be a quasi-compact quasi-separated scheme, and let  $\mathbf{Psh}(\mathcal{O}_X)$  and  $\mathbf{Sh}(\mathcal{O}_X)$  be the categories of presheaves of  $\mathcal{O}_X$ -modules and sheaves of  $\mathcal{O}_X$ -modules. The left exact reflection between sheaves and presheaves induces a left exact reflection  $\mathbf{Sh}(\mathcal{O}_X) \rightleftarrows \mathbf{Psh}(\mathcal{O}_X)$ . Furthermore, the subcategory  $\mathbf{QCoh}(\mathcal{O}_X) \subseteq \mathbf{Sh}(\mathcal{O}_X)$  of *quasicoherent* sheaves of  $\mathcal{O}_X$ -modules is coreflective by [5, Lemma II.3.2]. We thus have a model structure on the category of presheaves of  $\mathcal{O}_X$ -modules whose fibrant objects are the sheaves of  $\mathcal{O}_X$ -modules, and whose cofibrant–fibrant objects are the quasicoherent sheaves.

**Example 48** Recall that, if  $G$  is a topological group, then a *continuous*  $G$ -set is a set  $X$  endowed with an action  $G \times X \rightarrow X$  which is continuous for the discrete topology on  $X$ ; this is equally the condition that the stabiliser of each  $x \in X$  is an open subgroup of  $G$ . It follows easily that there is a coreflection

$$\mathbf{Cts}\text{-}G\text{-Set} \xleftarrow{\quad \top \quad} G\text{-Set}$$

between  $G$ -sets and continuous  $G$ -sets, where the right adjoint  $c$  sends a  $G$ -set  $X$  to the sub- $G$ -set  $cX = \{x \in X : \text{Stab}_x \text{ is open in } G\}$ . The counit map is, of course, simply the inclusion, and it follows easily from this description that the coreflector preserves pushouts along maps between continuous  $G$ -sets; so this adjunction is semi-right-exact.

Now let  $N$  be an open normal subgroup of  $G$ . The category of continuous  $G/N$ -sets can be identified with the full subcategory of continuous  $G$ -sets in which each element is stabilised by (at least)  $N$ , and in fact we have a reflection

$$\mathbf{Cts}\text{-}G/N\text{-Set} \xleftarrow{\quad \perp \quad} \mathbf{Cts}\text{-}G\text{-Set}$$

where the left adjoint quotients out a continuous  $G$ -set by the equivalence relation  $x \sim x'$  iff  $Nx = Nx'$ . We thus have a model structure on  $G\text{-Set}$  whose cofibrant objects are the continuous  $G$ -sets and whose cofibrant–fibrant objects are the continuous  $G/N$ -sets. The general fibrant objects are those  $G$ -sets in which every element with an open stabiliser is stabilised by at least  $N$ .

**Example 49** Let  $\Delta_3$  denote the full subcategory of  $\Delta$  on  $[0], \dots, [3]$ , and let  $\mathbf{sSet}_3 = [\Delta_3^{\text{op}}, \mathbf{Set}]$ . Left Kan extension, restriction and right Kan extension along the inclusion  $\Delta_3 \subseteq \Delta$  gives a chain of adjoints  $\text{sk}_3 \dashv \text{tr}_3 \dashv \text{cosk}_3 : \mathbf{sSet}_3 \rightarrow \mathbf{sSet}$  with both  $\text{sk}_3$  and  $\text{cosk}_3$  fully faithful. In particular,  $\text{sk}_3 : \mathbf{sSet}_3 \rightleftarrows \mathbf{sSet} : \text{tr}_3$  is a semi-right-exact coreflection. Now, as the data and axioms for a category only involve at most three composable arrows, the truncated nerve  $\text{tr}_3 N : \mathbf{Cat} \rightarrow \mathbf{sSet} \rightarrow \mathbf{sSet}_3$  is still fully faithful, and has a left adjoint  $L$  since  $N$  and  $\text{tr}_3$  do. So we also have a reflection  $\text{tr}_3 N : \mathbf{Cat} \rightleftarrows \mathbf{sSet}_3 : L$ .

So by the dual of Proposition 41, we have a model structure on  $\mathbf{sSet}$  whose cofibrant objects are the 3-truncated simplicial sets, and whose subcategory of fibrant-cofibrant objects is equivalent to  $\mathbf{Cat}$ . However, this equivalence does *not* identify a category in the usual way with its nerve, but rather with the 3-skeleton of its nerve. Indeed, the cofibrant-fibrant objects are simplicial sets  $X$  which are 3-truncated and satisfy the restricted Segal condition that the spine projections

$$X_2 \rightarrow X_1 \times_{X_0} X_1 \quad \text{and} \quad X_3 \rightarrow X_1 \times_{X_0} X_1 \times_{X_0} X_1 \tag{8.5}$$

are isomorphisms: in other words, the 3-skeleta of nerves of categories. More generally, the fibrant objects of this model structure are simplicial sets  $X$ , not necessarily 3-truncated, for which the Segal maps in (8.5) are invertible.

**Example 50** Let  $\mathcal{E}$  denote the category of small, strictly symmetric, strictly monoidal categories enriched over abelian groups. There is a full embedding of the category of commutative monoids into  $\mathcal{E}$  as *discrete* categories, and this has a right adjoint given by taking the set of objects. This right adjoint is clearly cocontinuous, and so we have a semi-right-exact coreflection

$$\mathbf{CMon} \begin{array}{c} \longleftarrow \top \\ \xrightarrow{\quad} \end{array} \mathcal{E}$$

On the other hand, we have the well-known construction of the Grothendieck group of a commutative monoid, giving a reflection

$$\mathbf{Ab} \begin{array}{c} \longleftarrow \perp \\ \xrightarrow{\quad} \end{array} \mathbf{CMon} .$$

We therefore have a model structure on  $\mathcal{E}$  whose cofibrant objects are commutative monoids and whose cofibrant-fibrant objects are abelian groups. The fibrant objects are the small, strictly symmetric, strictly monoidal  $\mathbf{Ab}$ -categories  $(\mathcal{C}, \otimes, I)$  in which every object is *strictly* invertible for the tensor product  $\otimes$ . Such categories  $\mathcal{C}$  with abelian group of objects  $M$  can be identified<sup>2</sup> with  $M$ -graded commutative rings  $C$ , via the correspondence

$$\mathcal{C}(x, y) \leftrightarrow C_{y \otimes x^{-1}} .$$

**Example 51** Let  $\mathcal{T}$  be any finitary algebraic theory, such as the theory of monoids, or groups, or rings, or  $k$ -vector spaces, and so on. In each case, there is a category with finite products  $\mathbb{T}$ —the *Lawvere theory* [23] associated to  $\mathcal{T}$ —whose objects are the natural numbers, and for which finite-product-preserving functors  $\mathbb{T} \rightarrow \mathcal{E}$  into any category with finite products are equivalent to  $\mathcal{T}$ -models in  $\mathcal{E}$ .

Now consider any one of the semi-right-exact coreflections  $i : \mathcal{A} \rightleftarrows \mathcal{E} : Q$  from the above examples (i.e., from Examples 43, 48, 49 or 50). Postcomposition with  $Q$  and  $i$  induces a semi-right-exact coreflection on functor categories

$$\mathcal{A}^{\mathbb{T}} \begin{array}{c} \longleftarrow Q^{\mathbb{T}} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} i^{\mathbb{T}} \end{array} \mathcal{E}^{\mathbb{T}}$$

<sup>2</sup> The second author learnt of this correspondence from James Dolan.

On the other hand, the category  $\mathbf{FP}(\mathbb{T}, \mathcal{A})$  of finite-product-preserving functors  $\mathbb{T} \rightarrow \mathcal{A}$  is reflective in  $\mathcal{A}^{\mathbb{T}}$ ; and so, applying the dual of Proposition 41, we obtain a model structure on  $\mathcal{E}^{\mathbb{T}}$  whose cofibrant objects are functors  $\mathbb{T} \rightarrow \mathcal{A}$  and whose fibrant-cofibrant objects are  $\mathcal{T}$ -models in  $\mathcal{A}$ . The general fibrant objects are functors  $\mathbb{T} \rightarrow \mathcal{E}$  whose postcomposition with  $Q: \mathcal{E} \rightarrow \mathcal{A}$  preserves finite products; these are equally those functors  $\mathbb{T} \rightarrow \mathcal{E}$  which preserve finite products *up to* a map which is inverted by  $Q$ .

Let us conclude the paper by taking stock of the preceding examples. As we explained in the introduction, this paper is really a prelude to further work explaining Bousfield (co)localisation of general model structures in terms of projective and injective liftings; in which context, of course, many serious examples already exist. In the one-dimensional context, there were no non-trivial examples at all, and so we felt compelled to provide some.

These examples are sufficiently natural that their existence is surely not without force. However, it is as yet unclear to us what this force may be. One possibility is that some aspects of model category theory do not trivialise in this setting, and provide interesting information. This may be true, for example, for the theory of monoidal model categories, or the theory of homotopy limits and colimits.

More interestingly, it could be that the existence of a one-dimensional model structure is alerting us to some as-yet unconsidered good property of the interaction between a reflective and a coreflective subcategory. It is suggestive that many of the *right proper* examples we have found come from topos theory and, in particular, relate either to the Galois theory of Grothendieck [18] or to the *petit topos-gros topos* dichotomy explored in, for example, [14,20]. It would be very interesting to see if these links can be made tighter. To this end, some natural questions to explore would be: can we give a topos-theoretic characterisation of all right proper one-dimensional model structures on a presheaf category  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ ? What information do Quillen adjunctions between such model structures carry?

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