

DIAGRAMMATIC CHARACTERISATION OF ENRICHED ABSOLUTE COLIMITS

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ABSTRACT. We provide a diagrammatic criterion for the existence of an absolute colimit in the context of enriched category theory.

An *absolute colimit* is one preserved by any functor; the class of absolute colimits was characterised for ordinary categories by Paré [4] and for enriched ones by Street [5]. For categories enriched over a monoidal category \mathcal{V} or bicategory \mathcal{W} , the appropriate colimits are the weighted colimits of [6], and Street's characterisation is in fact one of the class of *absolute weights*: those weights φ such that φ -weighted colimits are preserved by any functor. This is different to Paré's result, which gives a diagrammatic characterisation of when a particular cocone is absolutely colimiting. In this note, we give a result in the enriched context which is closer in spirit to Paré's than to Street's. This result is very useful in practice, but seems not to be in the literature; we set it down for future use.

1. The result

1.1. BACKGROUND. We work in the context of bicategory-enriched category theory; see [6], for example. \mathcal{W} will denote a bicategory whose homs are locally small, complete and cocomplete categories, and which is *biclosed*, meaning that for each 1-cell $A: x \rightarrow y$ in \mathcal{W} , the composition functors $A \otimes (-): \mathcal{W}(z, x) \rightarrow \mathcal{W}(z, y)$ and $(-) \otimes A: \mathcal{W}(y, z) \rightarrow \mathcal{W}(x, z)$ have right adjoints $[A, -]$ and $\langle A, - \rangle$ respectively.

A \mathcal{W} -category \mathcal{A} comprises a set $\text{ob } \mathcal{A}$ of objects; for each $a \in \text{ob } \mathcal{A}$ an object $\epsilon a \in \text{ob } \mathcal{W}$, the *extent* of a ; for each pair of objects a, b , a hom-object $\mathcal{C}(b, a) \in \mathcal{W}(\epsilon a, \epsilon b)$; and identity and composition 2-cells $\iota: I_{\epsilon a} \rightarrow \mathcal{C}(a, a)$ and $\mu: \mathcal{C}(c, b) \otimes \mathcal{C}(b, a) \rightarrow \mathcal{C}(c, a)$ satisfying the expected axioms. A \mathcal{W} -profunctor $M: \mathcal{A} \rightarrow \mathcal{B}$ is given by objects $M(b, a) \in \mathcal{W}(\epsilon a, \epsilon b)$ and action maps $\mu: \mathcal{B}(b', b) \otimes M(b, a) \otimes \mathcal{A}(a, a') \rightarrow M(b', a')$ satisfying unitality and associativity axioms. A *profunctor map* $M \rightarrow M': \mathcal{A} \rightarrow \mathcal{B}$ comprises maps $M(b, a) \rightarrow M'(b, a)$ compatible with the actions by \mathcal{A} and \mathcal{B} . The identity profunctor $\mathcal{A}: \mathcal{A} \rightarrow \mathcal{A}$ has components $\mathcal{A}(b, a)$ with action given by composition in \mathcal{A} . For profunctors $M: \mathcal{A} \rightarrow \mathcal{B}$ and $N: \mathcal{B} \rightarrow \mathcal{C}$ with \mathcal{B} small, the tensor product $N \otimes_{\mathcal{B}} M: \mathcal{A} \rightarrow \mathcal{C}$ has components given by coequalisers

$$\sum_{b, b'} N(c, b) \otimes \mathcal{B}(b, b') \otimes M(b', a) \rightrightarrows \sum_b N(c, b) \otimes M(b, a) \rightarrow (N \otimes_{\mathcal{B}} M)(c, a)$$

and actions by \mathcal{C} and \mathcal{A} inherited from N and M . Small \mathcal{W} -categories, profunctors and profunctor maps comprise a bicategory $\mathcal{W}\text{-Mod}$. There is a full embedding $\mathcal{W} \rightarrow \mathcal{W}\text{-Mod}$ sending X to the \mathcal{W} -category X with one object \star with $\epsilon(\star) = X$ and $X(\star, \star) = I_X$.

If \mathcal{A} and \mathcal{B} are \mathcal{W} -categories, then a \mathcal{W} -functor $F: \mathcal{A} \rightarrow \mathcal{B}$ comprises an extent-preserving assignation on objects, together with 2-cells $\mathcal{C}(b, a) \rightarrow \mathcal{D}(Fb, Fa)$ subject to two functoriality axioms. If $F: \mathcal{A} \rightarrow \mathcal{C}$ and $G: \mathcal{B} \rightarrow \mathcal{C}$ are \mathcal{W} -functors then there is an induced profunctor $\mathcal{C}(G, F): \mathcal{A} \leftrightarrow \mathcal{B}$ with components $\mathcal{C}(G, F)(b, a) = \mathcal{C}(Gb, Fa)$ and action derived from the action of F and G on homs and composition in \mathcal{C} .

Given profunctors $M: \mathcal{A} \leftrightarrow \mathcal{B}$, $N: \mathcal{B} \leftrightarrow \mathcal{C}$ and $L: \mathcal{A} \leftrightarrow \mathcal{C}$ with \mathcal{B} small, a profunctor map $u: N \otimes_{\mathcal{B}} M \rightarrow L$ is said to *exhibit M as $[N, L]$* if every map $f: N \otimes_{\mathcal{B}} K \rightarrow L$ is of the form $u \circ (N \otimes_{\mathcal{B}} \bar{f})$ for a unique $\bar{f}: K \rightarrow M$; while it is said to *exhibit N as $\langle M, L \rangle$* if every $f: K \otimes_{\mathcal{B}} M \rightarrow L$ is of the form $u \circ (\bar{f} \otimes_{\mathcal{B}} M)$ for a unique $\bar{f}: K \rightarrow N$.

Given $\varphi: \mathcal{A} \leftrightarrow \mathcal{B}$ in $\mathcal{W}\text{-Mod}$ and a functor $F: \mathcal{B} \rightarrow \mathcal{C}$, a φ -weighted colimit of F is a functor $Z: \mathcal{A} \rightarrow \mathcal{C}$ and profunctor map $a: \varphi \rightarrow \mathcal{C}(F, Z)$ such that for each $C \in \mathcal{C}$, the map

$$\varphi \otimes_{\mathcal{A}} \mathcal{C}(Z, C) \xrightarrow{a \otimes_{\mathcal{A}} 1} \mathcal{C}(F, Z) \otimes_{\mathcal{A}} \mathcal{C}(Z, C) \xrightarrow{\mu} \mathcal{C}(F, C) \quad (1)$$

exhibits $\mathcal{C}(Z, C)$ as $[\varphi, \mathcal{C}(F, C)]$. A functor $G: \mathcal{C} \rightarrow \mathcal{D}$ preserves this colimit just when the composite $\varphi \rightarrow \mathcal{C}(F, Z) \rightarrow \mathcal{D}(GF, GZ)$ exhibits GZ as a φ -weighted colimit of GF ; the colimit is *absolute* when it is preserved by all functors out of \mathcal{C} . [5] proves that φ -weighted colimits are absolute if and only if φ admits a right adjoint in $\mathcal{W}\text{-Mod}$.

Dually, given $\psi: \mathcal{B} \leftrightarrow \mathcal{A}$ in $\mathcal{W}\text{-Mod}$ and a functor $F: \mathcal{B} \rightarrow \mathcal{C}$, a ψ -weighted limit of F is a functor $Z: \mathcal{A} \rightarrow \mathcal{C}$ and map $b: \psi \rightarrow \mathcal{C}(Z, F)$ such that for each $C \in \mathcal{C}$, the map

$$\mathcal{C}(C, Z) \otimes_{\mathcal{A}} \psi \xrightarrow{1 \otimes_{\mathcal{A}} b} \mathcal{C}(C, Z) \otimes_{\mathcal{A}} \mathcal{C}(Z, F) \xrightarrow{\mu} \mathcal{C}(C, F)$$

exhibits $\mathcal{C}(C, Z)$ as $\langle \psi, \mathcal{C}(C, Z) \rangle$. Absoluteness of limits is defined as before; now every limit weighted by $\psi: \mathcal{B} \leftrightarrow \mathcal{A}$ is absolute if and only if ψ has a *left* adjoint in $\mathcal{W}\text{-Mod}$.

1.2. THEOREM. *Let $\varphi: \mathcal{A} \leftrightarrow \mathcal{B}$ admit the right adjoint $\psi: \mathcal{B} \leftrightarrow \mathcal{A}$ in $\mathcal{W}\text{-Mod}$, and let $F: \mathcal{B} \rightarrow \mathcal{C}$ and $Z: \mathcal{A} \rightarrow \mathcal{C}$ be \mathcal{W} -functors. There is a bijective correspondence between data of the following forms:*

- (a) A map $a: \varphi \rightarrow \mathcal{C}(F, Z)$ exhibiting Z as a φ -weighted colimit of F ;
- (b) A map $b: \psi \rightarrow \mathcal{C}(Z, F)$ exhibiting Z as a ψ -weighted limit of F ;
- (c) Maps $a: \varphi \rightarrow \mathcal{C}(F, Z)$ and $b: \psi \rightarrow \mathcal{C}(Z, F)$ such that the following two squares commute in $\mathcal{W}\text{-Mod}(\mathcal{A}, \mathcal{A})$ and $\mathcal{W}\text{-Mod}(\mathcal{B}, \mathcal{B})$:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\eta} & \psi \otimes_{\mathcal{B}} \varphi \\ \downarrow Z & & \downarrow b \otimes_{\mathcal{B}} a \\ \mathcal{C}(Z, Z) & \xleftarrow{\mu} & \mathcal{C}(Z, F) \otimes_{\mathcal{B}} \mathcal{C}(F, Z) \end{array} \quad \begin{array}{ccc} \varphi \otimes_{\mathcal{A}} \psi & \xrightarrow{\varepsilon} & \mathcal{B} \\ \downarrow a \otimes_{\mathcal{A}} b & & \downarrow F \\ \mathcal{C}(F, Z) \otimes_{\mathcal{A}} \mathcal{C}(Z, F) & \xrightarrow{\mu} & \mathcal{C}(F, F) \end{array} \quad (2)$$

Proof. Suppose first given (a); consider the composite profunctor map

$$\varphi \otimes_{\mathcal{A}} \mathcal{C}(Z, F) \xrightarrow{a \otimes_{\mathcal{A}} 1} \mathcal{C}(F, Z) \otimes_{\mathcal{A}} \mathcal{C}(Z, F) \xrightarrow{\mu} \mathcal{C}(F, F) \quad (3)$$

Evaluating in the second variable at any $a \in \mathcal{A}$ yields the map (1) exhibiting $\mathcal{C}(Z, Fa)$ as $[\varphi, \mathcal{C}(F, Fa)]$; it follows easily that (3) exhibits $\mathcal{C}(Z, F)$ as $[\varphi, \mathcal{C}(F, F)]$. Applying this universality to the composite $\varepsilon \circ F: \varphi \otimes_{\mathcal{A}} \psi \rightarrow \mathcal{B} \rightarrow \mathcal{C}(F, F)$ yields a unique map $b: \psi \rightarrow \mathcal{C}(Z, F)$ making the right square of (2) commute; we must show that the left one does too. Arguing as before shows that

$$\varphi \otimes_{\mathcal{A}} \mathcal{C}(Z, Z) \xrightarrow{a \otimes_{\mathcal{A}} 1} \mathcal{C}(F, Z) \otimes_{\mathcal{A}} \mathcal{C}(Z, Z) \xrightarrow{\mu} \mathcal{C}(F, Z) \quad (4)$$

exhibits $\mathcal{C}(Z, Z)$ as $[\varphi, \mathcal{C}(F, Z)]$. It thus suffices to show that the left square of (2) commutes after applying the functor $\varphi \otimes_{\mathcal{A}} (-)$ and postcomposing with (4); which follows by a short calculation using commutativity in the right square and the triangle identities.

So from the data in (a) we may obtain that in (c), and the assignation is injective, since b is uniquely determined by universality of a and commutativity on the right of (2). For surjectivity, suppose given a and b as in (c); we must show that a exhibits Z as a φ -weighted colimit of F , in other words, that for each $C \in \mathcal{C}$, the map (1) exhibits $\mathcal{C}(Z, C)$ as $[\varphi, \mathcal{C}(F, C)]$, or in other words, that for each map $f: \varphi \otimes_{\mathcal{A}} K \rightarrow \mathcal{C}(F, C)$, there is a unique map $\bar{f}: K \rightarrow \mathcal{C}(Z, C)$ such that $f = \mu \circ (a \otimes_{\mathcal{A}} \bar{f}): \varphi \otimes_{\mathcal{A}} K \rightarrow \mathcal{C}(F, Z) \otimes_{\mathcal{A}} \mathcal{C}(Z, C) \rightarrow \mathcal{C}(F, C)$. For existence, we let \bar{f} be the composite

$$K \cong \mathcal{A} \otimes_{\mathcal{A}} K \xrightarrow{\eta \otimes_{\mathcal{A}} 1} \psi \otimes_{\mathcal{B}} \varphi \otimes_{\mathcal{A}} K \xrightarrow{b \otimes_{\mathcal{B}} f} \mathcal{C}(Z, F) \otimes_{\mathcal{B}} \mathcal{C}(F, C) \xrightarrow{\mu} \mathcal{C}(Z, C); \quad (5)$$

now rewriting with the right-hand square of (2) and using the triangle identities and F 's preservation of units shows that $f = \mu \circ (a \otimes_{\mathcal{A}} \bar{f})$. For uniqueness, let $g: K \rightarrow \mathcal{C}(Z, C)$ also satisfy $f = \mu \circ (a \otimes_{\mathcal{A}} g)$. Substituting into (5) shows that \bar{f} is the composite

$$K \cong \mathcal{A} \otimes_{\mathcal{A}} K \xrightarrow{\eta \otimes_{\mathcal{A}} 1} \psi \otimes_{\mathcal{B}} \varphi \otimes_{\mathcal{A}} K \xrightarrow{b \otimes_{\mathcal{B}} a \otimes_{\mathcal{A}} g} \mathcal{C}(Z, F) \otimes_{\mathcal{B}} \mathcal{C}(F, Z) \otimes_{\mathcal{A}} \mathcal{C}(Z, C) \xrightarrow{\mu} \mathcal{C}(Z, C);$$

which by rewriting with the left square of (2) and using Z 's preservation of identities is equal to g . This proves the equivalence (a) \Leftrightarrow (c); now (a) \Leftrightarrow (b) follows by duality. ■

1.3. EXAMPLES. We first consider examples wherein \mathcal{W} is the one-object bicategory corresponding to a monoidal category \mathcal{V} .

- Let $\mathcal{V} = \mathbf{Set}$, and let φ be the weight for splittings of idempotents. The result recovers the bijection, for an idempotent $e: A \rightarrow A$, between: maps $p: A \rightarrow B$ coequalising e and 1_A ; maps $i: B \rightarrow A$ equalising e and 1_A ; and pairs (i, p) with $pi = 1_A$ and $ip = e$.
- Let $\mathcal{V} = \mathbf{Set}_*$, and let φ be the weight for an initial object. The result recovers the bijection in a pointed category between: initial objects; terminal objects; and objects X with $1_X = 0_X$.
- Let $\mathcal{V} = \mathbf{Ab}$, and let φ be the weight for binary coproducts. The result recovers the bijection, for objects A, B in a pre-additive category, between: coproduct diagrams $i_1: A \rightarrow Z \leftarrow B: i_2$; product diagrams $p_1: A \leftarrow Z \rightarrow B: p_2$; and tuples (i_1, i_2, p_1, p_2) such that $p_j i_k = \delta_{ik}$ and $i_1 p_1 + i_2 p_2 = 1_Z$.

- Let $\mathcal{V} = \mathbf{V}\text{-Lat}$, and let φ be the weight for J -fold coproducts (for J a small set). The result recovers the bijection, for objects $(A_j : j \in J)$ in a sup-lattice enriched category, between: coproduct diagrams $(i_j : A_j \rightarrow Z)_{j \in J}$; product diagrams $(p_j : Z \rightarrow A_j)_{j \in J}$; and families $(i_j)_{j \in J}$ and $(p_j)_{j \in J}$ with $p_j i_k = \delta_{jk}$ and $\bigvee_j i_j p_j = 1_Z$.
- Let $\mathcal{V} = k\text{-Vect}$ for k a field of characteristic zero, let G be a finite group, and let $\varphi : k \rightarrow kG$ be the trivial right kG -module k . By Burnside's Lemma, φ has right adjoint $kG \rightarrow k$ given by the trivial left kG -module k . Now the result recovers the bijection, for a G -representation A in a k -linear category, between: maps $p : A \rightarrow Z$ exhibiting Z as an object of coinvariants of A ; maps $i : Z \rightarrow A$ exhibiting Z as an object of invariants of A ; and pairs of maps (i, p) with $pi = 1_Z$ and $ip = \frac{1}{|G|} \sum_{g \in G} g$.

We conclude with two examples where \mathcal{W} is a genuine bicategory.

- Let (\mathcal{C}, j) be a subcanonical site, and let \mathcal{W} denote the full sub-bicategory of $\mathbf{Span}(\mathbf{Sh}(\mathcal{C}))^{\text{op}}$ on objects of the form $\mathcal{C}(-, X)$. To any prestack $p : \mathcal{E} \rightarrow \mathcal{C}$ over \mathcal{C} , we may (as in [1]) associate a \mathcal{W} -category with objects those of \mathcal{E} , extents $\epsilon(a) = p(a)$, and hom-object from a to b given by the span $\mathcal{C}(-, pa) \leftarrow \mathcal{E}(a, b) \rightarrow \mathcal{C}(-, pb)$ in $\mathbf{Sh}(\mathcal{C})$; here $\mathcal{E}(a, b)(x)$ is the set of all triples (f, g, θ) with $f : pa \leftarrow x \rightarrow pb : g$ in \mathcal{C} and $\theta : f^*(a) \rightarrow g^*(b)$ in \mathcal{E}_x (note that $\mathcal{E}(a, b)$ is a sheaf by the prestack condition).

For any cover $(f_i : U_i \rightarrow U)_{i \in I}$ in \mathcal{C} , we have a \mathcal{W} -category $R[f]$ with object set I , extents $\epsilon(i) = U_i$ and hom-objects $R[f](j, i) = \mathcal{C}(-, U_j) \leftarrow \mathcal{C}(-, U_j \times_U U_i) \rightarrow \mathcal{C}(-, U_i)$. There is a profunctor $\varphi : U \rightarrow R[f]$ with components given by the spans $\varphi(i, \star) = \mathcal{C}(-, U_i) \leftarrow \mathcal{C}(-, U_i) \rightarrow \mathcal{C}(-, U)$. Writing $\psi : R[f] \rightarrow U$ for the reverse profunctor, it is not hard to see that $\varphi \dashv \psi$ (in fact they are adjoint pseudoinverse).

The result now says the following. Given a prestack $p : \mathcal{E} \rightarrow \mathcal{C}$, a cover $(f_i : U_i \rightarrow U)$ in \mathcal{C} , and a family of spans $p_{ij} : a_i \leftarrow a_{ij} \rightarrow a_j : q_{ij}$ in \mathcal{E} whose legs are cartesian over the projections $U_i \leftarrow U_i \times_U U_j \rightarrow U_j$, there is a bijection between: cocones $(h_i : a_i \rightarrow a)$ in \mathcal{E} over the f_i 's that are colimiting for the diagram comprised of the p_{ij} 's and q_{ij} 's; universal objects $a \in \mathcal{E}_U$ equipped with vertical maps $f_i^*(a) \rightarrow a_i$ fitting into double pullback squares

$$\begin{array}{ccccc} f_i^*(a) & \longleftarrow & \cdot & \longrightarrow & f_j^*(a) \\ \downarrow & & \downarrow & & \downarrow \\ a_i & \xleftarrow{p_{ij}} & a_{ij} & \xrightarrow{q_{ij}} & a_j \end{array} ;$$

and objects $a \in \mathcal{E}_U$ equipped with a family of maps $(h_i : a_i \rightarrow a)$ cartesian over the f_i 's. This generalises [6, Proposition 5.2(b)]¹.

- Let \mathcal{W} denote the bicategory whose objects are sets, and whose hom-category $\mathcal{W}(X, Y)$ is the category of finitary functors $\mathbf{Set}/Y \rightarrow \mathbf{Set}/X$; note that $\mathcal{W}(X, Y) \simeq$

¹The proposition numbering here is taken from the TAC reprint.

$[\mathbf{Fam}(Y) \times X, \mathbf{Set}]$, where $\mathbf{Fam}(Y)$ has as objects, finite lists of elements of Y , and as maps $(y_0, \dots, y_m) \rightarrow (z_0, \dots, z_n)$, functions $f: [m] \rightarrow [n]$ such that $y_i = z_{f(i)}$. To any cartesian multicategory M (i.e., a *Gentzen multicategory* in the sense of [3]) we may associate a \mathcal{W} -category \mathcal{M} whose objects of extent X are X -indexed families of objects of M , and whose hom-object between families $(a_x)_{x \in X}$ and $(b_y)_{y \in Y}$ is the presheaf

$$\mathcal{M}((b_y), (a_x))(y_0, \dots, y_m; x) = M(b_{y_0}, \dots, b_{y_m}; a_x)$$

in $[\mathbf{Fam}(Y) \times X, \mathbf{Set}]$; reindexing along maps in Y makes use of the cartesianness of the multicategory structure. Composition and units in \mathcal{M} follow from those in M .

Given a finite set $X = \{x_0, \dots, x_n\}$, let $\varphi: 1 \rightarrow X$ be the \mathcal{W} -profunctor whose unique component is the representable $y(x_0, \dots, x_n; \star) \in [\mathbf{Fam}(X) \times 1, \mathbf{Set}]$. This has a right adjoint $\psi: X \rightarrow 1$ whose unique component is the presheaf $\Sigma_{x \in X} y(\star; x) \in [\mathbf{Fam}(1) \times X, \mathbf{Set}]$. The result now establishes a bijection, for any finite family (a_0, \dots, a_n) of objects in a cartesian multicategory M , between data of the following three forms: first, an object a and a multimap $i \in M(a_0, \dots, a_n; a)$, composition with which induces bijections between $M(b_0, \dots, b_k, a, c_0, \dots, c_\ell; d)$ and $M(b_0, \dots, b_k, a_0, \dots, a_n, c_0, \dots, c_\ell; d)$; second, an object a and unary maps $p_j \in M(a; a_j)$, composition with which establishes bijections between $M(b_0, \dots, b_k; a)$ and $\Pi_j M(b_0, \dots, b_k; a_j)$; third, an object a and maps i and p_j as above such that $p_j \circ i = \pi_j \in M(a_0, \dots, a_n; a_j)$ and $i \circ (p_0, \dots, p_n) = 1_a \in M(a; a)$. This generalises [2, Proposition 3.5].

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