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The Catalan simplicial set

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Abstract

The Catalan numbers are well known to be the answer to many different counting problems, and so there are many different families of sets whose cardinalities are the Catalan numbers. We show how such a family can be given the structure of a simplicial set. We show how the low-dimensional parts of this simplicial set classify, in a precise sense, the structures of monoid and of monoidal category. This involves aspects of combinatorics, algebraic topology, quantum groups, logic, and category theory.

1. *Introduction*

The nth Catalan number C_n, given explicitly by $\binom{2n}{n}/(n+1)$, is well known to be the answer to many different counting problems; for example, it is the number of bracketings of an $(n+1)$-fold product. Thus there are many $\mathbb{N}$-indexed families of sets whose cardinalities are the Catalan numbers; Stanley [**16**, **17**] describes at least 205 such.

A Catalan family of sets may bear extra structure that is invisible in the mere sequence of Catalan numbers. For example, one presentation of the nth Catalan set is as the set of functions $f\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$ which preserve order and satisfy $f(k) \leqslant k$ for each k. The set of such functions is a monoid under composition and in this way we obtain the *Catalan monoids* [**15**] which are of importance to combinatorial semigroup theory. For another example, a result due to Tamari [**19**] makes each Catalan set into a lattice, whose ordering is most clearly understood in terms of bracketings of words, as the order generated by the basic inequality $(xy)z \leqslant x(yz)$ under substitution.

The first objective of this paper is to describe another kind of structure borne by Catalan families of sets. We shall show how to define functions between them in such a way as to produce a simplicial set $\mathbb{C}$, which is the "Catalan simplicial set" of the title. The simplicial structure can be defined in various ways, but the most elegant makes use of what seems to be a new presentation of the Catalan sets that relies heavily on the Boolean algebra **2**.

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Simplicial sets are abstract, combinatorial entities, most often used as models of spaces in homotopy theory, but flexible enough to also serve as models of higher categories [**12**, **20**]. Therefore, we might hope that the Catalan simplicial set had some natural role to play in homotopy theory or higher category theory. Our second objective in this paper is to affirm this hope, by showing that the Catalan simplicial set has a *classifying property* with respect to certain kinds of categorical structure. More precisely, we will consider simplicial maps from $\mathbb{C}$ into the nerves of various kinds of higher category (the *nerve* of such a structure is a simplicial set which encodes its cellular data). We will see that:

(a) maps from $\mathbb{C}$ to the nerve of a monoidal category $\mathscr{V}$ are the same thing as monoids in $\mathscr{V}$;
(b) maps from $\mathbb{C}$ to the nerve of a bicategory $\mathscr{B}$ are the same thing as monads in $\mathscr{B}$;
(c) maps from $\mathbb{C}$ to the *pseudo* nerve of the monoidal bicategory Cat of categories and functors are the same thing as monoidal categories;
(d) maps from $\mathbb{C}$ to the *lax* nerve of the monoidal bicategory Cat are the same thing as *skew-monoidal categories*.

Skew-monoidal categories generalise Mac Lane's notion of monoidal category [**14**] by dropping the requirement of invertibility of the associativity and unit constraints; they were introduced recently by Szlachányi [**18**] in his study of bialgebroids, which are themselves an extension of the notion of quantum group. The result in (d) can be seen as a coherence result for the notion of skew-monoidal category, providing an abstract justification for the axioms. Thus the work presented here lies at the interface of several mathematical disciplines:

(i) combinatorics, in the form of the Catalan numbers;
(ii) algebraic topology, via simplicial sets and nerves;
(iii) quantum groups, through recent work on bialgebroids;
(iv) logic, through the distinguished role of the Boolean algebra **2**; and
(v) category theory.

Nor is this the end of the story. Monoidal categories and skew-monoidal categories can be generalised to notions of *monoidale* and *skew monoidale* in a monoidal bicategory; this has further relevance for quantum algebra, since Lack and Street showed in [**11**] that quantum categories in the sense of [**3**] can be described using skew monoidales. In a sequel to this paper we will generalise (c) and (d) to prove that:

(e) maps from $\mathbb{C}$ to the pseudo nerve of a monoidal bicategory $\mathscr{W}$ are the same thing as monoidales in $\mathscr{W}$; and
(f) maps from $\mathbb{C}$ to the lax nerve of a monoidal bicategory $\mathscr{W}$ are the same thing as skew monoidales in $\mathscr{W}$.

The results (a)–(f) use only the lower dimensions of the Catalan simplicial set and we expect that its higher dimensions in fact encode *all* of the coherence that a higher-dimensional monoidal object should satisfy. We therefore hope also to show that:

(g) maps from $\mathbb{C}$ to the pseudo nerve of the monoidal tricategory Bicat of bicategories are the same thing as monoidal bicategories;
(h) maps from $\mathbb{C}$ to the homotopy-coherent nerve of the monoidal simplicial category ∞-Cat of ∞-categories are the same thing as monoidal ∞-categories in the sense of [**13**];

together with appropriate skew analogues of these results.

Finally, a note on the genesis of this work. We have chosen to present the Catalan simplicial set as basic, and its classifying properties as derived. This belies the method of its discovery, which was to look for a simplicial set with the classifying property (d); the link with the Catalan numbers only came to light later. The notion that a classifying object as in (d) might exist is based on an old idea of Michael Johnson's on how to capture not only associativity but also unitality constraints simplicially. He reminded us of this in a recent talk [**9**] to the Australian Category Seminar.

2. *The Catalan simplicial set*

In this section we define and investigate the Catalan simplicial set. We begin by recalling some basic definitions. We write Δ for the *simplicial category*, whose objects are non-empty finite ordinals $[n] = \{0, \ldots, n\}$ and whose morphisms are order-preserving functions, and write SSet for the category of presheaves on Δ. Objects X of SSet are called *simplicial sets*; we think of them as glueings-together of discs, with the n-dimensional discs in that glueing labelled by the set $X_n := X([n])$ of *n-simplices* of X. We write $\delta_i \colon [n-1] \to [n]$ and $\sigma_i \colon [n+1] \to [n]$ for the maps of Δ defined by

$$\delta_i(x) = \begin{cases} x & \text{if } x < i \\ x+1 & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma_i(x) = \begin{cases} x & \text{if } x \leqslant i \\ x-1 & \text{otherwise.} \end{cases}$$

The action of these morphisms on a simplicial set X yields functions $d_i \colon X_n \to X_{n-1}$ and $s_i \colon X_n \to X_{n+1}$, which we call *face* and *degeneracy* maps. An $(n+1)$-simplex x is called *degenerate* when it is in the image of some s_i, and *non-degenerate* otherwise. The face and degeneracy maps of a simplicial set satisfy the following *simplicial identities*:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{for } i < j \\ s_i s_j &= s_{j+1} s_i && \text{for } i \leqslant j \end{aligned} \qquad d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j \\ \mathrm{id} & \text{for } i = j, j+1 \\ s_j d_{i-1} & \text{for } i > j+1; \end{cases}$$

and in fact, a simplicial set may be completely specified by giving its sets of n-simplices, together with face and degeneracy maps satisfying the simplicial identities.

Definition 2·1. The *Catalan simplicial set* $\mathbb{C}$ has its n-simplices given by *Dyck words* of length $2n+2$; these are strings comprised of $(n+1)$ U's and $(n+1)$ D's such that the ith U precedes the ith D for each $1 \leqslant i \leqslant n+1$. The face maps $d_i \colon \mathbb{C}_n \to \mathbb{C}_{n-1}$ act on a Dyck word by deleting the ith U and ith D; the degeneracy maps $s_i \colon \mathbb{C}_{n-1} \to \mathbb{C}_n$ act on a Dyck word by repeating the ith U and ith D.

The sets of Dyck words of length $2n$ are a Catalan family of sets—corresponding to (i) or (r) in Stanley's enumeration [**16**, exercise 6·19]—and so $|\mathbb{C}_n| = C_{n+1}$, the $(n+1)$st Catalan number.

Remark 2·2. The sets of n-simplices of $\mathbb{C}$ are not quite a Catalan family, due to the dimension shift causing us to omit the 0th Catalan number. We may rectify this by viewing $\mathbb{C}$ as an *augmented* simplicial set. An augmented simplicial set is a presheaf on Δ_+, the category of all finite ordinals and order-preserving maps; it is equally given by a simplicial set X together with a set X_{-1} of (-1)-simplices and an "augmentation" map $d_0 \colon X_0 \to X_{-1}$ satisfying $d_0 d_0 = d_0 d_1 \colon X_1 \to X_{-1}$. By allowing n to range over $\{-1\} \cup \mathbb{N}$ in the definition of the Catalan simplicial set $\mathbb{C}$, it becomes an augmented simplicial set with the property that its sets of $(n-1)$-simplices (for $n \in \mathbb{N}$) are a Catalan family.

In order to understand the Catalan simplicial set as a simplicial set, we must understand the face and degeneracy relations between its simplices. In low dimensions, we see directly that $\mathbb{C}$ has:

(i) a unique 0-simplex UD, which we write as $\star$;
(ii) two 1-simplices $UUDD$ and $UDUD$, the first of which is $s_0(\star)$ and the second of which is non-degenerate; we write these as $e = s_0(\star)\colon \star \to \star$ and $c\colon \star \to \star$;
(iii) five 2-simplices: three degenerate ones $UUUDDD$, $UUDDUD$ and $UDUUDD$, and two non-degenerate ones $UUDUDD$ and $UDUDUD$. We depict these, and their faces, by:

$$\begin{tikzcd}
& \star \arrow[dr, "e"] \arrow[d, phantom, "{s_0(e) \atop =s_1(e)}"] & & & \star \arrow[dr, "c"] \arrow[d, phantom, "s_0(c)"] & & & \star \arrow[dr, "e"] \arrow[d, phantom, "s_1(c)"] & \\
\star \arrow[ur, "e"] \arrow[rr, "e"'] & & \star & \star \arrow[ur, "e"] \arrow[rr, "c"'] & & \star & \star \arrow[ur, "c"] \arrow[rr, "c"'] & & \star \\
& & \star \arrow[dr, "c"] \arrow[d, phantom, "t"] & & & \star \arrow[dr, "e"] \arrow[d, phantom, "i"] & & & \\
& \star \arrow[ur, "c"] \arrow[rr, "c"'] & & \star & \star \arrow[ur, "e"] \arrow[rr, "c"'] & & \star\,. & &
\end{tikzcd} \tag{2·1}$$

In higher dimensions, the simplices of $\mathbb{C}$ will be determined by *coskeletality*. A simplicial set is called *r-coskeletal* when every n-boundary with $n > r$ has a unique filler; here, an *n-boundary* in a simplicial set is a collection of $(n-1)$-simplices $(x_0, \ldots, x_n)$ satisfying $d_j(x_i) = d_i(x_{j+1})$ for all $0 \leqslant i \leqslant j < n$; a *filler* for such a boundary is an n-simplex x with $d_i(x) = x_i$ for $i = 0, \ldots, n$.

PROPOSITION 2·3. *The Catalan simplicial set is* 2*-coskeletal.*

Proof. For each natural number n, let $\mathbb{K}_n$ be the set of binary relations $R \subset \{0, \ldots, n\}^2$ such that (i) $i \mathrel{R} j$ implies $i < j$; and (ii) $i < j < k$ and $i \mathrel{R} k$ implies $i \mathrel{R} j$ and $j \mathrel{R} k$. For each $n \geqslant 0$, there is a bijection $\mathbb{C}_n \to \mathbb{K}_n$ which sends a Dyck word W to the set of those pairs $i < j$ such that the $(j+1)$st D precedes the $(i+1)$st U in W. Transporting the simplicial structure of $\mathbb{C}$ along these bijections yields an isomorphic simplicial set $\mathbb{K}$ and it suffices to prove that this is 2-coskeletal.

We may identify the faces of an n-simplex $R \in \mathbb{K}_n$ with the restrictions of R to the $(n+1)$ distinct n-element subsets of $\{0, \ldots, n\}$. An arbitrary collection $(R_0, \ldots, R_n)$ of such relations, seen as elements of $\mathbb{K}_{n-1}$, comprises an n-boundary just when each R_i and R_j agree on the intersections of their domains. In this situation, there is a a unique relation $R \subset \{0, \ldots, n\}^2$ restricting back to the given R_i's, and satisfying (i) since each R_i does. If $n > 2$, then each triple $0 \leqslant i < j < k \leqslant n$ will lie entirely inside the domain of some R_ℓ, and so the relation R will satisfy (ii) since each R_ℓ does, and thus constitute an element of $\mathbb{K}_n$. Thus for $n > 2$, each n-boundary of $\mathbb{K} \cong \mathbb{C}$ has a unique filler.

We now give one further description of the Catalan simplicial set, perhaps the most appealing: we will exhibit it as the monoidal nerve of a particularly simple monoidal category, namely the poset $\mathbf{2} = \bot \leqslant \top$, seen as a monoidal category with tensor product given by disjunction. We first explain what we mean by this. Recall that if $\mathscr{A}$ is a category, then its *nerve* $\mathrm{N}(\mathscr{A})$ is the simplicial set whose 0-simplices are objects of $\mathscr{A}$, and whose n-simplices for $n > 0$ are strings of n composable morphisms. Since the face and degeneracy maps are obtained from identities and composition in $\mathscr{A}$, the nerve in fact encodes the entire category structure of $\mathscr{A}$.

Suppose now that $\mathscr{A}$ is a *monoidal category* in the sense of [**14**]—thus, equipped with a tensor product functor $\otimes \colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$, a unit object $I \in \mathscr{A}$, and families of natural isomorphisms $\alpha_{ABC} \colon (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$, $\lambda_A \colon I \otimes A \cong A$ and $\rho_A \colon A \cong A \otimes I$, satisfying certain coherence axioms which we recall in detail in Section 4 below. In this situation, the nerve of $\mathscr{A}$ as a category fails to encode any information concerning the monoidal structure. However, by viewing $\mathscr{A}$ as a one-object bicategory (= weak 2-category), we may form a different nerve which *does* encode this extra information.

Definition 2·4. Let $\mathscr{A}$ be a monoidal category. The *monoidal nerve* of $\mathscr{A}$ is the simplicial set $\mathrm{N}_\otimes(\mathscr{A})$ defined as follows:

(i) there is a unique 0-simplex, denoted $\star$;
(ii) a 1-simplex is an object $A \in \mathscr{A}$; its two faces are necessarily $\star$;
(iii) a 2-simplex is a map $A_{12} \otimes A_{01} \to A_{02}$ in $\mathscr{A}$; its three faces are A_{12}, A_{02} and A_{01};
(iv) a 3-simplex is a commuting diagram

$$\begin{tikzcd}
(A_{23} \otimes A_{12}) \otimes A_{01} \arrow[rr, "\alpha"] \arrow[d, "A_{123} \otimes 1"'] & & A_{23} \otimes (A_{12} \otimes A_{01}) \arrow[d, "1 \otimes A_{012}"] \\
A_{13} \otimes A_{01} \arrow[r, "A_{013}"'] & A_{03} & A_{23} \otimes A_{02} \arrow[l, "A_{023}"]
\end{tikzcd} \tag{2·2}$$

in $\mathscr{A}$; its four faces are A_{123}, A_{023}, A_{013} and A_{012};
(v) higher-dimensional simplices are determined by 3-coskeletality.

The degeneracy of the unique 0-simplex is the unit object $I \in \mathscr{A}$; the two degeneracies $s_0(A), s_1(A)$ of a 1-simplex are the respective coherence constraints $\rho_A^{-1} \colon A \otimes I \to A$ and $\lambda_A \colon I \otimes A \to A$; the three degeneracies of a 2-simplex are simply the assertions that certain diagrams commute, which is so by the axioms for a monoidal category. Higher degeneracies are determined by coskeletality.

Note that, because the monoidal nerve arises from viewing a monoidal category as a one-object bicategory, we have a dimension shift: objects and morphisms of $\mathscr{A}$ become 1- and 2-simplices of the nerve, rather than 0- and 1-simplices.

PROPOSITION 2·5. *The simplicial set $\mathbb{C}$ is uniquely isomorphic to the monoidal nerve of the poset $\mathbf{2} = \bot \leqslant \top$, seen as a monoidal category under disjunction.*

Proof. In any monoidal nerve $\mathrm{N}_\otimes(\mathscr{A})$, each 3-dimensional boundary has at most one filler, existing just when the diagram (2·2) associated to the boundary commutes. Since every diagram in a poset commutes, the nerve $\mathrm{N}_\otimes(\mathbf{2})$, like $\mathbb{C}$, is 2-coskeletal. It remains to show that $\mathbb{C} \cong \mathrm{N}_\otimes(\mathbf{2})$ in dimensions 0, 1, 2. In dimension 0 this is trivial. In dimension 1, any isomorphism must send $s_0(\star) = e \in \mathbb{C}_1$ to $s_0(\star) = \bot \in \mathrm{N}_\otimes(\mathbf{2})_1$ and hence must send c to $\top$. In dimension 2, the 2-simplices of $\mathrm{N}_\otimes(\mathbf{2})$ are of the form

$$\begin{tikzcd}
& \star \arrow[dr, "x_{12}"] & \\
\star \arrow[ur, "x_{01}"] \arrow[rr, "x_{02}"'] & & \star
\end{tikzcd}$$

where $x_{12} \vee x_{01} \leqslant x_{02}$ in $\mathrm{N}_\otimes(\mathbf{2})$. Thus in $\mathrm{N}_\otimes(\mathbf{2})$, as in $\mathbb{C}$, there is at most one 2-simplex with a given boundary, and by examination of (2·1), we see that the same possibilities arise on both sides; thus there is a unique isomorphism $\mathbb{C}_2 \cong \mathrm{N}_\otimes(\mathbf{2})_2$ compatible with the face maps, as required.

We conclude this section by investigating the non-degenerate simplices of the Catalan simplicial set; these will be of importance in the following sections, where they will play the role of basic coherence data in higher-dimensional monoidal structures. We will see that these non-degenerate simplices form a Motzkin family of sets. The *Motzkin numbers* [**5**] $1, 1, 2, 4, 9, \ldots$ are defined by the recurrence relations

$$M_0 = 1 \qquad \text{and} \qquad M_{n+1} = M_n + \textstyle\sum_{k=0}^{n-1} M_k M_{n-1-k},$$

and an $\mathbb{N}$-indexed family of sets is a *Motzkin family of sets* if there are a Motzkin number of elements in each dimension. For example, if we define a *Motzkin word* to be a string in the alphabet $\{U, C, D\}$ which, on striking out every C, becomes a Dyck word, then the sets $\mathbb{M}_n$ of Motzkin words of length n are a Motzkin family of sets—corresponding to Stanley [**16**, Exercise 6·38 item (b) or (d)].

PROPOSITION 2·6. *The family* $(\mathrm{nd}\,\mathbb{C}_n : n \in \mathbb{N})$ *of non-degenerate simplices of* $\mathbb{C}$ *is a Motzkin family of sets.*

Proof. It suffices to construct a bijection $\mathrm{nd}\,\mathbb{C}_n \cong \mathbb{M}_n$ for each n. In one direction, we have a map $\mathrm{nd}\,\mathbb{C}_n \to \mathbb{M}_n$ sending a non-degenerate Dyck word W to the Motzkin word $M_1 \ldots M_n$ defined as follows: if the ith and $(i+1)$st U's are adjacent in W, then $M_i = U$; if the ith and $(i+1)$st D's are adjacent in W, then $M_i = D$; otherwise $M_i = C$. (Note that the first two cases are disjoint; a Dyck word W satisfying both would have to be in the image of the ith degeneracy map.)

In the other direction, suppose given a Motzkin word $M = M_1 \ldots M_n$. Let $a_1 < \cdots < a_k$ enumerate all i for which M_i is D or C, and let $b_1 < \cdots < b_k$ enumerate all i for which M_i is U or C. The inverse mapping $\mathbb{M}_n \to \mathrm{nd}\,\mathbb{C}_n$ now sends M to the Dyck word

$$U^{a_1} D^{b_1} U^{a_2-a_1} D^{b_2-b_1} \cdots U^{a_k-a_{k-1}} D^{b_k-b_{k-1}} U^{n+1-a_k} D^{n+1-b_k}\ .$$

That these two mappings are mutually inverse is the content of the equivalence between the Motzkin families (M1) and (M4) of [**5**].

Using this result, we may re-derive a well-known combinatorial identity relating the Catalan and Motzkin numbers.

COROLLARY 2·7. *For each* $n \geqslant 0$, *we have* $C_{n+1} = \sum_k \binom{n}{k} M_k$.

Proof. Recall that the *Eilenberg–Zilber lemma* [**6**, Section II·3] states that every simplex $x \in X_n$ of a simplicial set X is the image under a unique surjection $\phi \colon [n] \twoheadrightarrow [k]$ in Δ of a unique non-degenerate simplex $y \in X_k$. Since there are $\binom{n}{k}$ order-preserving surjections $[n] \twoheadrightarrow [k]$,

$$C_{n+1} = |\mathbb{C}_n| = \textstyle\sum_{\phi\colon [n]\twoheadrightarrow[k]} |\mathrm{nd}\,\mathbb{C}_k| = \sum_k \binom{n}{k} |\mathrm{nd}\,\mathbb{C}_k| = \sum_k \binom{n}{k} M_k$$

as required.

3. *First classifying properties*

We now begin to investigate the *classifying properties* of the Catalan simplicial set, by looking at the structure picked out by maps from $\mathbb{C}$ into the nerves of certain kinds of categorical structure. For our first classifying property, recall that a *monoid* in a monoidal category $\mathscr{A}$ is given by an object $A \in \mathscr{A}$ and morphisms $\mu \colon A \otimes A \to A$ and $\eta \colon I \to A$ rendering commutative the three diagrams

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "\alpha"] \arrow[d, "\mu\otimes 1"'] & & A \otimes (A \otimes A) \arrow[d, "1\otimes\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A & A \otimes A \arrow[l, "\mu"]
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "\rho_A"] \arrow[d, equal] & A \otimes I \arrow[d, "1\otimes\eta"] \\
A & A \otimes A \arrow[l, "\mu"]
\end{tikzcd}
\quad
\begin{tikzcd}
I \otimes A \arrow[r, "\lambda_A"] \arrow[d, "\eta\otimes 1"'] & A \\
A \otimes A \arrow[ur, "\mu"'] &
\end{tikzcd} .$$

PROPOSITION 3·1. *If $\mathscr{A}$ is a monoidal category, then to give a map $f \colon \mathbb{C} \to \mathrm{N}_\otimes(\mathscr{A})$ of simplicial sets is equally to give a monoid in $\mathscr{A}$.*

Proof. Since $\mathrm{N}_\otimes(\mathscr{A})$ is 3-coskeletal, a simplicial map $f \colon \mathbb{C} \to \mathrm{N}_\otimes(\mathscr{A})$ is uniquely determined by where it sends non-degenerate simplices of dimension $\leqslant 3$. We have already described the non-degenerate simplices in dimensions $\leqslant 2$, while in dimension 3, there are four such, given by

$$a = (t, t, t, t) \qquad \ell = (i, s_1(c), t, s_1(c))$$
$$r = (s_0(c), t, s_0(c), i) \qquad k = (i, s_1(c), s_0(c), i) \; .$$

Here, we take advantage of 2-coskeletality of $\mathbb{C}$ to identify a 3-simplex x with its tuple $(d_0(x), d_1(x), d_2(x), d_3(x))$ of 2-dimensional faces. Thus to give $f \colon \mathbb{C} \to \mathrm{N}_\otimes(\mathscr{A})$ is to give:

(i) in dimension 0, no data: f must send $\star$ to $\star$;

(ii) in dimension 1, an object $A \in \mathscr{A}$, the image of the non-degenerate simplex $c \in \mathbb{C}_1$;

(iii) in dimension 2, morphisms $\mu \colon A \otimes A \to A$ and $\eta' \colon I \otimes I \to A$, the images of the non-degenerate simplices $t, i \in \mathbb{C}_2$;

(iv) in dimension 3, commutative diagrams

$$f(a) = \begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "\alpha"] \arrow[d, "\mu\otimes 1"'] & & A \otimes (A \otimes A) \arrow[d, "1\otimes\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A & A \otimes A \arrow[l, "\mu"]
\end{tikzcd}$$

$$f(\ell) = \begin{tikzcd}
(I \otimes I) \otimes A \arrow[rr, "\alpha"] \arrow[d, "\eta'\otimes 1"'] & & I \otimes (I \otimes A) \arrow[d, "1\otimes\lambda_A"] \\
A \otimes A \arrow[r, "\mu"'] & A & I \otimes A \arrow[l, "\lambda_A"]
\end{tikzcd}$$

$$f(r) = \begin{tikzcd}
(A \otimes I) \otimes I \arrow[rr, "\alpha"] \arrow[d, "\rho_A^{-1}\otimes 1"'] & & A \otimes (I \otimes I) \arrow[d, "1\otimes\eta'"] \\
A \otimes I \arrow[r, "\rho_A^{-1}"'] & A & A \otimes A \arrow[l, "\mu"]
\end{tikzcd}$$

$$f(k) = \begin{tikzcd}
(I \otimes I) \otimes I \arrow[rr, "\alpha"] \arrow[d, "\eta'\otimes 1"'] & & I \otimes (I \otimes I) \arrow[d, "1\otimes\eta'"] \\
A \otimes I \arrow[r, "\rho_A^{-1}"'] & A & I \otimes A \arrow[l, "\lambda_A"]
\end{tikzcd}$$

the images as displayed of the non-degenerate 3-simplices of $\mathbb{C}$.

On defining $\eta = \eta' \circ \rho_A \colon I \to I \otimes I \to A$, we obtain a bijective correspondence between the data (A, μ, η') for a simplicial map $\mathbb{C} \to \mathrm{N}_\otimes(\mathscr{A})$ and the data (A, μ, η) for a monoid in $\mathscr{A}$. Under this correspondence, the axiom $f(a)$ for (A, μ, η') is clearly the same as the associativity axiom for (A, μ, η); a short calculation with the axioms for a monoidal category shows that $f(\ell)$ and $f(r)$ correspond likewise with the unit axioms for a monoid. This leaves only $f(k)$; but it is easy to show that this is automatically satisfied in any monoidal category. Thus monoids in $\mathscr{A}$ correspond bijectively with simplicial maps $\mathbb{C} \to \mathrm{N}_\otimes(\mathscr{A})$ as claimed.

Remark 3·2. A generalisation of this classifying property concerns maps from $\mathbb{C}$ into the nerve of a *bicategory* $\mathscr{B}$ in the sense of [**1**]. Bicategories are "many object" versions of monoidal categories, and the nerve of a bicategory is a "many object" version of the monoidal nerve of Definition 2·4. An easy modification of the preceding argument shows that simplicial maps $\mathbb{C} \to \mathrm{N}(\mathscr{B})$ classify monads in the bicategory $\mathscr{B}$.

4. *Higher classifying properties*

The category Cat of small categories and functors bears a monoidal structure given by cartesian product, and monoids with respect to this are precisely small *strict* monoidal categories—those for which the associativity and unit constraints α, λ and ρ are all identities. It follows by Proposition 3·1 that simplicial maps $\mathbb{C} \to \mathrm{N}_\otimes(\mathrm{Cat})$ classify small strict monoidal categories. The purpose of this section is to show that, in fact, we may also classify both:

(i) not-necessarily-strict monoidal categories; and
(ii) *skew-monoidal* categories in the sense of [**18**];

by simplicial maps from $\mathbb{C}$ into suitably modified nerves of Cat, where the modifications at issue involve changing the simplices from dimension 3 upwards. The 3-simplices will no longer be commutative diagrams as in (2·2), but rather diagrams commuting up to a natural transformation, invertible in the case of (i) but not necessarily so for (ii). The 4-simplices will be, in both cases, suitably commuting diagrams of natural transformations, while higher simplices will be determined by coskeletality as before. Note that, to obtain these new classification results, we do not need to change $\mathbb{C}$ itself, only what we map it into. The change is from something 3-coskeletal to something 4-coskeletal, which means that the non-degenerate 4-simplices of $\mathbb{C}$ come into play. As we will see, these encode precisely the coherence axioms for monoidal or skew-monoidal structure.

Before continuing, let us make precise the definition of skew-monoidal category. As explained in the introduction, this notion was introduced by Szlachányi in [**18**] to describe structures arising in quantum algebra, and generalises Mac Lane's notion of monoidal category by dropping the requirement that the coherence constraints be invertible.

Definition 4·1. A *skew-monoidal category* is a category $\mathscr{A}$ equipped with a unit element $I \in \mathscr{A}$, a tensor product $\otimes \colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$, and natural families of (not neccesarily invertible) constraint maps

$$\begin{gathered} \alpha_{ABC} \colon (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \\ \lambda_A \colon I \otimes A \longrightarrow A \quad \text{and} \quad \rho_A \colon A \longrightarrow A \otimes I \end{gathered} \tag{4·1}$$

subject to the commutativity of the following diagrams—wherein tensor is denoted by juxtaposition—for all $A, B, C, D \in \mathscr{A}$:

$$\begin{tikzcd}
 & (AB)(CD) \arrow[dr, "\alpha"] & \\
((AB)C)D \arrow[ur, "\alpha"] \arrow[d, "\alpha 1"'] & \text{(S1)} & A(B(CD)) \\
(A(BC))D \arrow[r, "\alpha"'] & & A((BC)D) \arrow[u, "1\alpha"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(AI)B \arrow[r, "\alpha"] & A(IB) \arrow[d, "1\lambda"] \\
AB \arrow[u, "\rho 1"] \arrow[r, "\text{id}"'] \arrow[ur, phantom, "\text{(S2)}"] & AB
\end{tikzcd}$$

$$\begin{tikzcd}
 & I(AB) \arrow[dr, "\lambda"] & \\
(IA)B \arrow[ur, "\alpha"] \arrow[rr, "\lambda 1"'] & \text{(S3)} & AB
\end{tikzcd}
\quad
\begin{tikzcd}
 & (AB)I \arrow[dr, "\alpha"] & \\
AB \arrow[ur, "\rho"] \arrow[rr, "1\rho"'] & \text{(S4)} & A(BI)
\end{tikzcd}
\quad
\begin{tikzcd}
 & II \arrow[dr, "\lambda"] & \\
I \arrow[ur, "\rho"] \arrow[rr, "\text{id}"'] & \text{(S5)} & I\,.
\end{tikzcd}$$

A skew-monoidal category in which α, λ and ρ are invertible is exactly a monoidal category; the axioms (S1)–(S5) are then Mac Lane's original five axioms [**14**], justified by the fact that they imply the commutativity of *all* diagrams of constraint maps. In the skew case, this justification no longer applies, as the axioms no longer force every diagram of constraint maps to commute; for example, we need not have $1_{I\otimes I} = \rho_I \circ \lambda_I \colon I \otimes I \to I \otimes I$. The classification of skew-monoidal structure by maps out of the Catalan simplicial set can thus be seen as an alternative justification of the axioms in the absence of such a result.

Before giving our classification result, we describe the modified nerves of Cat which will be involved. The possibility of taking natural transformations as 2-cells makes Cat not just a monoidal category, but a *monoidal bicategory* in the sense of [**7**]. Just as one can form a nerve of a monoidal category by viewing it as a one-object bicategory, so one can form a nerve of a monoidal bicategory by viewing it as a one-object tricategory (= weak 3-category), and in fact, various nerve constructions are possible—see [**4**]. The following definitions are specialisations of some of these nerves to the case of Cat.

Definition 4·2. The *lax nerve* $\mathrm{N}_\ell(\mathrm{Cat})$ of the monoidal bicategory Cat is the simplicial set defined as follows:

(i) there is a unique 0-simplex, denoted $\star$;
(ii) a 1-simplex is a (small) category $\mathscr{A}_{01}$;
(iii) a 2-simplex is a functor $A_{012} \colon \mathscr{A}_{12} \times \mathscr{A}_{01} \to \mathscr{A}_{02}$;
(iv) a 3-simplex is a natural transformation

$$\begin{tikzcd}
(\mathscr{A}_{23} \times \mathscr{A}_{12}) \times \mathscr{A}_{01} \arrow[rr, "\cong"] \arrow[d, "A_{123}\times 1"'] & & \mathscr{A}_{23} \times (\mathscr{A}_{12} \times \mathscr{A}_{01}) \arrow[d, "1\times A_{012}"] \\
\mathscr{A}_{13} \times \mathscr{A}_{01} \arrow[r, "A_{013}"'] & \mathscr{A}_{03} \arrow[u, phantom, "\overset{A_{0123}}{\Longrightarrow}"] & \mathscr{A}_{23} \times \mathscr{A}_{02} \arrow[l, "A_{023}"]
\end{tikzcd}$$

with 1-cell components

$$(A_{0123})_{a_{23},a_{12},a_{01}} \colon A_{013}(A_{123}(a_{23}, a_{12}), a_{01}) \to A_{023}(a_{23}, A_{012}(a_{12}, a_{01}))\ ;$$

(v) a 4-simplex is a quintuple $(A_{1234}, A_{0234}, A_{0134}, A_{0124}, A_{0123})$ of appropriately-formed natural transformations making the pentagon

$$\begin{tikzcd}
 & A_{024}(A_{234}(a_{34}, a_{23}), A_{012}(a_{12}, a_{01})) \arrow[ddr, "A_{0234}"] & \\
A_{014}(A_{124}(A_{234}(a_{34}, a_{23}), a_{12}), a_{01}) \arrow[ur, "A_{0124}"] \arrow[dd, "A_{014}(A_{1234}, 1)"'] & & \\
 & & A_{034}(a_{34}, A_{023}(a_{23}, A_{012}(a_{12}, a_{01}))) \\
A_{014}(A_{134}(a_{34}, A_{123}(a_{23}, a_{12})), a_{01}) \arrow[dr, "A_{0134}"'] & & \\
 & A_{034}(a_{34}, A_{013}(A_{123}(a_{23}, a_{12}), a_{01})) \arrow[uur, "A_{034}(1, A_{0123})"'] &
\end{tikzcd}$$

commute in $\mathscr{A}_{04}$ for all $(a_{01}, a_{12}, a_{23}, a_{34}) \in \mathscr{A}_{01} \times \mathscr{A}_{12} \times \mathscr{A}_{23} \times \mathscr{A}_{34}$;

(vi) higher-dimensional simplices are determined by 4-coskeletality, and face and degeneracy maps are defined as before.

The *pseudo nerve* $\mathrm{N}_p(\mathrm{Cat})$ is defined identically except that the natural transformations occurring in dimensions 3 and 4 are required to be invertible.

We are now ready to give our higher classifying property of the Catalan simplicial set.

PROPOSITION 4·3. *To give a simplicial map* $f\colon \mathbb{C} \to \mathrm{N}_p(\mathrm{Cat})$ *is equally to give a small monoidal category; to give a simplicial map* $f\colon \mathbb{C} \to \mathrm{N}_\ell(\mathrm{Cat})$ *is equally to give a small skew-monoidal category.*

Proof. First we prove the second statement. Since $\mathrm{N}_\ell(\mathrm{Cat})$ is 4-coskeletal, a simplicial map into it is uniquely determined by where it sends non-degenerate simplices of dimension at most four. In dimensions $\leqslant 3$, to give $f\colon \mathbb{C} \to \mathrm{N}_\ell(\mathrm{Cat})$ is to give:

(i) in dimension 0, no data: f must send $\star$ to $\star$;
(ii) in dimension 1, a small category $\mathscr{A} = f(c)$;
(iii) in dimension 2, a functor $\otimes = f(t)\colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ and an object $I \in \mathscr{A}$ picked out by the functor $f(i)\colon 1 \times 1 \to \mathscr{A}$;
(iv) in dimension 3, natural transformations

$$f(a) = \begin{tikzcd}
(\mathscr{A} \times \mathscr{A}) \times \mathscr{A} \arrow[rr, "\cong"] \arrow[d, "\otimes\times 1"'] & {} \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & \mathscr{A} \times (\mathscr{A} \times \mathscr{A}) \arrow[d, "1\times\otimes"] \\
\mathscr{A} \times \mathscr{A} \arrow[r, "\otimes"'] & \mathscr{A} & \mathscr{A} \times \mathscr{A} \arrow[l, "\otimes"]
\end{tikzcd}$$

$$f(\ell) = \begin{tikzcd}
(1 \times 1) \times \mathscr{A} \arrow[rr, "\cong"] \arrow[d, "f(i)\times 1"'] & {} \arrow[d, phantom, "\overset{\lambda}{\Rightarrow}"] & 1 \times (1 \times \mathscr{A}) \arrow[d, "1\times\cong"] \\
\mathscr{A} \times \mathscr{A} \arrow[r, "\otimes"'] & \mathscr{A} & 1 \times \mathscr{A} \arrow[l, "\cong"]
\end{tikzcd}$$

$$f(r) = \begin{tikzcd}
(\mathscr{A} \times 1) \times 1 \arrow[rr, "\cong"] \arrow[d, "\cong\times 1"'] & {} \arrow[d, phantom, "\overset{\rho}{\Rightarrow}"] & \mathscr{A} \times (1 \times 1) \arrow[d, "1\times f(i)"] \\
\mathscr{A} \times 1 \arrow[r, "\cong"'] & \mathscr{A} & \mathscr{A} \times \mathscr{A} \arrow[l, "\otimes"]
\end{tikzcd}$$

$$f(k) = \begin{tikzcd} (1\times 1)\times 1 \arrow[rr, "\cong"] \arrow[d, "f(i)\times 1"'] \arrow[drr, phantom, "\overset{\kappa}{\Rightarrow}"] & & 1\times(1\times 1) \arrow[d, "1\times f(i)"] \\ \mathscr{A}\times 1 \arrow[r, "\cong"'] & \mathscr{A} & 1\times\mathscr{A} \arrow[l, "\cong"] \end{tikzcd}$$

which are equally well natural families α, λ and ρ as in (4·1) together with a map $\kappa_\star\colon I \to I$.

So the data in dimensions $\leqslant 3$ for a simplicial map $\mathbb{C} \to \mathrm{N}_\ell(\mathrm{Cat})$ is the data $(\mathscr{A}, \otimes, I, \alpha, \lambda, \rho)$ for a small skew-monoidal category augmented with a map $\kappa_\star\colon I \to I$ in $\mathscr{A}$. It remains to consider the action on non-degenerate 4-simplices of $\mathbb{C}$. There are nine such, given by:

$$\begin{aligned} A1 &= (a, a, a, a, a) & A6 &= (s_0(i), \ell, k, r, s_2(i)) \\ A2 &= (r, s_1(t), a, s_1(t), \ell) & A7 &= (k, \ell, s_0s_1(c), r, k) \\ A3 &= (\ell, \ell, s_2(t), a, s_2(t)) & A8 &= (r, s_1(t), s_0(t), r, k) \\ A4 &= (s_0(t), a, s_0(t), r, r) & A9 &= (k, \ell, s_2(t), s_1(t), \ell), \\ A5 &= (s_1(i), s_2(i), k, s_0(i), s_1(i)) \end{aligned}$$

where, as before, we take advantage of coskeletality of $\mathbb{C}$ to identify a 4-simplex with its tuple of 3-dimensional faces. The images of these simplices each assert the commutativity of a pentagon of natural transformations involving α, ρ, λ or κ; explicitly, they assert that for any $A, B, C, D \in \mathscr{A}$, the following pentagons commute in $\mathscr{A}$:

$$\begin{tikzcd} & & (AB)(CD) \arrow[drr, "\alpha"] & & \\ ((AB)C)D \arrow[urr, "\alpha"] \arrow[dr, "\alpha 1"'] & & \text{(A1)} & & A(B(CD)) \\ & (A(BC))D \arrow[rr, "\alpha"'] & & A((BC)D) \arrow[ur, "1\alpha"'] & \end{tikzcd}$$

$$\begin{tikzcd} & & AB \arrow[drr, "1"] & & \\ AB \arrow[urr, "1"] \arrow[dr, "\rho 1"'] & & \text{(A2)} & & AB \\ & (AI)B \arrow[rr, "\alpha"'] & & A(IB) \arrow[ur, "1\lambda"'] & \end{tikzcd}$$

$$\begin{tikzcd} & & I(AB) \arrow[drr, "\lambda"] & & \\ (IA)B \arrow[urr, "\alpha"] \arrow[dr, "\lambda 1"'] & & \text{(A3)} & & AB \\ & AB \arrow[rr, "1"'] & & AB \arrow[ur, "1"'] & \end{tikzcd}$$

$$\begin{tikzcd} & & (AB)I \arrow[drr, "\alpha"] & & \\ AB \arrow[urr, "\rho"] \arrow[dr, "1"'] & & \text{(A4)} & & A(BI) \\ & AB \arrow[rr, "1"'] & & AB \arrow[ur, "1\rho"'] & \end{tikzcd}$$

$$\begin{tikzcd} & & I \arrow[drr, "1"] & & \\ I \arrow[urr, "1"] \arrow[dr, "1"'] & & \text{(A5)} & & I \\ & I \arrow[rr, "\kappa_\star"'] & & I \arrow[ur, "1"'] & \end{tikzcd}$$

$$\begin{tikzcd} & & II \arrow[drr, "\lambda_I"] & & \\ I \arrow[urr, "\rho_I"] \arrow[dr, "1"'] & & \text{(A6)} & & I \\ & I \arrow[rr, "\kappa_\star"'] & & I \arrow[ur, "1"'] & \end{tikzcd}$$

$$\begin{tikzcd} & & II \arrow[drr, "\lambda_I"] & & \\ I \arrow[urr, "\rho_I"] \arrow[dr, "\kappa_\star"'] & & \text{(A7)} & & I \\ & I \arrow[rr, "1"'] & & I \arrow[ur, "\kappa_\star"'] & \end{tikzcd}$$

$$\begin{tikzcd} & & AI \arrow[drr, "1"] & & \\ A \arrow[urr, "\rho"] \arrow[dr, "\rho"'] & & \text{(A8)} & & AI \\ & AI \arrow[rr, "1"'] & & AI \arrow[ur, "1\kappa"'] & \end{tikzcd}$$

$$\begin{tikzcd} & & IA \arrow[drr, "\lambda"] & & \\ IA \arrow[urr, "1"] \arrow[dr, "\kappa 1"'] & & \text{(A9)} & & A\,. \\ & IA \arrow[rr, "1"'] & & IA \arrow[ur, "\lambda"'] & \end{tikzcd}$$

Note first that (A5) forces $\kappa_\star = 1_I \colon I \to I$. Now (A1)–(A4) express the axioms (S1)–(S4), both (A6) and (A7) express axiom (S5), whilst (A8) and (A9) are trivially satisfied. Thus the 4-simplex data of a simplicial map $\mathbb{C} \to \mathrm{N}_\ell(\mathrm{Cat})$ exactly express the skew-monoidal axioms and the fact that the additional datum $\kappa_\star \colon I \to I$ is trivial; whence a simplicial map $\mathbb{C} \to \mathrm{N}_\ell(\mathrm{Cat})$ is precisely a small skew-monoidal category.

The same proof now shows that a simplicial map $\mathbb{C} \to \mathrm{N}_p(\mathrm{Cat})$ is precisely a small monoidal category, under the identification of monoidal categories with skew-monoidal categories whose constraint maps are invertible.

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