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Commutativity

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ABSTRACT

We describe a general framework for notions of commutativity based on enriched category theory. We extend Eilenberg and Kelly's tensor product for categories enriched over a symmetric monoidal base to a tensor product for categories enriched over a normal duoidal category; using this, we re-find notions such as the commutativity of a finitary algebraic theory or a strong monad, the commuting tensor product of two theories, and the Boardman–Vogt tensor product of symmetric operads.

1. Introduction

This article is a category-theoretic investigation into the notion of *commutativity*. We first meet commutativity in elementary algebra: two elements a, b of a monoid M are said to commute if $ab = ba$, while M itself is called commutative if all its elements commute pairwise. This immediately yields other notions of commutativity: for groups (on forgetting the inverses), for rings (on forgetting the additive structure) and for Lie algebras (on passing to the universal enveloping algebra).

Later on, we encounter more sophisticated forms of commutativity not directly reducible to that for monoids. For example, a pair of operations f, g of arities m, n in an algebraic theory $\mathcal{T}$ are said to *commute* if the two mn-ary operations

$$\begin{aligned} & f(g(x_{11},\dots,x_{1n}),\dots,g(x_{m1},\dots,x_{mn})) \\ \text{and}\qquad & g(f(x_{11},\dots,x_{m1}),\dots,f(x_{1n},\dots,x_{mn})) \end{aligned}$$

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are equal, while $\mathcal{T}$ itself is *commutative* when all of its operations commute pairwise; typical commutative theories are those for join-semilattices, for commutative monoids and for modules over a commutative ring R. An important related notion in this context is the *commuting tensor product* $\mathcal{S} \odot \mathcal{T}$ of theories $\mathcal{S}$ and $\mathcal{T}$; this has the property that $\mathcal{S} \odot \mathcal{T}$-models in a category $\mathcal{E}$ correspond with $\mathcal{S}$-models in the category of $\mathcal{T}$-models in $\mathcal{E}$, and also with $\mathcal{T}$-models in the category of $\mathcal{S}$-models in $\mathcal{E}$. There is a corresponding notion of commutativity for operations in *symmetric operads* in the sense of [42], and the analogue of the commuting tensor product in this context is the *Boardman–Vogt* tensor product of [6].

Yet another kind of generalised commutativity arises in the context of the *sesquicategories* of [48]; these may be defined succinctly as comprising a category $\mathcal{C}$ together with a lifting of $\mathbf{Hom}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ through the forgetful functor $\mathbf{Cat} \to \mathbf{Set}$. To give such a lifting is to equip $\mathcal{C}$ with 2-cells that admit vertical composition and whiskering on each side with 1-cells, but which need not satisfy the *interchange* axiom, which requires that for any pair of 2-cells in the configuration

$$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "h"{name=U2}] \arrow[r, bend right, "k"'{name=D2}] \arrow[from=U2, to=D2, phantom, "\Downarrow\beta"] & C \end{tikzcd}$$

we should have $\beta g \circ h\alpha = k\alpha \circ \beta f \colon hf \Rightarrow kg$. If we declare the pair (α, β) to commute just when they *do* satisfy interchange, then a sesquicategory will be commutative, in the sense of all of its composable pairs commuting, precisely when it is a 2-category. A related example involves the *premonoidal categories* of [46], which bear the same relation to (non-strict) monoidal categories as sesquicategories do to 2-categories.

The objective of this paper is to describe an abstract framework for commutativity that encompasses each of the examples given above, and others besides. As a starting point, we observe that each of our examples is concerned with a kind of structure—monoids, algebraic theories, operads, sesquicategories—that can be viewed as a monoid in a particular monoidal category; an ordinary monoid, is, of course, a monoid in the cartesian monoidal $\mathbf{Set}$, while a finitary algebraic theory can be seen as a monoid in the substitution monoidal category $[\mathbb{F}, \mathbf{Set}]$, where $\mathbb{F}$ is the category of functions between finite cardinals. Consequently, the key notion of our abstract theory will be the definition, for a suitable monoidal category $(\mathcal{V}, \circ, I)$ and a monoid C therein, of what it means for a pair of generalised elements

$$\begin{tikzcd} A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd} \tag{1.1}$$

of C to commute. As explained in [27], other aspects of the theory flow easily once this definition is made: for example, C itself is commutative just when the generalised element 1_C commutes with itself, while the commuting tensor product of monoids A and B is the universal monoid $A \odot B$ in which A and B commute; other notions such as centralizers and centres also admit expression in this generality.

When $(\mathcal{V}, \circ, I)$ is a *braided* monoidal category, it is easy to say when (1.1) should be a commuting cospan—namely, just when the left-hand diagram in:

$$\begin{tikzcd}[column sep=small] & A \circ B \arrow[r, "f \circ g"] & C \circ C \arrow[dr, "m"] & \\ A \circ B \arrow[ur, "1"] \arrow[dr, "c"'] & & & C \\ & B \circ A \arrow[r, "g \circ f"'] & C \circ C \arrow[ur, "m"'] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & A \circ B \arrow[r, "f \circ g"] & C \circ C \arrow[dr, "m"] & \\ A * B \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & C \\ & B \circ A \arrow[r, "g \circ f"'] & C \circ C \arrow[ur, "m"'] & \end{tikzcd} \tag{1.2}$$

commutes in $\mathcal{V}$; here c is the braiding of $\mathcal{V}$ and m is the multiplication of the monoid C. With $\mathcal{V} = (\mathbf{Set}, \times, 1)$, this recovers the case of classical monoids; but it does not account for examples—such as finitary algebraic theories—wherein $\mathcal{V}$ is *not* braided monoidal. The key novelty of our treatment is in how we extend the basic commutativity notion in (1.1) to cases such as these. Rather than a braiding, we assume that $\mathcal{V}$ is equipped with a second monoidal structure whose tensor $*\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, unit $J\colon 1 \to \mathcal{V}$ and associated coherences are normal opmonoidal with respect to the first; this makes $(\mathcal{V}, *, J, \circ, I)$ into a *normal duoidal* [4] or 2-*fold monoidal* [2] category. As we recall in Section 2.2 below, a normal duoidal structure gives rise to natural families of maps

$$\sigma\colon A * B \to A \circ B \qquad \text{and} \qquad \tau\colon A * B \to A \circ B$$

which in suitable circumstances can serve as a surrogate for a braiding; in particular, we may generalise the notion of commuting cospan (1.1) from the braided to the normal duoidal context by replacing 1 and c as to the left of (1.2) by σ and τ as to the right. Note that this is a true generalisation: any braided monoidal $\mathcal{V}$ bears a canonical normal duoidal structure with $* = \circ$ and $J = I$ for which σ and τ reduce exactly to 1 and c.

This more general framework for commutativity is sufficient to capture all of the leading examples. For example, the substitution monoidal category $([\mathbb{F}, \mathbf{Set}], \circ, I)$, wherein monoids are finitary algebraic theories, becomes normal duoidal when equipped with the second monoidal structure $(*, J)$ given by *Day convolution* [9] with respect to product; as we will see in Section 5, the resultant theory of commutativity for finitary algebraic theories is precisely the classical one outlined above. One of the basic contentions of this paper is consequently that *normal duoidal categories are an appropriate environment for describing a theory of commutativity.*

In fact, this theory becomes more perspicuous if we adopt a broader perspective. A monoid in a monoidal category $\mathcal{V}$ is equally well a one-object $\mathcal{V}$-*enriched category* [31], and the meaning of our theory of commutativity may be clarified by generalising it from $\mathcal{V}$-monoids to $\mathcal{V}$-categories; the basic notion of commuting cospan (1.1) is then replaced by a notion of *bifunctor* between $\mathcal{V}$-categories—which we now explain.

Consider first the case where $\mathcal{V} = \mathbf{Set}$; here a $\mathcal{V}$-category is just an ordinary (locally small) category, and we are familiar with the fact that a *bifunctor* from $\mathcal{A}, \mathcal{B}$ to $\mathcal{C}$ is simply a functor $T\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$. An important basic exercise (see [39, §II.3, Proposition 1]) shows that giving the data of a bifunctor is equivalent to giving families of functors $T(a, –)\colon \mathcal{B} \to \mathcal{C}$ for $a \in \mathcal{A}$ and $T(–, b)\colon \mathcal{A} \to \mathcal{C}$ for $b \in \mathcal{B}$ that agree on objects—so $T(–, b)(a) = T(a, –)(b) = T(a, b)$, say—and which on arrows satisfy the *commutativity* condition that, for all $f\colon a \to a'$ in $\mathcal{A}$ and $g\colon b \to b'$ in $\mathcal{B}$, the following square should commute in $\mathcal{C}$:

$$\begin{tikzcd} T(a,b) \arrow[r, "T(f,b)"] \arrow[d, "T(a,g)"'] & T(a',b) \arrow[d, "T(a',g)"] \\ T(a,b') \arrow[r, "T(f,b')"'] & T(a',b')\ . \end{tikzcd} \tag{1.3}$$

More generally, for any braided monoidal $\mathcal{V}$, there is a well-established notion of $\mathcal{V}$-bifunctor; as explained in [13, Chapter III, §4], it may again be described in two ways. On the one hand, the existence of the braiding means that there is an easily-defined notion of *tensor product* for $\mathcal{V}$-categories [31, §1.4], and a $\mathcal{V}$-bifunctor from $\mathcal{A}, \mathcal{B}$ to $\mathcal{C}$ may now be defined simply as a $\mathcal{V}$-functor $T\colon \mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$. On the other hand, we may once again specify a bifunctor in terms of families of $\mathcal{V}$-functors $T(a, –)\colon \mathcal{B} \to \mathcal{C}$ and $T(–, b)\colon \mathcal{A} \to \mathcal{C}$ which match up on objects, and which satisfy a commutativity condition like (1.3), but now expressed in terms of commuting diagrams of hom-objects in $\mathcal{V}$:

$$\begin{tikzcd}[column sep=small]
 & \mathcal{A}(a,a')\circ\mathcal{B}(b,b') \arrow[r, "T(-,b')\circ T(a,-)"] & \mathcal{C}(Tab',Ta'b')\circ\mathcal{C}(Tab,Tab') \arrow[dr, "m"] & \\
\mathcal{A}(a,a')\circ\mathcal{B}(b,b') \arrow[ur, "1"] \arrow[dr, "c"'] & & & \mathcal{C}(Tab,Ta'b') \\
 & \mathcal{B}(b,b')\circ\mathcal{A}(a,a') \arrow[r, "T(a',-)\circ T(-,b)"'] & \mathcal{C}(Ta'b,Ta'b')\circ\mathcal{C}(Tab,Ta'b). \arrow[ur, "m"'] &
\end{tikzcd} \tag{1.4}$$

Comparing (1.4) to the left-hand diagram of (1.2), we see that the latter is simply the one-object case of the former, so that, for braided monoidal categories, the commuting cospans of (1.1) are *the same* as $\mathcal{V}$-bifunctors between one-object $\mathcal{V}$-categories.

Given this, there is now an obvious way of generalising the notion of $\mathcal{V}$-bifunctor from the braided monoidal to the normal duoidal context: we define a bifunctor from $\mathcal{A}, \mathcal{B}$ to $\mathcal{C}$ in terms of families of one-variable $\mathcal{V}$-functors $T(-,b)$ and $T(a,-)$ which agree on objects, and which satisfy the commutativity condition obtained from (1.4) by replacing 1 and c therein with σ and τ—just as we did in (1.2). Under reasonable hypotheses on $\mathcal{V}$, such $\mathcal{V}$-bifunctors $\mathcal{A}, \mathcal{B} \to \mathcal{C}$ can be represented by $\mathcal{V}$-functors $\mathcal{A} \odot \mathcal{B} \to \mathcal{C}$—and this tensor product $\mathcal{A} \odot \mathcal{B}$ of $\mathcal{V}$-categories is now the many-object version of the *commuting tensor product* discussed above. Under further reasonable hypotheses on $\mathcal{V}$, these commuting tensor products can be made into part of a monoidal biclosed structure on the 2-category of small $\mathcal{V}$-categories, generalising the one existing in the braided monoidal case.

The many-object perspective clarifies not only the basic commutativity notion and the commuting tensor product, but also various further aspects of the theory. For example, as we will see in Section 4.2, we may exploit the internal homs $[-,-]$ of the commuting tensor product to construct *centralizers* of monoid maps and *centres* of monoids. Similarly, we will see in Section 5.3 that in the case $\mathcal{V} = [\mathbb{F}, \mathbf{Set}]$, we may realise the category of models of a theory $\mathcal{T}$ in a category $\mathcal{E}$ with finite powers as an internal hom $[\mathcal{T}, \mathcal{E}]$ in $\mathcal{V}$-$\mathbf{Cat}$; the correspondence between $\mathcal{S} \odot \mathcal{T}$-models in $\mathcal{E}$, $\mathcal{S}$-models in $\mathcal{T}$-models in $\mathcal{E}$, and $\mathcal{T}$-models in $\mathcal{S}$-models in $\mathcal{E}$, is then a direct consequence of the isomorphisms $[\mathcal{S} \odot \mathcal{T}, \mathcal{E}] \cong [\mathcal{S}, [\mathcal{T}, \mathcal{E}]] \cong [\mathcal{T}, [\mathcal{S}, \mathcal{E}]]$ associated to the symmetric monoidal closed structure on $\mathcal{V}$-$\mathbf{Cat}$.

It may also be useful to note what we do *not* do in this paper. First, we will say nothing about notions of commutativity in the context of *semi-abelian* categories [7]. This is because, as far as we have been able to tell, examples from this sphere simply do not fit into our framework. The intersection of our theory and the semi-abelian theory is essentially the content of [27], which shows how to define notions such as commuting tensor product and commutative object in a category *given*, as in (1.1), a suitable commutation relation on cospans. We exploit some of these results in Section 4 below, but note that this only relates to the one-object case of our theory; for the many-object case we must argue from scratch.

A second point we do not touch on, purely for reasons of space, is the generalisation of our theory from enrichment over monoidal categories to enrichment over *bicategories* [50]. While it may sound esoteric, such a generalisation would allow us, for example, to exploit the work of [14] in order to describe not only the Boardman–Vogt tensor product of symmetric operads but also that of *symmetric multicategories* [35].

The final point we do not deal with, again for reasons of space, is the generalisation of our theory from one-dimensional to two-dimensional enrichment. The basic example is the category of small 2-categories, which as well as its commuting tensor product (= cartesian product) also admits a "pseudo-commuting" tensor product known as the *Gray tensor product* [20], together with lax and oplax variants thereof [21]. Likewise, when we generalise from algebraic theories to *two-dimensional* algebraic theories [5], we have not just a commuting tensor product but also pseudo, lax, and oplax variants. We hope to deal with both this generalisation and the preceding one in future work.

The remainder of this paper is laid out as follows. We begin in Section 2 by gathering together the necessary background on duoidal categories. Section 3 then introduces the key notions of our theory of commutativity: the notions of sesquifunctor and bifunctor for categories enriched over a normal duoidal $\mathcal{V}$, and a description of the monoidal closed structure this induces on $\mathcal{V}$-**Cat** for a well-behaved $\mathcal{V}$. Section 4 then goes on to indicate how these results specialise to the important case of one-object $\mathcal{V}$-categories. This concludes the abstract theory; the remainder of the paper is devoted to examples.

Section 5 considers the case of *finitary algebraic theories*, showing that our framework suffices to re-find the classical notions of commutativity described above; Section 6 then considers *symmetric operads*, in particular showing how the Boardman–Vogt tensor product mentioned above falls out of our theory. Section 7 breaks off from our main development to describe a process by which arbitrary duoidal categories can be *normalized* into normal duoidal ones; we then make use of this construction in giving our final three examples. In Section 8, we study *strong monads* on a monoidal category [34]; in Section 9, we generalise this to the *Freyd-categories* of [46,38]; while finally in Section 10, we incorporate the example of sesquicategories into our general framework.

2. Background on duoidal categories

2.1. Duoidal categories

As explained in the introduction, the ambient setting for our theory of commutativity is that of a *duoidal category*. These were introduced in a slightly degenerate form in [2] under the name 2-*fold monoidal categories*; the fully general definition may be found, for example, in [1] under the name 2-*monoidal category*. The term "duoidal" is due to [4].

Definition 1. A *duoidal category* is a monoidale (= pseudomonoid) in the monoidal 2-category of monoidal categories, oplax monoidal functors and oplax monoidal natural transformations.

A duoidal structure on a category $\mathcal{V}$ thus comprises two monoidal structures $(\circ, I)$ and $(*, J)$ (whose unit and associativity constraints we leave unnamed) such that the functors $*\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and $J\colon \mathbf{1} \to \mathcal{V}$ and the associated coherence transformations are oplax monoidal with respect to $\circ$. The oplax monoidal constraint data of $*$ and J comprise a natural family of *interchange maps*

$$\xi\colon (X \circ Y) * (Z \circ W) \to (X * Z) \circ (Y * W)$$

together with arrows $\mu\colon I * I \to I$, $\upsilon\colon J \to I$ and $\gamma\colon J \to J \circ J$ satisfying axioms which, among other things, make (I, υ, μ) into a $*$-monoid and (J, υ, γ) into a $\circ$-comonoid. These data and axioms are equally those required to make $\circ$ and I and the associated constraints *lax* monoidal with respect to $(*, J)$, so that a duoidal category is alternatively a monoidale in the 2-category of monoidal categories and lax monoidal functors.

Example 2. Any braided monoidal category $(\mathcal{V}, \otimes, I)$ can be made into a duoidal category by taking $\circ = * = \otimes$ and $I = J$, with υ taken to be the identity, μ and γ given by unit constraints, and the interchange maps ξ constructed from associativities and the braiding. Conversely, if the duoidal $\mathcal{V}$ has all its constraint maps invertible, then the monoidal structures $\circ$ and $*$ are isomorphic and braided; see [29, Remark 5.1] or [1, Proposition 6.11].

We will give more examples relevant to our theory from Section 5 onwards. In these examples, the two monoidal structures $\circ$ and $*$ are generally thought of as *composition* and *convolution* respectively; it is almost always the case that the convolution tensor has associated internal homs, and will often be the case that it is braided in a manner compatible with $\circ$. The following definition formalises these concepts.

Definition 3. Let $\mathcal{V}$ be a duoidal category. We say that:

(i) $\mathcal{V}$ is $*$-*biclosed* if each functor $(-) * X$ and $X * (-) \colon \mathcal{V} \to \mathcal{V}$ has a right adjoint, written as $[X, -]_\ell$ and $[X, -]_r$ and called *left* and *right* hom, respectively.

(ii) $\mathcal{V}$ is $*$-*braided* if the $*$-monoidal structure is given a braiding c, with respect to which the $\circ$-monoidal structure maps are braided monoidal; we may say $*$-*symmetric* if the given braiding is in fact a symmetry.

Spelling out (ii), the compatibility of the $*$-braiding and the $\circ$-monoidal structure amounts to the requirement that (I, υ, μ) be a commutative $*$-monoid in $\mathcal{V}$, and that each diagram of the following form should commute:

$$\begin{tikzcd}
(X \circ Y) * (Z \circ W) \arrow[r, "\xi"] \arrow[d, "c"'] & (X * Z) \circ (Y * W) \arrow[d, "c \circ c"] \\
(Z \circ W) * (X \circ Y) \arrow[r, "\xi"] & (Z * X) \circ (W * Y) \, .
\end{tikzcd} \tag{2.1}$$

For instance, if $(\mathcal{V}, \otimes)$ is a braided monoidal category, then the associated duoidal category $(\mathcal{V}, \otimes, \otimes)$ is $*$-braided if and only if the braiding on $\otimes$ is a symmetry; this is proven in [1, Proposition 6.13].

2.2. Normality

It turns out that not every duoidal category will be appropriate for our theory; we must assume essentially the same degeneracy with respect to units as appears in [2]. Recall that an opmonoidal functor F is called *normal* if the unit comparison map $FI \to I$ is invertible.

Definition 4. A duoidal category $\mathcal{V}$ is called *normal* if the opmonoidal functors $* \colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and $J \colon \mathbf{1} \to \mathcal{V}$ are normal.

In elementary terms, the normality of the duoidal $\mathcal{V}$ amounts to the requirements that the morphisms $\upsilon \colon J \to I$ and $\mu \colon I * I \to I$ be invertible; in fact, invertibility of υ easily implies that of μ and also of γ. In a normal duoidal category, there are maps $\sigma \colon X * Y \to X \circ Y$ and $\tau \colon X * Y \to Y \circ X$ given by

$$\begin{aligned}
\sigma &= X * Y \xrightarrow{\cong} (X \circ I) * (I \circ Y) \xrightarrow{\xi} (X * I) \circ (I * Y) \xrightarrow{\cong} X \circ Y \\
\tau &= X * Y \xrightarrow{\cong} (I \circ X) * (Y \circ I) \xrightarrow{\xi} (I * Y) \circ (X * I) \xrightarrow{\cong} Y \circ X \, ,
\end{aligned} \tag{2.2}$$

where the unnamed isomorphisms are built from unit constraints for $\circ$ and $*$ and the inverse of $\upsilon \colon J \to I$. These maps play a central role in the theory that follows. For the canonical normal duoidal structure on a braided monoidal category, σ and τ are the identity map and the braiding respectively.

More generally, any normal duoidal $\mathcal{V}$ possesses the following families of maps, which are the *linear distributivities* of [8] (there called "weak distributivities"). While the statements of our main definitions and results will not make use of these, the proofs will.

$$\begin{aligned}
\delta^\ell_\ell &\colon X * (Y \circ Z) \xrightarrow{\cong} (X \circ I) * (Y \circ Z) \xrightarrow{\xi} (X * Y) \circ (I * Z) \xrightarrow{\cong} (X * Y) \circ Z \\
\delta^\ell_r &\colon X * (Y \circ Z) \xrightarrow{\cong} (I \circ X) * (Y \circ Z) \xrightarrow{\xi} (I * Y) \circ (X * Z) \xrightarrow{\cong} Y \circ (X * Z) \\
\delta^r_\ell &\colon (X \circ Y) * Z \xrightarrow{\cong} (X \circ Y) * (Z \circ I) \xrightarrow{\xi} (X * Z) \circ (Y * I) \xrightarrow{\cong} (X * Z) \circ Y \\
\delta^r_r &\colon (X \circ Y) * Z \xrightarrow{\cong} (X \circ Y) * (I \circ Z) \xrightarrow{\xi} (X * I) \circ (Y * Z) \xrightarrow{\cong} X \circ (Y * Z) \, .
\end{aligned}$$

Note that when $\mathcal{V}$ is $*$-braided, the maps δ^ℓ_ℓ and δ^ℓ_r above, and similarly δ^r_ℓ and δ^r_r, may be derived from each other using the braiding. A particular case of this is that the maps σ and τ are related through the braiding c by a commuting diagram

$$\begin{tikzcd} & X * Y \arrow[dl, "\sigma"'] \arrow[d, "c"] \arrow[dr, "\tau"] & \\ X \circ Y & Y * X \arrow[l, "\tau"'] \arrow[r, "\sigma"] & Y \circ X . \end{tikzcd} \tag{2.3}$$

2.3. Bimonoids and duoids

Some kinds of structure definable using a braiding on a monoidal category can be defined more generally using the interchange maps of a duoidal structure; two examples relevant for us are the notions of *bialgebra* and of *commutative monoid*. The key to the generalisation is the fact that, since the $\circ$-monoidal structure of a duoidal $\mathcal{V}$ is lax $*$-monoidal, it lifts to a $\circ$-monoidal structure on the category of $*$-monoids in $\mathcal{V}$.

Definition 5. (See [1, Definitions 6.25 and 6.28].) Let $\mathcal{V}$ be a duoidal category.

(i) A *bimonoid* in $\mathcal{V}$ is a $\circ$-comonoid in the category of $*$-monoids in $\mathcal{V}$. The category of bimonoids $\mathbf{Bimon}(\mathcal{V})$ is the category $\mathbf{Comon}_\circ(\mathbf{Mon}_*(\mathcal{V}))$.
(ii) A *duoid* in $\mathcal{V}$ is a $\circ$-monoid in the category of $*$-monoids in $\mathcal{V}$. The category of duoids $\mathbf{Duoid}(\mathcal{V})$ is the category $\mathbf{Mon}_\circ(\mathbf{Mon}_*(\mathcal{V}))$.

If $\mathcal{V}$ is $*$-braided, then we declare a bimonoid or duoid to be $*$-*commutative* if its underlying $*$-monoid is commutative.

Spelling these definitions out in more detail, a bimonoid is thus an object A equipped with $*$-monoid and $\circ$-comonoid structures $e\colon J \to A \leftarrow A * A\colon m$ and $u\colon I \leftarrow A \to A \circ A\colon d$ which are such that $e\colon J \to A$ is a map of $\circ$-comonoids, $u\colon A \to I$ is a map of $*$-monoids, and the following *bialgebra* diagram commutes:

$$\begin{tikzcd} A * A \arrow[r, "m"] \arrow[d, "d*d"'] & A \arrow[r, "d"] & A \circ A \\ (A \circ A) * (A \circ A) \arrow[rr, "\xi"] & & (A * A) \circ (A * A) . \arrow[u, "m \circ m"'] \end{tikzcd}$$

On the other hand, a duoid is an object A equipped with $\circ$-monoid and $*$-monoid structures $e\colon I \to A \leftarrow A \circ A\colon m$ and $\iota\colon J \to A \leftarrow A * A\colon \nu$, which are such that $e\colon I \to A$ is a map of $*$-monoids, $\iota\colon J \to A$ is a map of $\circ$-monoids, and the following *duoid* diagram commutes:

$$\begin{tikzcd} (A \circ A) * (A \circ A) \arrow[r, "\xi"] \arrow[d, "m*m"'] & (A * A) \circ (A * A) \arrow[r, "\nu\circ\nu"] & A \circ A \arrow[d, "m"] \\ A * A \arrow[rr, "\nu"'] & & A . \end{tikzcd} \tag{2.4}$$

When the duoidal structure on $\mathcal{V}$ is induced by a braided monoidal structure, bimonoids are bialgebras in $\mathcal{V}$ in the usual sense, while duoids reduce by the Eckmann–Hilton argument to commutative monoids.

3. Commutativity: the general theory

In this section, we introduce our abstract framework for commutativity. As explained in the introduction, the central notion is that of a *bifunctor* between categories enriched over a normal duoidal category $\mathcal{V}$; we introduce this, and describe circumstances under which bifunctors between small $\mathcal{V}$-categories are represented by a monoidal closed structure on the 2-category of $\mathcal{V}$-categories.

3.1. Sesquifunctors

To start with we assume only that $(\mathcal{V}, \circ, I)$ is a monoidal category; shortly, we will add a second monoidal structure making $\mathcal{V}$ normal duoidal, but even then our convention will be that a $\mathcal{V}$*-category* is one enriched in $(\mathcal{V}, \circ, I)$. We write $\mathcal{V}$-**Cat** and $\mathcal{V}$-**CAT** for the 2-categories of small and large $\mathcal{V}$-categories, together with the $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations between them; see [31, §1.2] for the full definitions.

The notion of $\mathcal{V}$-bifunctor we introduce is expressed in terms of families of one-variable $\mathcal{V}$-functors satisfying a commutativity or bifunctoriality condition. While the commutativity condition requires a normal duoidal structure, the rest of the definition does not; we begin, therefore, with this.

Definition 6. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be $\mathcal{V}$-categories.

- A *sesquifunctor* $T\colon \mathcal{A}, \mathcal{B} \to \mathcal{C}$ comprises families $(T(a,–)\colon \mathcal{B} \to \mathcal{C})_{a\in\mathcal{A}}$ and $(T(–,b)\colon \mathcal{A} \to \mathcal{C})_{b\in\mathcal{B}}$ of $\mathcal{V}$-functors such that $T(a,–)(b) = T(–,b)(a) = Tab$, say.
- A *sesquitransformation* $\alpha\colon S \Rightarrow T\colon \mathcal{A}, \mathcal{B} \to \mathcal{C}$ comprises families of $\mathcal{V}$-natural transformations $\alpha_{a–}\colon S(a,–) \Rightarrow T(a,–)$ and $\alpha_{–b}\colon S(–,b) \Rightarrow T(–,b)$ such that $(\alpha_{a–})_b = (\alpha_{–b})_a = \alpha_{ab}\colon I \to \mathcal{C}(Sab, Tab)$.

Given a sesquifunctor $T\colon \mathcal{A}, \mathcal{B} \to \mathcal{C}$ and $\mathcal{V}$-functors $F\colon \mathcal{A}' \to \mathcal{A}$ and $G\colon \mathcal{B}' \to \mathcal{B}$ and $H\colon \mathcal{C} \to \mathcal{C}'$, there is a composite sesquifunctor $HT(F,G)\colon \mathcal{A}', \mathcal{B}' \to \mathcal{C}'$ with components $HT(Fa', G–)\colon \mathcal{B}' \to \mathcal{C}'$ and $HT(F–, Gb')\colon \mathcal{A}' \to \mathcal{C}'$; with the obvious extension of this composition to transformations, we obtain a 2-functor

$$\mathbf{SESQ}(–,–;–)\colon \mathcal{V}\text{-}\mathbf{CAT}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{CAT}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{CAT} \to \mathbf{CAT}\ .$$

There are also higher arity analogues of sesquifunctors and sesquitransformations; the general pattern may be deduced if we describe the next simplest case:

Definition 7. A *ternary sesquifunctor* $T\colon \mathcal{A}, \mathcal{B}, \mathcal{C} \to \mathcal{D}$ between $\mathcal{V}$-categories comprises families of sesquifunctors $T(a,–,–)\colon \mathcal{B}, \mathcal{C} \to \mathcal{D}$ and $T(–,b,–)\colon \mathcal{A}, \mathcal{C} \to \mathcal{D}$ and $T(–,–,c)\colon \mathcal{A}, \mathcal{B} \to \mathcal{D}$ which are compatible on objects, in the sense that the $\mathcal{V}$-functors $T(a,b,–)$ and $T(a,–,c)$ and $T(–,b,c)$ are unambiguously defined.

These higher arity sesquifunctors and transformations also compose; for example, given sesquifunctors $T\colon \mathcal{C}, \mathcal{D} \to \mathcal{E}$ and $S\colon \mathcal{A}, \mathcal{B} \to \mathcal{C}$, the ternary sesquifunctor $T(S,1)\colon \mathcal{A}, \mathcal{B}, \mathcal{D} \to \mathcal{E}$ has components $T(S(a,–),1)$ and $T(S(–,b),1)$ and $T(S–,d)$. Taken together, these compositions make the totality of $\mathcal{V}$-categories, sesquifunctors and sesquitransformations into a *symmetric 2-multicategory* **SESQ**. This structure on sesquifunctors is studied in some detail, and in a somewhat more general context, in [52]; what is relevant here is that the 2-multicategory **Sesq** of sesquifunctors between small $\mathcal{V}$-categories is often represented by a monoidal structure on $\mathcal{V}$-**Cat**. The following result summarises the key points; in (c), we write $\mathcal{I}$ for the $\mathcal{V}$-category with a single object $*$ and $\mathcal{I}(*,*) = I$.

Proposition 8. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be $\mathcal{V}$-categories, with $\mathcal{A}$ and $\mathcal{B}$ small.*

*(a) If $\mathcal{V}$-**Cat** has conical colimits, preserved by the inclusion into $\mathcal{V}$-**CAT**, then the 2-functor $\mathbf{SESQ}(\mathcal{A},\mathcal{B};\text{–})\colon \mathcal{V}\text{-}\mathbf{CAT} \to \mathbf{CAT}$ admits a small representation $\mathcal{A}\,\square\,\mathcal{B}$.*
(b) If $\mathcal{V}$ is complete, then the 2-functors $\mathbf{SESQ}(\mathcal{B},\text{–};\mathcal{C})\colon \mathcal{V}\text{-}\mathbf{CAT}^{\mathrm{op}} \to \mathbf{CAT}$ and $\mathbf{SESQ}(\text{–},\mathcal{B};\mathcal{C})\colon \mathcal{V}\text{-}\mathbf{CAT}^{\mathrm{op}} \to \mathbf{CAT}$ admit a common representing object $[\![\mathcal{B},\mathcal{C}]\!]$, which is small whenever $\mathcal{C}$ is so.
(c) If (a) and (b) hold, then $(\mathcal{V}\text{-}\mathbf{Cat}, \square, \mathcal{I})$ is a symmetric monoidal closed 2-category.

Proof. For (a), the universal sesquifunctor $V\colon \mathcal{A},\mathcal{B} \to \mathcal{A}\,\square\,\mathcal{B}$ is the pushout:

$$\begin{tikzcd}[column sep=small]
& \sum_{a,b\in\mathcal{A}\times\mathcal{B}} \mathcal{I} \arrow[dl, "\ell"'] \arrow[dr, "r"] & \\
\sum_{a\in\mathcal{A}} \mathcal{B} \arrow[dr, "\langle V(a,-)\rangle_{a\in\mathcal{A}}"'] & & \sum_{b\in\mathcal{B}} \mathcal{A} \arrow[dl, "\langle V(-,b)\rangle_{b\in\mathcal{B}}"] \\
& \mathcal{A}\,\square\,\mathcal{B} &
\end{tikzcd} \tag{3.1}$$

in $\mathcal{V}$-**Cat**. Here, the (a, b)-components of ℓ and r pick out the object b in the ath copy of $\mathcal{B}$, respectively the object a in the bth copy of $\mathcal{A}$.

For (b), we take $[\![\mathcal{B},\mathcal{C}]\!]$ to have $\mathcal{V}$-functors $F\colon \mathcal{B} \to \mathcal{C}$ as objects, and hom-objects given by $[\![\mathcal{B},\mathcal{C}]\!]\,(F,G) = \prod_{a\in\mathcal{B}} \mathcal{C}(Fa, Ga)$ with componentwise composition. Clearly $[\![\mathcal{B},\mathcal{C}]\!]$ is small if $\mathcal{C}$ is so. The universal sesquifunctor $\varepsilon\colon\ [\![\mathcal{B},\mathcal{C}]\!]\,,\mathcal{B} \to \mathcal{C}$ has $\varepsilon(F,-) = F\colon \mathcal{B} \to \mathcal{C}$ and $\varepsilon(-,b) = \mathrm{ev}_b\colon\ [\![\mathcal{B},\mathcal{C}]\!] \to \mathcal{C}$; the universal $\mathcal{B}, [\![\mathcal{B},\mathcal{C}]\!] \to \mathcal{C}$ is dual.

For (c), we obtain the unit constraints λ and ρ of the desired monoidal structure directly on taking $\mathcal{A} = \mathcal{I}$ or $\mathcal{B} = \mathcal{I}$ in (3.1), while the symmetry constraints are immediate from (3.1)'s symmetry in $\mathcal{A}$ and $\mathcal{B}$. As for associativity, let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be small $\mathcal{V}$-categories; it is easy to see that composition with the universal map $\varepsilon\colon\ [\![\mathcal{C},\mathcal{D}]\!]\,,\mathcal{C} \to \mathcal{D}$ induces bijections, 2-natural in $\mathcal{D}$, of the form:

$$\mathbf{SESQ}(\mathcal{A},\mathcal{B};[\![\mathcal{C},\mathcal{D}]\!]) \xrightarrow{\cong} \mathbf{SESQ}(\mathcal{A},\mathcal{B},\mathcal{C};\mathcal{D}) \ , \tag{3.2}$$

where on the right we have the category of ternary sesquifunctors and sesquitransformations. Since $\mathbf{SESQ}(\mathcal{A},\mathcal{B};[\![\mathcal{C},\mathcal{D}]\!]) \cong \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A}\,\square\,\mathcal{B},[\![\mathcal{C},\mathcal{D}]\!]) \cong \mathcal{V}\text{-}\mathbf{CAT}((\mathcal{A}\,\square\,\mathcal{B})\,\square\,\mathcal{C},\mathcal{D})$, we see that $(\mathcal{A}\,\square\,\mathcal{B})\,\square\,\mathcal{C}$ classifies ternary sesquifunctors; by symmetry, so too does $\mathcal{A}\,\square(\mathcal{B}\,\square\,\mathcal{C})$, whence there is a unique isomorphism $\alpha\colon (\mathcal{A}\,\square\,\mathcal{B})\,\square\,\mathcal{C} \cong \mathcal{A}\,\square(\mathcal{B}\,\square\,\mathcal{C})$ commuting with the universal maps. The triangle axiom is now easy to verify, while the pentagon axiom follows by arguing that each vertex of the pentagon represents quaternary sesquifunctors, and each edge commutes with the universal maps. □

Remark 9. The observation in (c) above that the maps (3.2) are bijective is part of the fact that **SESQ** is a *closed* 2-*multicategory*; as explained in [41], this, together with the *weak representability* of multimaps exhibited in (a), allows the associativity of the monoidal structure to be derived in a purely formal manner. We will use a similar argument in the proof of Proposition 18 below.

When $\mathcal{V} = \mathbf{Set}$, the symmetric monoidal closed structure on **Cat** this proposition yields is the "funny tensor product" [48], whose internal hom $[\![\mathcal{B},\mathcal{C}]\!]$ is the category of functors and not-necessarily-natural transformations $\mathcal{B} \to \mathcal{C}$. This and the cartesian structure are in fact the only symmetric monoidal closed structures on **Cat**; see [15].

3.2. Bifunctors

We are now ready to state the central definition of our theory: that of a commuting sesquifunctor, or *bifunctor*. In order to do so, we henceforth assume that $\mathcal{V}$ is a normal duoidal category $(\mathcal{V}, *, J, \circ, I)$; we reiterate that, in this context, "$\mathcal{V}$-category" will mean "category enriched in $(\mathcal{V}, \circ, I)$".

Definition 10. Let $\mathcal{V}$ be a normal duoidal category, and let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be $\mathcal{V}$-categories. A sesquifunctor $T\colon \mathcal{A}, \mathcal{B} \to \mathcal{C}$ is said to *commute*, or to be a *bifunctor*, if for each $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$, the diagram

$$\begin{tikzcd}[column sep=small]
 & \mathcal{A}(a,a') \circ \mathcal{B}(b,b') \arrow[r, "T(-,b')\circ T(a,-)"] & \mathcal{C}(Tab',Ta'b') \circ \mathcal{C}(Tab,Tab') \arrow[dr, "m"] & \\
\mathcal{A}(a,a') * \mathcal{B}(b,b') \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & \mathcal{C}(Tab,Ta'b') \\
 & \mathcal{B}(b,b') \circ \mathcal{A}(a,a') \arrow[r, "T(a',-)\circ T(-,b)"'] & \mathcal{C}(Ta'b,Ta'b') \circ \mathcal{C}(Tab,Ta'b) \arrow[ur, "m"'] &
\end{tikzcd} \tag{3.3}$$

commutes in $\mathcal{V}$; here σ and τ are the maps defined in (2.2).

When the normal duoidal structure on $\mathcal{V}$ comes from a braided monoidal structure, the diagram (3.3) reduces to the (1.4) of the introduction, and so our bifunctors reduce to those of [13, Chapter III, §4]; in particular, when $\mathcal{V} = \mathbf{Set}$, the bifunctoriality of $T\colon \mathcal{A}, \mathcal{B} \to \mathcal{C}$ is the familiar requirement that each (1.3) should commute in $\mathcal{C}$.

Proposition 11. *Let $\mathcal{V}$ be a normal duoidal category, and suppose given $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-sesquifunctors as in:*

$$F\colon \mathcal{A}' \to \mathcal{A} \qquad G\colon \mathcal{B}' \to \mathcal{B} \qquad T\colon \mathcal{A}, \mathcal{B} \to \mathcal{C} \qquad H\colon \mathcal{C} \to \mathcal{C}' \ .$$

(i) If T commutes, then so does $HT(F,G)$.
(ii) If HT commutes and H is faithful, then T commutes.
(iii) If $\mathcal{V}$ is $*$-braided, then T commutes if and only if $T^c\colon \mathcal{B}, \mathcal{A} \to \mathcal{C}$ does so.

In (ii), we call a functor *faithful* if the morphisms in $\mathcal{V}$ expressing its action on hom-objects are monomorphic. In (iii), we write T^c for the sesquifunctor with components $T^c(-,a) = T(a,-)$ and $T^c(b,-) = T(-,b)$.

Proof. Parts (i) and (ii) are immediate on observing that pre- and postcomposing (3.3) for T by $F_{aa'} * G_{bb'}$ and $H_{Tab,Ta'b'}$ yields (3.3) for $HT(F,G)$. Part (iii) is verified by precomposing (3.3) with $c\colon \mathcal{B}(b,b') * \mathcal{A}(a,a') \to \mathcal{A}(a,a') * \mathcal{B}(b,b')$ and using (2.3). □

It follows from this proposition that we have a 2-functor

$$\mathbf{BIFUN}(-,-;-)\colon \mathcal{V}\text{-}\mathbf{CAT}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{CAT}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{CAT} \longrightarrow \mathbf{CAT}$$

sending $\mathcal{A}, \mathcal{B}, \mathcal{C}$ to the category of bifunctors and sesquitransformations $\mathcal{A}, \mathcal{B} \to \mathcal{C}$. Clearly, this is a locally full sub-2-functor of $\mathbf{SESQ}(-,-;-)$; and more generally, we can show that the symmetric 2-multicategory $\mathbf{SESQ}$ of $\mathcal{V}$-categories and n-ary sesquifunctors has a locally full sub-2-multicategory $\mathbf{BIFUN}$ of $\mathcal{V}$-categories and n-ary bifunctors; here, for example, a *trifunctor* $T\colon \mathcal{A}, \mathcal{B}, \mathcal{C} \to \mathcal{D}$ is a ternary sesquifunctor such that each of $T(a,-,-)$ and $T(-,b,-)$ and $T(-,-,c)$ are bifunctors. When $\mathcal{V}$ is an $*$-braided duoidal category, the sub-2-multicategory **BIFUN** of **SESQ** is symmetric by Proposition 11(iii); Example 26 below shows that it is not so in general.

3.3. Commuting tensor product of $\mathcal{V}$-categories

Just as with **Sesq**, the 2-multicategory **Bifun** of bifunctors between small $\mathcal{V}$-categories is often represented by a monoidal structure on $\mathcal{V}$-**Cat**, whose tensor product we call the *commuting tensor product*. Eilenberg and Kelly show in [13, Chapter III, §4] how to construct this monoidal structure when $\mathcal{V}$ is a symmetric monoidal category: the commuting tensor product $\mathcal{A} \odot \mathcal{B}$ of $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ has object-set $\mathrm{ob}\,\mathcal{A} \times \mathrm{ob}\,\mathcal{B}$, hom-objects $(\mathcal{A} \odot \mathcal{B})((a,b),(a',b')) = \mathcal{A}(a,a') \otimes \mathcal{B}(b,b')$, and composition maps defined using the composition in $\mathcal{A}$ and $\mathcal{B}$ and the symmetry isomorphisms.

This construction still works in the braided monoidal case [29, Remark 5.2], but when the normal duoidal structure on $\mathcal{V}$ does not come from a braided monoidal one, the construction of the commuting tensor product is completely different, and will require stronger assumptions on $\mathcal{V}$. Recall that a $\mathcal{V}$*-graph* A comprises a set $\mathrm{ob}\,A$ together with a family $A(a,a')_{a,a' \in \mathrm{ob}\,A}$ of objects of $\mathcal{V}$, while a map $f\colon A \to B$ of $\mathcal{V}$-graphs is given by a function $f\colon \mathrm{ob}\,\mathcal{A} \to \mathrm{ob}\,\mathcal{B}$ together with a family of maps $f_{aa'}\colon A(a,a') \to B(fa,fa')$ in $\mathcal{V}$. We write $\mathcal{V}$-**Gph** for the category of *small* $\mathcal{V}$-graphs—those with small object set—and $U\colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Gph}$ for the obvious forgetful functor. We will say that *free* $\mathcal{V}$*-categories exist* if this functor U has a left adjoint F.

Proposition 12. *Let $\mathcal{V}$ be normal duoidal and let $\mathcal{A},\mathcal{B}$ be small $\mathcal{V}$-categories. If $\mathcal{V}$-**Cat** has conical colimits, preserved by the inclusion into $\mathcal{V}$-**CAT**, and free $\mathcal{V}$-categories exist, then* $\mathbf{BIFUN}(\mathcal{A},\mathcal{B};\text{–})\colon \mathcal{V}\text{-}\mathbf{CAT} \to \mathbf{CAT}$ *has a small representation $\mathcal{A} \odot \mathcal{B}$.*

Proof. Consider the small $\mathcal{V}$-graph $U\mathcal{A} * U\mathcal{B}$ with object-set $\mathrm{ob}\,\mathcal{A} \times \mathrm{ob}\,\mathcal{B}$ and with homs $(U\mathcal{A} * U\mathcal{B})((a,b),(a',b')) = \mathcal{A}(a,a') * \mathcal{B}(b,b')$. For any sesquifunctor $T\colon \mathcal{A},\mathcal{B} \to \mathcal{C}$, we have a parallel pair of $\mathcal{V}$-graph morphisms $U\mathcal{A} * U\mathcal{B} \rightrightarrows U\mathcal{C}$ which on objects both send (a,b) to Tab, and on homs have their respective actions given by the upper and lower paths around (3.3). Applying this to the universal sesquifunctor $V\colon \mathcal{A},\mathcal{B} \to \mathcal{A} \,\square\, \mathcal{B}$ whose existence is guaranteed by Proposition 8(a), we obtain a pair of $\mathcal{V}$-graph morphisms $U\mathcal{A} * U\mathcal{B} \rightrightarrows U(\mathcal{A} \,\square\, \mathcal{B})$, corresponding under adjunction to a parallel pair $H,K\colon F(U\mathcal{A} * U\mathcal{B}) \rightrightarrows \mathcal{A} \,\square\, \mathcal{B}$ of $\mathcal{V}$-functors. Let

$$\begin{tikzcd} F(U\mathcal{A} * U\mathcal{B}) \arrow[r, shift left, "H"] \arrow[r, shift right, "K"'] & \mathcal{A} \,\square\, \mathcal{B} \arrow[r, two heads, "Q"] & \mathcal{A} \odot \mathcal{B} \end{tikzcd}$$

be their coequalizer in $\mathcal{V}$-**Cat**; we claim that $QV\colon \mathcal{A},\mathcal{B} \to \mathcal{A} \odot \mathcal{B}$ is the required universal bifunctor. This is to say that, for each $\mathcal{C} \in \mathcal{V}\text{-}\mathbf{CAT}$, the functor

$$\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A} \odot \mathcal{B}, \mathcal{C}) \xrightarrow{(-)\circ Q} \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A} \,\square\, \mathcal{B}, \mathcal{C}) \xrightarrow{(-)\circ V} \mathbf{SESQ}(\mathcal{A},\mathcal{B};\mathcal{C}) \tag{3.4}$$

is injective on objects and fully faithful, and has as its image precisely the bifunctors $\mathcal{A},\mathcal{B} \to \mathcal{C}$. Now Q is the coequalizer of two functors H,K which agree on objects; thus if $F,G\colon \mathcal{A} \odot \mathcal{B} \to \mathcal{C}$ and $\alpha\colon FQ \Rightarrow GQ$, then necessarily $\alpha H = \alpha K$, and so there is a unique $\bar{\alpha}\colon F \Rightarrow G$ with $\alpha = \bar{\alpha} Q$. So the first arrow in (3.4) is fully faithful; the second is too, being an isomorphism, and so (3.4) is itself fully faithful. On the other hand, a $\mathcal{V}$-functor $F\colon \mathcal{A} \,\square\, \mathcal{B} \to \mathcal{C}$ factors through Q just when $FH = FK$; transposing, this is equally to ask that the two composite morphisms $U\mathcal{A} * U\mathcal{B} \rightrightarrows U(\mathcal{A} \,\square\, \mathcal{B}) \to U\mathcal{C}$ are equal; but since F is a functor, these two composites are precisely the two sides of (3.3) for $T = FV$. It follows that F is in the image of the full embedding $(-) \circ Q$ just when FV is a bifunctor, as required. $\square$

3.4. Functor $\mathcal{V}$-categories

We now turn our attention to the internal homs associated to the commuting tensor product of $\mathcal{V}$-categories. It is easy to see that the unit $\mathcal{V}$-category $\mathcal{I}$ is a unit for this tensor, from which it follows that the underlying ordinary category of either internal hom $[\mathcal{B},\mathcal{C}]_\ell$ or $[\mathcal{B},\mathcal{C}]_r$ must be the ordinary category of $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations $\mathcal{B} \to \mathcal{C}$; which justifies our calling these internal homs *functor $\mathcal{V}$-categories*. The key to constructing these is a notion of enriched *end*, which generalises from the symmetric monoidal case the definition of [31, §2.1]. Before giving this, let us recall some necessary background on profunctors.

Definition 13. Let $(\mathcal{V}, \circ, I)$ be a monoidal category.

(i) If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-categories, then a *$\mathcal{V}$-profunctor* $M\colon \mathcal{A} \nrightarrow \mathcal{B}$ comprises a family of objects $M(b,a) \in \mathcal{V}$ together with actions $m\colon \mathcal{A}(a,a') \circ M(b,a) \to M(b,a')$ and $m\colon M(b,a) \circ \mathcal{B}(b',b) \to M(b',a)$ satisfying the usual associativity and unitality laws.
(ii) Given profunctors $M\colon \mathcal{A} \nrightarrow \mathcal{B}$ and $N\colon \mathcal{B} \nrightarrow \mathcal{C}$ and $L\colon \mathcal{A} \nrightarrow \mathcal{C}$, a family of maps $f_{abc}\colon M(b,a) \circ N(c,b) \to L(c,a)$ is *bilinear* if it is equivariant with respect to the left $\mathcal{A}$-action and right $\mathcal{C}$-action and satisfies the $\mathcal{B}$-bilinearity axiom $f(m \circ 1) = f(1 \circ m)\colon M(b',a) \circ \mathcal{B}(b,b') \circ N(c,b) \rightrightarrows L(c,a)$.
(iii) Given $\mathcal{V}$-functors $F\colon \mathcal{A} \to \mathcal{C}$ and $G\colon \mathcal{B} \to \mathcal{C}$, we write $\mathcal{C}(F,G)\colon \mathcal{B} \nrightarrow \mathcal{A}$ for the profunctor whose (a,b)-component is $\mathcal{C}(Fa, Gb)$ and whose actions are obtained using composition in $\mathcal{C}$ and the actions of F and G on homs. If we have a further $\mathcal{V}$-functor $H\colon \mathcal{D} \to \mathcal{C}$, then it is easy to see that the family of composition maps $\mathcal{C}(Gb, Hd) \circ \mathcal{C}(Fa, Gb) \to \mathcal{C}(Fa, Hd)$ is $\mathcal{B}$-bilinear.

As our notation suggests, when $\mathcal{V}$ is a normal duoidal category, we interpret these notions with respect to the $\circ$-monoidal structure.

Definition 14. Let $\mathcal{V}$ be a normal duoidal category and $M\colon \mathcal{A} \nrightarrow \mathcal{A}$ a $\mathcal{V}$-profunctor. A family of maps $p_a\colon K \to M(a,a)$ in $\mathcal{V}$ is called *left extranatural* or *right extranatural* if each diagram to the left, respectively right, in:

$$
\begin{tikzcd}[column sep=small]
& K \circ \mathcal{A}(a,b) \arrow[r, "p_b \circ 1"] & M(b,b) \circ \mathcal{A}(a,b) \arrow[dr, "m"] & \\
K * \mathcal{A}(a,b) \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & M(a,b) \\
& \mathcal{A}(a,b) \circ K \arrow[r, "1 \circ p_a"'] & \mathcal{A}(a,b) \circ M(a,a) \arrow[ur, "m"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \mathcal{A}(a,b) \circ K \arrow[r, "1 \circ q_a"] & \mathcal{A}(a,b) \circ M(a,a) \arrow[dr, "m"] & \\
\mathcal{A}(a,b) * K \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & M(a,b) \\
& K \circ \mathcal{A}(a,b) \arrow[r, "q_b \circ 1"'] & M(b,b) \circ \mathcal{A}(a,b) \arrow[ur, "m"'] &
\end{tikzcd}
\tag{3.5}
$$

commutes in $\mathcal{V}$. A *left end* for the profunctor M is a universal left extranatural family $u_a\colon U \to M(a,a)$; this means that each extranatural family $p_a\colon K \to M(a,a)$ is of the form $u_a \circ \bar{p}$ for a unique $\bar{p}\colon K \to U$. A *right end* for M is defined correspondingly.

Proposition 15. *If $\mathcal{V}$ is complete and $*$-biclosed, and $\mathcal{A}$ is a small $\mathcal{V}$-category, then any profunctor $M\colon \mathcal{A} \nrightarrow \mathcal{A}$ has both a left and right end.*

Proof. The left end is given by an equalizer $s, t\colon \Pi_a M(a,a) \rightrightarrows \Pi_{a,b}[\mathcal{A}(a,b), M(a,b)]_\ell$, where the (a,b)-components of s and t are the transposes of the two sides of the left hexagon in (3.5), interpreted for the family $\pi_a \colon \Pi_a M(a,a) \to M(a,a)$. The right end is given by a similar equalizer, but with the right hom $[-,-]_r$ replacing the left $[-,-]_\ell$, and with the right hexagon replacing the left in (3.5). □

We will exploit the notions of left and right end to construct the hom-objects of the functor $\mathcal{V}$-categories $[\mathcal{B},\mathcal{C}]_\ell$ and $[\mathcal{B},\mathcal{C}]_r$ associated to the commuting tensor product. The following lemma is the crucial step in describing the composition law.

Lemma 16. *Let $\mathcal{V}$ be normal duoidal, let $\mathcal{A}$ be a $\mathcal{V}$-category, and let $M, N, P \colon \mathcal{A} \to\!\!\!\!\!\!\to \mathcal{A}$ be profunctors. If $p_a \colon K \to M(a,a)$ and $q_a \colon L \to N(a,a)$ are left extranatural families, and $r_{abc} \colon M(b,a) \circ N(c,b) \to P(c,a)$ is a bilinear one, then the composite family*

$$K \circ L \xrightarrow{p_a \circ q_a} M(a,a) \circ N(a,a) \xrightarrow{r_{aaa}} P(a,a) \tag{3.6}$$

is left extranatural. The corresponding result holds for right extranaturality.

Proof. We consider the following diagram, wherein for typographical convenience we temporarily write $\mathcal{A}(a,b)$ as $\mathcal{A}_{ab}$, and so on:

$$\begin{tikzcd}
(K \circ L) * \mathcal{A}_{ab} \arrow[r, "\delta^r_r"] \arrow[d, "\delta^r_\ell"'] & K \circ (L * \mathcal{A}_{ab}) \arrow[r, "1\circ\sigma"] \arrow[d, "1\circ\tau"] & K \circ L \circ \mathcal{A}_{ab} \arrow[r, "1\circ q_b \circ 1"] & K \circ N_{bb} \circ \mathcal{A}_{ab} \arrow[d, "1\circ m"] \\
(K * \mathcal{A}_{ab}) \circ L \arrow[r, "\sigma\circ 1"] \arrow[d, "\tau\circ 1"'] & K \circ \mathcal{A}_{ab} \circ L \arrow[r, "1\circ 1\circ q_a"] \arrow[d, "p_b\circ 1\circ 1"'] & K \circ \mathcal{A}_{ab} \circ N_{aa} \arrow[r, "1\circ m"] \arrow[d, "p_b\circ 1\circ 1"] & K \circ N_{ab} \arrow[d, "p_b\circ 1"] \\
\mathcal{A}_{ab} \circ K \circ L \arrow[d, "1\circ p_a\circ 1"'] & M_{bb} \circ \mathcal{A}_{ab} \circ L \arrow[r, "1\circ 1\circ q_a"] \arrow[d, "m\circ 1"'] & M_{bb} \circ \mathcal{A}_{ab} \circ N_{aa} \arrow[r, "1\circ m"] \arrow[d, "m\circ 1"] & M_{bb} \circ N_{ab} \arrow[d, "r_{abb}"] \\
\mathcal{A}_{ab} \circ M_{aa} \circ L \arrow[r, "m\circ 1"] & M_{ab} \circ L \arrow[r, "1\circ q_a"] & M_{ab} \circ N_{aa} \arrow[r, "r_{aab}"] & P_{ab}\,.
\end{tikzcd}$$

The rectangular regions commute by left extranaturality of p and q, and the bottom right square by bilinearity of r. The top left square, in which we make use of the linear distributivities of Section 2.2, commutes as an easy consequence of the duoidal category axioms; while the remaining three squares commute by (ordinary) bifunctoriality of $\circ$. We conclude that the outside rectangle commutes, and the duoidal axioms and bifunctoriality of $\circ$ now show that this rectangle is the extranaturality hexagon (3.5) for (3.6). The case of right extranaturality is dual. □

Proposition 17. *Let $\mathcal{V}$ be a complete $*$-biclosed normal duoidal category. If $\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{V}$-categories with $\mathcal{B}$ small, then the 2-functors $\mathbf{BIFUN}(-,\mathcal{B};\mathcal{C}) \colon \mathcal{V}\text{-}\mathbf{CAT}^{\mathrm{op}} \to \mathbf{CAT}$ and $\mathbf{BIFUN}(\mathcal{B},-;\mathcal{C}) \colon \mathcal{V}\text{-}\mathbf{CAT}^{\mathrm{op}} \to \mathbf{CAT}$ admit representations $[\mathcal{B},\mathcal{C}]_\ell$ and $[\mathcal{B},\mathcal{C}]_r$, which are sub-$\mathcal{V}$-categories of $[\![\mathcal{B},\mathcal{C}]\!]$, and are small whenever $\mathcal{C}$ is so.*

Proof. The objects of $[\mathcal{B},\mathcal{C}]_\ell$ are $\mathcal{V}$-functors $\mathcal{B} \to \mathcal{C}$, and the hom-object $[\mathcal{B},\mathcal{C}]_\ell(F,G)$ is the left end of $\mathcal{C}(F,G)$, whose existence is guaranteed by Proposition 15. Clearly $[\mathcal{B},\mathcal{C}]_\ell$ will be a small $\mathcal{V}$-category whenever $\mathcal{C}$ is so. We write the universal extranatural families associated to the homs as $p_{F,G,a} \colon [\mathcal{B},\mathcal{C}]_\ell(F,G) \to \mathcal{C}(Fa,Ga)$.

Now, by universality, maps $I \to [\mathcal{B},\mathcal{C}]_\ell(F,G)$ correspond to left extranatural families $I \to \mathcal{C}(Fa,Ga)$, and these are easily seen to correspond to $\mathcal{V}$-natural transformations in the usual sense. In particular, for each $F \in [\mathcal{B},\mathcal{C}]_\ell$, the identity $\mathcal{V}$-natural transformation on F yields a unique map j_F rendering commutative each square as to the left in:

$$
\begin{tikzcd}
I \arrow[r, "j_F"] \arrow[d, "1"'] & {[\mathcal{B},\mathcal{C}]_\ell(F,F)} \arrow[d, "p_{F,F,a}"] \\
I \arrow[r, "j_{Fa}"'] & \mathcal{C}(Fa,Fa)
\end{tikzcd}
\qquad
\begin{tikzcd}
{[\mathcal{B},\mathcal{C}]_\ell(G,H) \circ [\mathcal{B},\mathcal{C}]_\ell(F,G)} \arrow[r, "m_{FGH}"] \arrow[d, "p_{G,H,a} \circ p_{F,G,a}"'] & {[\mathcal{B},\mathcal{C}]_\ell(F,H)} \arrow[d, "p_{F,H,a}"] \\
\mathcal{C}(Ga,Ha) \circ \mathcal{C}(Fa,Ga) \arrow[r, "m_{Fa,Ga,Ha}"'] & \mathcal{C}(Fa,Ha)\ .
\end{tikzcd}
$$

On the other hand, given $F, G, H \colon \mathcal{B} \to \mathcal{C}$ and $a \in A$, we may form the lower composite around the square right above, and by Lemma 16, these composites constitute a left extranatural family; so by universality we induce a unique m_{FGH} as displayed making the square commute for all $a \in \mathcal{B}$. This defines the unit and composition maps of $[\mathcal{B},\mathcal{C}]_\ell$; the associativity and unitality axioms follow immediately from those of $\mathcal{C}$ and the universal property of end. Note also that, by the definition of left end, the universal families $p_{F,G,b}$ assemble to give monomorphisms $[\mathcal{B},\mathcal{C}]_\ell(F,G) \rightarrowtail \prod_b \mathcal{C}(Fb,Gb) = [\![\mathcal{B},\mathcal{C}]\!]\,(F,G)$, and commutativity of the above diagrams immediately implies that these are part of an identity-on-objects $\mathcal{V}$-functor $[\mathcal{B},\mathcal{C}]_\ell \to [\![\mathcal{B},\mathcal{C}]\!]$.

For a $\mathcal{V}$-bifunctor $T \colon \mathcal{A}, \mathcal{B} \to \mathcal{C}$, the corresponding $\bar{T} \colon \mathcal{A} \to [\mathcal{B},\mathcal{C}]_\ell$ is given on objects by $\bar{T}(a) = T(a,–)$; on homs, the map $\bar{T}_{aa'} \colon \mathcal{A}(a,a') \to [\mathcal{B},\mathcal{C}]_\ell(T(a,–),T(a',–))$ corresponds to the family $T(–,b)_{aa'} \colon \mathcal{A}(a,a') \to \mathcal{C}(Tab,Tab')$, whose left extranaturality in b is expressed precisely by the bifunctoriality hexagon (3.3). The functoriality of $\bar{T}$ follows from that of each $T(–,b)$ and the universal property of end, and the assignation $T \mapsto \bar{T}$ is easily seen to be bijective, and 2-natural in $\mathcal{A}$.

Suppose now we are given a sesquitransformation $\alpha \colon T \Rightarrow S \colon \mathcal{A}, \mathcal{B} \to \mathcal{C}$. The corresponding $\bar{\alpha} \colon \bar{T} \Rightarrow \bar{S} \colon \mathcal{A} \to [\mathcal{B},\mathcal{C}]_\ell$ has components $\bar{\alpha}_a \colon I \to [\mathcal{B},\mathcal{C}]_\ell(T(a,–),S(a,–))$ corresponding to the family $\alpha_{ab} \colon I \to \mathcal{C}(Tab,Sab)$ whose left extranaturality in b is precisely its $\mathcal{V}$-naturality in b. The $\mathcal{V}$-naturality of $\bar{\alpha}$ itself is correspondingly the $\mathcal{V}$-naturality of the components α_{ab} in a. Once again, the assignation $\alpha \mapsto \bar{\alpha}$ is easily seen to be bijective, and 2-natural in $\mathcal{A}$. This shows that $[\mathcal{B},\mathcal{C}]_\ell$ represents $\mathbf{BIFUN}(–,\mathcal{B};\mathcal{C})$ as required; the case of $[\mathcal{B},\mathcal{C}]_r$ is dual. $\square$

3.5. The monoidal 2-category $\mathcal{V}$-Cat

By assembling the constructions of the preceding two sections, we are now able to give sufficient conditions for the commuting tensor product of $\mathcal{V}$-categories to make $\mathcal{V}$-**Cat** into a monoidal closed 2-category.

Proposition 18. *Let $\mathcal{V}$ be a complete $*$-biclosed normal duoidal category. If for all $\mathcal{C}, \mathcal{D} \in \mathcal{V}$-**Cat**, the commuting tensor product $\mathcal{C} \odot \mathcal{D}$ exists, then it forms part of a biclosed monoidal 2-category structure on $\mathcal{V}$-**Cat** with unit the $\mathcal{V}$-category $\mathcal{I}$. This monoidal structure is symmetric whenever $\mathcal{V}$ is $*$-braided.*

Proof. The universal property of the commuting tensor product gives a 2-functor $\odot \colon \mathcal{V}\text{-}\mathbf{Cat} \times \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cat}$ and 2-natural transformation $q \colon \square \Rightarrow \odot$ from the tensor product of Proposition 8 which is pointwise epimorphic and co-fully faithful. It is easy to see that any sesquifunctor $\mathcal{I}, \mathcal{A} \to \mathcal{B}$ or $\mathcal{A}, \mathcal{I} \to \mathcal{B}$ is a bifunctor, so that the unit coherence maps for $\square$ descend (uniquely) to $\odot$, as to the left in:

$$
\begin{tikzcd}
\mathcal{I} \square \mathcal{A} \arrow[r, "\lambda_\square"] \arrow[d, two heads, "q"'] & \mathcal{A} \arrow[d, two heads, "\mathrm{id}"] \\
\mathcal{I} \odot \mathcal{A} \arrow[r, dashed, "\lambda_\odot"] & \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \square \mathcal{I} \arrow[r, "\rho_\square"] \arrow[d, two heads, "q"'] & \mathcal{A} \arrow[d, two heads, "\mathrm{id}"] \\
\mathcal{A} \odot \mathcal{I} \arrow[r, dashed, "\rho_\odot"] & \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathcal{A} \square \mathcal{B}) \square \mathcal{C} \arrow[r, "\alpha_\square"] \arrow[d, two heads, "q(q \,\square\, \mathrm{id})"'] & \mathcal{A} \square (\mathcal{B} \square \mathcal{C}) \arrow[d, two heads, "q(\mathrm{id} \,\square\, q)"] \\
(\mathcal{A} \odot \mathcal{B}) \odot \mathcal{C} \arrow[r, dashed, "\alpha_\odot"] & \mathcal{A} \odot (\mathcal{B} \odot \mathcal{C})\ .
\end{tikzcd}
$$

Since $\square$ preserves colimits in each variable, the vertical maps in the square right above are also epimorphic and co-fully faithful; so to obtain the associativity 2-natural transformation it suffices to show that the

$\alpha_\Box$'s also descend as indicated. The key point is to show that for any $\mathcal{V}$-category $\mathcal{D}$, composition with the universal $\varepsilon\colon [\mathcal{C},\mathcal{D}]_\ell, \mathcal{C} \to \mathcal{D}$ induces 2-natural bijections

$$\mathbf{BIFUN}(\mathcal{A},\mathcal{B};[\mathcal{C},\mathcal{D}]_\ell) \to \mathbf{BIFUN}(\mathcal{A},\mathcal{B},\mathcal{C};\mathcal{D}) \ . \tag{3.7}$$

Given this, the 2-natural bijections $\mathbf{BIFUN}(\mathcal{A},\mathcal{B};[\mathcal{C},\mathcal{D}]_\ell) \cong \mathcal{V}\text{-}\mathbf{CAT}((\mathcal{A}\odot\mathcal{B})\odot\mathcal{C},\mathcal{D})$ then imply that $(\mathcal{A}\odot\mathcal{B})\odot\mathcal{C}$ represents trifunctors; a dual argument shows that $\mathcal{A}\odot(\mathcal{B}\odot\mathcal{C})$ does so too, so allowing us to conclude that $\alpha_\Box$ descends as required.

To show that (3.7) is invertible, consider a trifunctor $T\colon \mathcal{A},\mathcal{B},\mathcal{C} \to \mathcal{D}$. For each object $a \in \mathcal{A}$, the bifunctor $T(a,\text{–},\text{–})\colon \mathcal{B},\mathcal{C} \to \mathcal{D}$ is classified by a $\mathcal{V}$-functor $T'(a,\text{–})\colon \mathcal{B} \to [\mathcal{C},\mathcal{D}]_\ell$, while for each $b \in \mathcal{B}$, the bifunctor $T(\text{–},b,\text{–})\colon \mathcal{A},\mathcal{C} \to \mathcal{D}$ is classified by a $\mathcal{V}$-functor $T'(\text{–},b)\colon \mathcal{A} \to [\mathcal{C},\mathcal{D}]_\ell$; these now assemble to give a sesquifunctor $T'\colon \mathcal{A},\mathcal{B} \to [\mathcal{C},\mathcal{D}]_\ell$, which we claim is commutative. The evident evaluation $\mathcal{V}$-functors $\mathrm{ev}_c\colon [\mathcal{C},\mathcal{D}]_\ell \to \mathcal{D}$ are jointly faithful, and so by Proposition 11(ii), it suffices to show that each $\mathrm{ev}_c \circ T'\colon \mathcal{A},\mathcal{B} \to \mathcal{C}$ is a bifunctor; which is so since $\mathrm{ev}_c \circ T' = T(\text{–},\text{–},c)$.

Given this, we conclude by the above argument that both $(\mathcal{A}\odot\mathcal{B})\odot\mathcal{C}$ and $\mathcal{A}\odot(\mathcal{B}\odot\mathcal{C})$ classify trifunctors, whence the associativity constraints for $\Box$ descend to $\odot$. The pentagon and triangle axioms for $\odot$ now follow easily from those for $\Box$, and so $(\mathcal{V}\text{-}\mathbf{Cat},\odot,\mathcal{I})$ becomes a biclosed monoidal 2-category with internal homs $[\mathcal{A},\mathcal{B}]_\ell$ and $[\mathcal{A},\mathcal{B}]_r$. Finally, when $\mathcal{V}$ is $*$-braided, we see by Proposition 11(iii) that $\mathcal{A}\odot\mathcal{B}$ represents bifunctors $\mathcal{B},\mathcal{A} \to \mathcal{C}$ as well as ones $\mathcal{A},\mathcal{B} \to \mathcal{C}$; whence the symmetry isomorphisms of $\Box$ descend to $\odot$, which is thus symmetric monoidal. $\Box$

3.6. Commuting graph morphisms

A critical examination of Definition 10 reveals that the definition of $\mathcal{V}$-bifunctor $\mathcal{A},\mathcal{B} \to \mathcal{C}$ makes no use of the compositional structure of $\mathcal{A}$ or $\mathcal{B}$. Guided by this, we may define a more general notion of bimorphism whose codomain is still a $\mathcal{V}$-category but whose domain is given by a pair of mere $\mathcal{V}$-graphs:

Definition 19. Let $\mathcal{V}$ be a normal duoidal category, let A and B be $\mathcal{V}$-graphs and let $\mathcal{C}$ be a $\mathcal{V}$-category. A *graph sesquimorphism* $T\colon A, B \to U\mathcal{C}$ comprises families $(T(a,\text{–})\colon B \to U\mathcal{C})_{a\in A}$ and $(T(\text{–},b)\colon A \to U\mathcal{C})_{b\in B}$ of graph morphisms such that $T(a,\text{–})(b) = T(\text{–},b)(a) = Tab$, say. T is said to *commute*, or to be a *graph bimorphism*, if each instance of (3.3)—with A and B replacing $\mathcal{A}$ and $\mathcal{B}$—is commutative in $\mathcal{V}$.

In fact, when $\mathcal{V}$ is $*$-biclosed and complete and free $\mathcal{V}$-categories exist, the extra generality this provides is only apparent: graph bimorphisms $A, B \to U\mathcal{C}$ turn out to coincide with bifunctors $FA, FB \to \mathcal{C}$ from the corresponding free $\mathcal{V}$-categories. The key to proving this is the construction of a more general kind of functor $\mathcal{V}$-category, whose domain is a small $\mathcal{V}$-*graph* rather than a $\mathcal{V}$-category.

Proposition 20. *Let $\mathcal{V}$ be a complete $*$-biclosed normal duoidal category, let B be a small $\mathcal{V}$-graph and let $\mathcal{C}$ be a $\mathcal{V}$-category. There is a $\mathcal{V}$-category $[B,\mathcal{C}]_\ell$ and graph bimorphism $\varepsilon\colon U[B,\mathcal{C}]_\ell, B \to U\mathcal{C}$, $\mathcal{V}$-functorial in its first variable, that provides a representation for the functor* $\mathbf{BIMOR}(\text{–},B;U\mathcal{C})\colon \mathcal{V}\text{-}\mathbf{GPH}^{\mathrm{op}} \to \mathbf{SET}$.

Of course, there is a dual construction of $[B,\mathcal{C}]_r$, with correspondingly dual properties, which we do not trouble to state.

Proof. First, a *profunctor* between $\mathcal{V}$-graphs $M\colon A \twoheadrightarrow B$ comprises components $M(b,a)$ with left A-actions and right B-actions, but satisfying no associativity or unit axioms. Next, if $M\colon A \twoheadrightarrow A$ is a profunctor between graphs, then a *left* or *right extranatural* family $K \to M(a,a)$ is defined just as in Definition 14; the corresponding notions of *left* and *right end*, together with their construction in Proposition 15, carry over directly. We may thus define $[B,\mathcal{C}]_\ell$ to be the $\mathcal{V}$-category with:

- *Objects* being graph morphisms $F\colon B \to U\mathcal{C}$;
- *Hom-object* $[B, \mathcal{C}]_\ell(F, G)$ being the left end of $\mathcal{C}(F, G)\colon B \twoheadrightarrow B$;
- *Composition* derived as in Proposition 17, using the graph analogue of Lemma 16.

The universal property of $[B, \mathcal{C}]_\ell$ follows exactly as in the proof of Proposition 17. □

Proposition 21. *Let $\mathcal{V}$ be complete $*$-biclosed normal duoidal, let A and B be $\mathcal{V}$-graphs, and let $\eta_A\colon A \to UFA$ and $\eta_B\colon B \to UFB$ exhibit FA and FB as the free $\mathcal{V}$-categories on A and B. For any $\mathcal{V}$-category $\mathcal{C}$, composing with (η_A, η_B) establishes a bijection between $\mathcal{V}$-bifunctors $FA, FB \to \mathcal{C}$ and graph bimorphisms $A, B \to U\mathcal{C}$.*

Proof. If $T\colon FA, FB \to \mathcal{C}$ is a $\mathcal{V}$-bifunctor, then $UT\colon UFA, UFB \to U\mathcal{C}$ is a graph bimorphism, whence by (the graph analogue of) Proposition 11(i), so too is the composite $UT \circ (\eta_A, \eta_B)\colon A, B \to U\mathcal{C}$. Suppose conversely that $S\colon A, B \to U\mathcal{C}$ is a graph bimorphism. This corresponds by Proposition 20 to a graph morphism $A \to U[B, \mathcal{C}]_\ell$, and so to a $\mathcal{V}$-functor $S'\colon FA \to [B, \mathcal{C}]_\ell$. The composite $\varepsilon \cdot (US', 1)\colon UFA, B \to U\mathcal{C}$ is now a graph bimorphism, $\mathcal{V}$-functorial in its first variable, whose precomposition with $(\eta_A, 1)$ is S. Repeating the same argument using $[UFA, \mathcal{C}]_r$ in place of $[B, \mathcal{C}]_\ell$ yields a graph bimorphism $UFA, UFB \to U\mathcal{C}$, $\mathcal{V}$-functorial in each variable—thus, a $\mathcal{V}$-bifunctor $FA, FB \to \mathcal{C}$—whose precomposition with (η_A, η_B) is S, as required. □

3.7. Change of base

In the final part of this section, we briefly explore the interaction of commuting tensor products with *change of base*. Recall that, if $F\colon \mathcal{V} \to \mathcal{W}$ is a (lax) monoidal functor between monoidal categories, then there is an induced 2-functor $F_*\colon \mathcal{V}\text{-}\mathbf{CAT} \to \mathcal{W}\text{-}\mathbf{CAT}$, which sends a $\mathcal{V}$-category $\mathcal{A}$ to the $\mathcal{W}$-category $F_*\mathcal{A}$ with the same set of objects, and with homs $(F_*\mathcal{A})(X, Y) = F(\mathcal{A}(X, Y))$.

If $\mathcal{V}$ and $\mathcal{W}$ are braided monoidal and F is a braided monoidal functor, then F_* easily becomes a monoidal 2-functor with respect to the tensor products on $\mathcal{V}$-**CAT** and $\mathcal{W}$-**CAT**, one which is *strong* monoidal whenever F is so. We wish to extend this fact from the braided monoidal to the normal duoidal context. To this end, given two duoidal categories $(\mathcal{V}, *, \circ)$ and $(\mathcal{W}, *, \circ)$, we define a *duoidal functor* $F\colon \mathcal{V} \to \mathcal{W}$ to be a functor F which is monoidal with respect to both $*$ and $\circ$ in such a way that the constraint maps $FX \circ FY \to F(X \circ Y)$ and $I \to F(I)$ for the $\circ$-monoidal structure are monoidal natural transformations with respect to $*$. We call F *strong* if both underlying monoidal functors are so.

Example 22. If $\mathcal{V}$ is any duoidal category, then $\mathcal{V}(J, -)\colon \mathcal{V} \to \mathbf{Set}$ is a duoidal functor when **Set** is seen as cartesian duoidal as in Example 2.

Proposition 23. *Let $F\colon \mathcal{V} \to \mathcal{W}$ be a duoidal functor between normal duoidal categories. The induced $F_*\colon \mathcal{V}\text{-}\mathbf{CAT} \to \mathcal{W}\text{-}\mathbf{CAT}$ is lax monoidal with respect to the commuting tensor products on $\mathcal{V}$-**CAT** and $\mathcal{W}$-**CAT**, insofar as these are defined; if moreover $\mathcal{V}$, $\mathcal{W}$ and F are $*$-braided, then F_* is symmetric monoidal.*

Proof. Since sesquifunctors are defined by families of partial functors, it is clear that the action of F_* on morphisms can be extended to an action

$$F_*\colon \mathbf{SESQ}_{\mathcal{V}}(\mathcal{A}, \mathcal{B}; \mathcal{C}) \longrightarrow \mathbf{SESQ}_{\mathcal{W}}(F_*\mathcal{A}, F_*\mathcal{B}; F_*\mathcal{C}) \tag{3.8}$$

which is 2-natural in $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Given a sesqui-$\mathcal{V}$-functor $T\colon \mathcal{A}, \mathcal{B} \to \mathcal{C}$, applying F to the bifunctor diagram (3.3) for T and precomposing with the comparison map $F\mathcal{A}(a, a') * F\mathcal{B}(b, b') \to F(\mathcal{A}(a, a') * \mathcal{B}(b, b'))$

yields the corresponding bifunctor diagram for $F_*(T)$; in particular, commutativity of the former implies commutativity of the latter, and so the functors (3.8) restrict to ones

$$F_* \colon \mathbf{BIFUN}_{\mathcal{V}}(\mathcal{A}, \mathcal{B}; \mathcal{C}) \longrightarrow \mathbf{BIFUN}_{\mathcal{W}}(F_*\mathcal{A}, F_*\mathcal{B}; F_*\mathcal{C}) \ .$$

More generally, we may show that F_* extends to a morphism of 2-multicategories $\mathbf{BIFUN}_{\mathcal{V}} \to \mathbf{BIFUN}_{\mathcal{W}}$, and as a direct consequence of this, F_* is lax monoidal with respect to the representing monoidal structures $(\odot, \mathcal{I})$ on $\mathcal{V}$-**CAT** and $\mathcal{W}$-**CAT**, insofar as these are defined. Finally, if $\mathcal{V}$, $\mathcal{W}$ and F are $*$-braided, then F_* is a morphism of *symmetric* 2-multicategories, and so is *symmetric* monoidal wherever the tensor product is defined. $\square$

In order for change of base along a duoidal functor $F \colon \mathcal{V} \to \mathcal{W}$ to be *strong* monoidal with respect to $\odot$, we will have to assume more than merely that F is strong duoidal. The reason is that the construction of the commuting tensor product of $\mathcal{V}$-categories involves free $\mathcal{V}$-categories and colimits of $\mathcal{V}$-categories, and there is no reason why F being strong duoidal should force change of base to preserve these. The simplest way of ensuring such preservation is to ask for F to be a left adjoint.

More precisely, by a *duoidal adjunction* between duoidal categories, we mean an adjunction $F \dashv G \colon \mathcal{W} \to \mathcal{V}$ for which F and G are duoidal functors and the unit and counit $1 \Rightarrow GF$ and $FG \Rightarrow 1$ are monoidal natural transformations with respect to both $\circ$ and $*$. By the considerations of [30], the F in this situation must be *strong* duoidal, and the duoidal constraint cells for G determined as the mates under adjunction of those for F.

Example 24. If $\mathcal{V}$ is a normal duoidal category admitting copowers of the unit J, then the functor $\mathcal{V}(J, –) \colon \mathcal{V} \to \mathbf{Set}$ has a left adjoint $(–) \cdot J$. If both monoidal structures $\circ$ and $*$ on $\mathcal{V}$ are closed on at least one side, then this left adjoint is easily seen to be strong duoidal, and so part of a duoidal adjunction $(–) \cdot J \dashv \mathcal{V}(J, –) \colon \mathcal{V} \to \mathbf{Set}$.

Proposition 25. *If $F \dashv G \colon \mathcal{W} \to \mathcal{V}$ is a duoidal adjunction, then F_* preserves any commuting tensor product of $\mathcal{V}$-categories that exists; hence, under the hypotheses of Proposition 18, F_* becomes a strong monoidal 2-functor $\mathcal{V}$-**Cat** $\to$ $\mathcal{W}$-**Cat**.*

Proof. The adjunction $F \dashv G$ induces a 2-adjunction $F_* \dashv G_* \colon \mathcal{W}$-**Cat** $\to \mathcal{V}$-**Cat**, and so 2-natural isomorphisms

$$\mathbf{SESQ}_{\mathcal{W}}(F_*\mathcal{A}, F_*\mathcal{B}; \mathcal{C}) \cong \mathbf{SESQ}_{\mathcal{V}}(\mathcal{A}, \mathcal{B}; G_*\mathcal{C}) \ . \tag{3.9}$$

We have, moreover, a bijection between maps $FA * FB \to C$ in $\mathcal{W}$ and ones $A * B \to GC$ in $\mathcal{V}$; under this bijection, the diagram (3.3) for a sesquifunctor $T \colon F_*\mathcal{A}, F_*\mathcal{B} \to \mathcal{C}$ transposes to the corresponding diagram for $\tilde{T} \colon \mathcal{A}, \mathcal{B} \to G_*\mathcal{C}$, and so the isomorphisms (3.9) restrict to ones

$$\mathbf{BIFUN}_{\mathcal{W}}(F_*\mathcal{A}, F_*\mathcal{B}; \mathcal{C}) \cong \mathbf{BIFUN}_{\mathcal{V}}(\mathcal{A}, \mathcal{B}; G_*\mathcal{C}) \ .$$

Thus if $\mathcal{A}, \mathcal{B}$ in $\mathcal{W}$-**CAT** admit the tensor product $\mathcal{A} \odot \mathcal{B}$, then $\mathcal{W}$-$\mathbf{CAT}(F_*(\mathcal{A} \odot \mathcal{B}), –)$ is isomorphic to $\mathbf{BIFUN}(F_*\mathcal{A}, F_*\mathcal{B}; –)$, and so $F_*(\mathcal{A} \odot \mathcal{B})$ is a tensor product of $F_*(\mathcal{A})$ and $F_*(\mathcal{B})$. The second assertion can be easily deduced from the first and the construction of the associativity and unit constraints for $\odot$ as in Section 3.5. $\square$

Note that, in the situation of this proposition, the adjunction $F_* \dashv G_* \colon \mathcal{W}$-**Cat** $\to \mathcal{V}$-**Cat** becomes an adjunction of monoidal 2-categories. In fact, an alternative way of proving the last clause of the preceding proposition would have been to run this argument backwards: by exploiting the 2-functoriality of the assignation $\mathcal{V} \mapsto \mathcal{V}$-**Cat** from normal duoidal categories to monoidal 2-categories, we could deduce that any duoidal adjunction is sent to an adjunction of monoidal 2-categories, which by [30] must have a strong left adjoint.

4. Commutativity: the one-object case

In this section, we indicate how our theory of commutativity specialises to the case of one-object $\mathcal{V}$-categories—that is, of $\circ$-monoids in $\mathcal{V}$.

4.1. The basic notions

Given a $\circ$-monoid A in the normal duoidal $\mathcal{V}$, we will write ΣA for the corresponding one-object $\mathcal{V}$-category. With this convention, we see that a sesquifunctor $T\colon \Sigma A, \Sigma B \to \Sigma C$ is given simply by a cospan $f\colon A \to C \leftarrow B\colon g$ of $\circ$-monoid morphisms. As anticipated in the introduction, such a cospan will be said to *commute*, in the sense of corresponding to a bifunctor, just when the hexagon

$$\begin{tikzcd} & A \circ B \arrow[r, "f \circ g"] & C \circ C \arrow[dr, "m"] & \\ A * B \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & C \\ & B \circ A \arrow[r, "g \circ f"'] & C \circ C \arrow[ur, "m"'] & \end{tikzcd} \tag{4.1}$$

commutes in $\mathcal{V}$. Correspondingly, an n-ary sesquifunctor $\Sigma A_1, \ldots, \Sigma A_n \to \Sigma B$ amounts to an n-tuple of $\circ$-monoid morphisms $f_i\colon A_i \to B$, and such an n-tuple commutes just when the cospan (f_i, f_j) does so for each $1 \leqslant i < j \leqslant n$.

More generally, the one-object case of a graph sesquimorphism $A, B \to U\mathcal{C}$ in the sense of Definition 19 is that of a cospan $f\colon A \to UC \leftarrow B\colon g$, where C is a $\circ$-monoid and A, B are mere objects in $\mathcal{V}$. The commutativity of such a cospan is the requirement that the same (4.1) should commute in $\mathcal{V}$. In this context, Proposition 21 expresses that a cospan of this kind is commutative precisely when the transpose cospan $FA \to C \leftarrow FB$ of $\circ$-monoids is so.

Note that, if $T\colon \Sigma A, \Sigma B \to \Sigma C$ corresponds to a cospan $f\colon A \to C \leftarrow B\colon g$, then $T^c\colon \Sigma B, \Sigma A \to \Sigma C$ corresponds to the reverse cospan $g\colon B \to C \leftarrow A\colon f$; using this fact, we can fulfil a promise made in Section 3.2 above by exhibiting a $\mathcal{V}$ for which the notion of bifunctor is not symmetric in its input arguments.

Example 26. Let $\mathcal{V}$ be the category of positively graded k-vector spaces, equipped with the tensor product

$$(X \otimes Y)_n = \sum_{n=r+s} X_r \otimes Y_s \qquad \text{and} \qquad I_n = \begin{cases} k & \text{if } n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

If we fix $q \neq 0$ in k, there is as in [1, Section 2.3] a braiding $c_q\colon X \otimes Y \to Y \otimes X$ given on homogeneous elements by $c_q(x \otimes y) = q^{rs}(y \otimes x)$ for $x \in X_r$ and $y \in Y_s$; using this, we may see $\mathcal{V}$ as a normal duoidal category in the canonical way.

In this situation, a $\circ$-monoid is a graded k-algebra. If C is any graded k-algebra, and A and B are the graded k-algebras freely generated by elements $a \in A_r$ and $b \in B_s$, then a cospan of $\circ$-monoid maps $f\colon A \to C \leftarrow B\colon g$, corresponding to a sesquifunctor T, will commute just when $f(a)g(b) = q^{rs}g(b)f(a)$. On the other hand, the cospan $g\colon B \to C \leftarrow A\colon f$, corresponding to the transpose sesquifunctor T^c, will commute just when $g(b)f(a) = q^{rs}f(a)g(b)$. In particular, if $q \neq \pm 1$, then these two commutativities are, in general, distinct.

4.2. Commuting tensor product and centralizers

Whenever $\mathcal{V}$ has a terminal object, the set-of-objects functor $\mathcal{V}$-**Cat** $\to$ **Set** has a right adjoint, and so preserves colimits and sends $\square$ to $\times$; it follows that one-object $\mathcal{V}$-categories are closed under the commuting tensor product as constructed in Section 3.3.

In fact, constructing the commuting tensor product of $\circ$-monoids A and B does not require the full strength of the hypotheses of Proposition 12: it suffices that free $\circ$-monoids should exist and that $\mathbf{Mon}_\circ(\mathcal{V})$ should admit finite colimits. Indeed, one first forms the coproduct $\iota_1 \colon A \to A + B \leftarrow B \colon \iota_2$ in $\mathbf{Mon}_\circ(\mathcal{V})$; then the parallel pair $A * B \rightrightarrows U(A+B)$ in $\mathcal{V}$ given by the two sides of the hexagon (4.1) for the cospan (ι_1, ι_2); and finally obtains the commuting tensor product of A and B as the coequalizer in $\mathbf{Mon}_\circ(\mathcal{V})$ of the transposed maps $F(A * B) \rightrightarrows A + B$.

By contrast to the above, one-object $\mathcal{V}$-categories are not closed under taking functor $\mathcal{V}$-categories; indeed, objects of either internal hom $[\Sigma A, \Sigma B]_\ell$ or $[\Sigma A, \Sigma B]_r$ are arbitrary $\circ$-monoid morphisms $A \to B$. Nonetheless, each endo-hom-object of $[\Sigma A, \Sigma B]_\ell$ or $[\Sigma A, \Sigma B]_r$ is a $\circ$-monoid in $\mathcal{V}$ and in fact a sub-$\circ$-monoid of B—and may be seen as providing a general notion of *centralizer*:

Definition 27. Let $\mathcal{V}$ be a complete $*$-biclosed normal duoidal category. If $f \colon A \to B$ is a $\circ$-monoid morphism in $\mathcal{V}$, then the *left centralizer* of f is the sub-$\circ$-monoid $C_\ell(f) := [\Sigma A, \Sigma B]_\ell(f, f)$ of B; the *right centralizer* $C_r(f)$ is defined dually. The *left* or *right centre* of a $\circ$-monoid A is the left or right centralizer of 1_A.

Note that, when $\mathcal{V}$ is $*$-braided, the two notions of centralizer and centre coincide by Proposition 11(iii), and in this case, we drop the modifiers "left" and "right". Unwinding the proof of Proposition 17, we find that the left centralizer of $f \colon A \to B$ may be constructed as follows. We take $g = 1_B$ in (4.1), transpose both paths under the adjunction $A * (-) \dashv [A, -]_\ell$ to obtain a parallel pair $B \rightrightarrows [A, B]_\ell$, and take the equalizer of this pair to obtain $C_\ell(f)$. The construction of $C_r(f)$ is dual.

When $\mathcal{V} = \mathbf{Set}$, the centralizer of a monoid morphism $f \colon N \to M$ is, as expected, the set $\{m \in M : mn = nm \text{ for all } n \in N\}$. For a general $\mathcal{V}$, our nomenclature is justified by the following result, which is an immediate consequence of the universal characterisation of functor $\mathcal{V}$-categories.

Proposition 28. *Let $\mathcal{V}$ be a complete $*$-biclosed normal duoidal category and let $f \colon A \to C \leftarrow B \colon g$ be a cospan of $\circ$-monoid morphisms. The following are equivalent:*

(i) *(f, g) is a commuting cospan;*
(ii) *f factors through the left centralizer $C_\ell(g) \rightarrowtail C$;*
(iii) *g factors through the right centralizer $C_r(f) \rightarrowtail C$.*

By adapting the proof of Proposition 18, we see that, if $\mathcal{V}$ is complete $*$-biclosed normal duoidal, and all commuting tensor products of $\circ$-monoids exist, then we obtain a monoidal structure $\odot$ on the category $\mathbf{Mon}_\circ(\mathcal{V})$ of $\circ$-monoids in $\mathcal{V}$. This monoidal structure has the two properties that (i) its unit object I is initial; (ii) for each $A, B \in \mathbf{Mon}_\circ(\mathcal{V})$, the two maps

$$A \xrightarrow{\cong} A \odot I \xrightarrow{A \odot !} A \odot B \xleftarrow{! \odot B} I \odot B \xleftarrow{\cong} B \tag{4.2}$$

are jointly epimorphic. Monoidal structures with these two properties were studied in detail in [27, §2], and characterised in terms of properties of the "generalised commutation relation" on cospans that induces them.

4.3. Commutative $\circ$-monoids

In the one-object case, there is a natural definition of *commutative* $\circ$-monoid; we now give this together with a number of alternative characterisations of the notion.

Definition 29. Let A be a $\circ$-monoid in the normal duoidal $\mathcal{V}$. We say that A is *commutative* if $1_A \colon A \to A \leftarrow A \colon 1_A$ is a commuting cospan; equivalently, if the following square commutes in $\mathcal{V}$:

$$\begin{tikzcd} & A \circ A \arrow[dr, "m"] & \\ A * A \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & A\,. \\ & A \circ A \arrow[ur, "m"'] & \end{tikzcd}$$

Comparing this definition with Section 2.3, we see that, in a normal duoidal category, both *duoids* and *commutative $\circ$-monoids* are generalisations of commutative monoids in a braided monoidal category. The following proposition shows that, in fact, they are the same generalisation.

Proposition 30. *Let $\mathcal{V}$ be a normal duoidal category. The forgetful functor from duoids to $\circ$-monoids $U \colon \mathbf{Duoid}(\mathcal{V}) \to \mathbf{Mon}_\circ(\mathcal{V})$ is injective on objects and fully faithful, and its image comprises the commutative $\circ$-monoids. Moreover, if $\mathcal{V}$ is $*$-braided, then every duoid in $\mathcal{V}$ is $*$-commutative.*

Proof. Let the maps $e \colon I \to A \leftarrow A \circ A \colon m$ and $\iota \colon J \to A \leftarrow A * A \colon \nu$ exhibit A as a duoid. Since $e \colon I \to A$ is a map of $*$-monoids, we have $\iota = ev \colon J \to I \to A$ and so ι is determined by e. Moreover, by precomposing the axiom (2.4) by the map

$$A * A \cong (A \circ I) * (I \circ A) \xrightarrow{(A \circ e)*(e \circ A)} (A \circ A) * (A \circ A)\ , \tag{4.3}$$

the lower and upper sides become $\nu \colon A * A \to A$ and $m\sigma \colon A * A \to A \circ A \to A$ respectively, so that ν is determined by m. Thus U is injective on objects, and the formulae $\iota = ev$ and $\nu = m\sigma$ now imply easily that it is fully faithful too.

Replacing $(A \circ e) * (e \circ A)$ by $(e \circ A) * (A \circ e)$ in (4.3), we see that any duoid A verifies $\nu = m\tau$ as well as $\nu = m\sigma$; so $m\tau = m\sigma$, which is the condition for the underlying $\circ$-monoid to be commutative. Moreover, if $\mathcal{V}$ is $*$-braided, then by (2.3) we have $\nu = m\sigma = m\tau c = \nu c \colon A * A \to A$, so that the duoid A is necessarily $*$-commutative.

All that remains is to show that *every* commutative $\circ$-monoid A is in the image of U. Of course, the $*$-monoid structure on A must be given by $ev \colon J \to A$ and $m\sigma = m\tau \colon A * A \to A$; it is now direct that this is an $*$-monoid, and that $e \colon I \to A$ is a map of $*$-monoids. To verify (2.4), we first verify that the following diagrams commute in any normal duoidal category:

$$\begin{tikzcd} (A \circ B) * (C \circ D) \arrow[r, "\delta^\ell_\ell"] \arrow[d, "\xi"'] & ((A \circ B) * C) \circ D \arrow[r, "\delta^r_r \circ D"] & (A \circ (B * C)) \circ D \arrow[d, "(A \circ \tau) \circ D"] \\ (A * C) \circ (B * D) \arrow[r, "\sigma \circ \sigma"] & (A \circ C) \circ (B \circ D) \arrow[r, "\cong"] & (A \circ (C \circ B)) \circ D \end{tikzcd}$$

$$\begin{tikzcd}
(A \circ B) * (C \circ D) \arrow[r, "\delta^\ell_\ell"] \arrow[d, equal] & ((A \circ B) * C) \circ D \arrow[r, "\delta^r_r \circ D"] & (A \circ (B * C)) \circ D \arrow[d, "(A\circ\sigma)\circ D"] \\
(A \circ B) * (C \circ D) \arrow[r, "\sigma"] & (A \circ B) \circ (C \circ D) \arrow[r, "\cong"] & (A \circ (B \circ C)) \circ D\ .
\end{tikzcd}$$

On taking $A = B = C = D$ and postcomposing with the quaternary multiplication map $m.(m \circ A).((A \circ m) \circ A)\colon (A \circ (A \circ A)) \circ A \to A$, commutativity of A implies that the two upper paths become equal; whence the two lower paths do too. But these two paths are easily seen to be the two sides of (2.4), which thus commutes. □

In the situation where we have the commuting monoidal structure on $\circ$-monoids, we may give a further characterisation of the commutative $\circ$-monoids.

Proposition 31. *Let $\mathcal{V}$ be a normal duoidal category for which $\mathbf{Mon}_\circ(\mathcal{V})$ admits the commuting monoidal structure $\odot$. The forgetful functor $U\colon \mathbf{Mon}_\odot(\mathbf{Mon}_\circ(\mathcal{V})) \to \mathbf{Mon}_\circ(\mathcal{V})$ is injective on objects and fully faithful, and its image comprises the commutative $\circ$-monoids. If $\mathcal{V}$ is $*$-braided, then every $\odot$-monoid in $\mathbf{Mon}_\circ(\mathcal{V})$ is commutative.*

Proof. This relies solely on the two properties of the $\odot$-monoidal structure on $\mathbf{Mon}_\circ(\mathcal{V})$ noted in Section 4.2 above: that the unit is initial, and that the maps (4.2) are jointly epimorphic. A full proof of the result from these assumptions is given in [27, Theorem 2.8.2]; we reproduce it here for completeness. Of course, the unit of any $\odot$-monoid A in $\mathbf{Mon}_\circ(\mathcal{V})$ is necessarily the unique map $\eta\colon I \to A$ from the initial object, and by the epimorphicity of (4.2), the multiplication μ must be the unique map fitting into a diagram

$$\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, "\cong"'] & A & A \arrow[l, "1"'] \arrow[d, "\cong"] \\
I \odot A \arrow[r, "\eta\odot A"'] & A \odot A \arrow[u, dashed, "\mu"] & A \odot I\ . \arrow[l, "A\odot\eta"]
\end{tikzcd} \tag{4.4}$$

So U is injective on objects, and clearly faithful; for fullness, we observe that any map $f\colon A \to B$ of $\circ$-monoids between $\odot$-monoids must satisfy $\mu.(f \odot f) = f.\mu\colon A \odot A \to B$, since both sides precompose with the jointly epimorphic cospan $A \to A \odot A \leftarrow A$ to yield the cospan $f\colon A \to B \leftarrow A\colon f$. Commutativity of (4.4), together with the universal property of $\odot$, shows that any $\odot$-monoid in $\mathbf{Mon}_\circ(\mathcal{V})$ has a commutative underlying $\circ$-monoid; it remains to show that every commutative $\circ$-monoid lies in the image of U. The commutativity and the universal property of $\odot$ yields the existence of a map μ as in (4.4) for which the unique map $\eta\colon I \to A$ is a unit; to show that (A, η, μ) is a $\odot$-monoid, it thus remains to check the associativity axiom $\mu.(\mu \odot A) = \mu.(A \odot \mu)\colon A \odot A \odot A \to A$; which is clear on precomposition with the three jointly epimorphic maps $A \to A \odot A \odot A$.

Finally, when $\mathcal{V}$ is $*$-braided, the $\odot$-tensor product on $\mathbf{Mon}_\circ(\mathcal{V})$ admits a symmetry c; now for any $\odot$-monoid (A, η, μ), we have that $\mu = \mu c\colon A \odot A \to A$, since both sides precompose with the jointly epimorphic cospan $A \to A \odot A \leftarrow A$ to yield the identity cospan $1_A\colon A \to A \leftarrow A\colon 1_A$. □

5. Example: algebraic theories

This concludes our development of the general theory of commutativity; in the remainder of the paper, we apply it to a range of examples, starting in this section with (finitary) *algebraic theories*. In this context, there are well-known notions of *commuting tensor product* of theories and of *commutative* algebraic theory [16]; our objective is to see how these arise as particular instances of our general notions.

5.1. Algebraic theories and commutativity

There are many ways of presenting algebraic theories—see [24] for an overview—but for our purposes, we take the following perspective. We write $\mathcal{F}$ for the presheaf category $[\mathbb{F}, \mathbf{Set}]$, where $\mathbb{F}$ is the category of finite cardinals and mappings; now restriction and left Kan extension along the inclusion $I \colon \mathbb{F} \to \mathbf{Set}$ exhibits $\mathcal{F}$ as equivalent to the category $\mathbf{End}_\omega(\mathbf{Set})$ of filtered-colimit-preserving endofunctors of $\mathbf{Set}$, and under this equivalence, the composition monoidal structure on $\mathbf{End}_\omega(\mathbf{Set})$ transports to the *substitution* monoidal structure on $\mathcal{F}$, whose unit object is the inclusion I, and whose binary tensor is given by

$$(A \circ B)(n) = \int^{m \in \mathbf{F}} Am \times (Bn)^m \ . \tag{5.1}$$

By an *algebraic theory*, we mean a $\circ$-monoid (T, η, μ) in $\mathcal{F}$. We call elements $\alpha \in Tn$ the *n-ary operations* of T, write $\pi_0, \ldots, \pi_{n-1} \in Tn$ for the elements in the image of $\eta_n \colon n \to Tn$, and, given $f \in Tm$ and $g_1, \ldots, g_n \in Tn$, write $f(g_1, \ldots, g_n)$ for the image under $\mu_n \colon \int^{m \in \mathbb{F}} Tm \times (Tn)^m \to Tn$ of the element $(f, g_1, \ldots, g_n)$.

Definition 32. Let $\mathcal{E}$ be a category with finite powers, and let T be an algebraic theory. A *model* of T in $\mathcal{E}$ is an object $X \in \mathcal{E}$ together with functions $[\![\alpha]\!] : X^n \to X$ for each $\alpha \in Tn$ such that $[\![\pi_i]\!] = \pi_i \colon X^n \to X$ for each $0 \leqslant i < n$, and such that

$$[\![f(g_1, \ldots, g_n)]\!] = X^m \xrightarrow{([\![g_1]\!], \ldots, [\![g_n]\!])} X^n \xrightarrow{[\![f]\!]} X$$

for all $f \in Tn$ and $g_1, \ldots, g_n \in Tm$. A *homomorphism of models* from X to Y is a map $f \colon X \to Y$ in $\mathcal{E}$ such that $f . [\![\alpha]\!]_X = [\![\alpha]\!]_Y . f^n$ for all $\alpha \in Tn$. We write $\mathbf{Mod}(T; \mathcal{E})$ for the category of T-models and model homomorphisms in $\mathcal{E}$.

For any $\mathcal{E}$ with finite powers, it is easy to see that the category $\mathbf{Mod}(T; \mathcal{E})$ of the preceding definition again has finite powers, created by the evident forgetful functor to $\mathcal{E}$. Any finite-power-preserving functor $\mathcal{E} \to \mathcal{G}$ induces a finite-power-preserving functor $\mathbf{Mod}(T; \mathcal{E}) \to \mathbf{Mod}(T; \mathcal{G})$; similarly, any morphism $S \to T$ of finitary monads induces a finite-power-preserving functor $\mathbf{Mod}(T; \mathcal{E}) \to \mathbf{Mod}(S; \mathcal{E})$ commuting with the forgetful functors to $\mathcal{E}$.

Definition 33. Let S and T be algebraic theories and let $\mathcal{E}$ be a category with finite powers. By a *commuting S-T-model* in $\mathcal{E}$, we mean a T-model in $\mathbf{Mod}(S; \mathcal{E})$, or equivalently, an S-model in $\mathbf{Mod}(T; \mathcal{E})$. We write $\mathbf{Mod}(S, T; \mathcal{E})$ for either of the isomorphic categories $\mathbf{Mod}(S; \mathbf{Mod}(T; \mathcal{E})) \cong \mathbf{Mod}(T; \mathbf{Mod}(S; \mathcal{E}))$.

With the above definitions in place, we can now define the (commuting) tensor product of theories, and the notion of commutative algebraic theory.

Definition 34. The *tensor product* of algebraic theories S and T is a theory U equipped with an isomorphism $\mathbf{Mod}(U; \mathbf{Set}) \cong \mathbf{Mod}(S, T; \mathbf{Set})$ over $\mathbf{Set}$. An algebraic theory T is *commutative* if the diagonal $\mathbf{Mod}(T; \mathbf{Set}) \to \mathbf{Mod}(T; \mathbf{Set}) \times_{\mathbf{Set}} \mathbf{Mod}(T; \mathbf{Set})$ factors through $\mathbf{Mod}(T, T; \mathbf{Set})$.

5.2. The duoidal category $\mathcal{F}$

Our goal now is to make the substitution monoidal structure on $\mathcal{F}$ into part of a normal duoidal one in such a way that our general commutativity notions reduce to those of Definition 34; to obtain the second tensor product, we make use of *Day convolution* [9].

Definition 35. Suppose that $\mathcal{A}$ and $\mathcal{V}$ are monoidal categories.

(i) A *convolution tensor* of $F, G \in [\mathcal{A}, \mathcal{V}]$ is a functor $F * G \in [\mathcal{A}, \mathcal{V}]$ equipped with a universal natural family of maps $u_{AB} \colon FA \otimes GB \to (F*G)(A \otimes B)$; this means that any natural family $k_{AB} \colon FA \otimes GB \to H(A \otimes B)$ is of the form $k_{AB} = \bar{k}_{A\otimes B}.u_{AB}$ for a unique map $\bar{k} \colon F * G \to H$ in $[\mathcal{A}, \mathcal{V}]$.

(ii) A *nullary convolution tensor* is a functor $J \in [\mathcal{A}, \mathcal{V}]$ together with a universal map $j \colon I \to J(I)$; this means that each map $k \colon I \to H(I)$ in $\mathcal{V}$ is of the form $\bar{k}_I.j$ for a unique $\bar{k} \colon J \to H$ in $[\mathcal{A}, \mathcal{V}]$.

When $\mathcal{A}$ is small, and $\mathcal{V}$ is biclosed, complete and cocomplete, all convolution tensors exist, and underlie a biclosed monoidal structure on $[\mathcal{A}, \mathcal{V}]$, which will be braided or symmetric whenever the monoidal structures on $\mathcal{A}$ and $\mathcal{V}$ are so. The unit of this monoidal structure is the copower $J = \mathcal{A}(I, \text{–}) \cdot I$, while binary tensors and internal homs are given by the formulae:

$$F * G = \int^{A,B\in\mathcal{A}} \mathcal{A}(A \otimes B, \text{–}) \cdot FA \otimes GB$$
$$[F, G]_\ell = \int_{A\in\mathcal{A}} [FA, G(A \otimes \text{–})]_\ell \quad \text{and} \quad [F, G]_r = \int_{B\in\mathcal{A}} [FB, G(\text{–} \otimes B)]_r \ .$$

In the case of $\mathcal{F} = [\mathbb{F}, \mathbf{Set}]$, applying Day convolution with respect to product in both $\mathbb{F}$ and $\mathbf{Set}$ yields a symmetric monoidal closed structure $(*, J)$ on $\mathcal{F}$.

Proposition 36. *$(\mathcal{F}, *, J, \circ, I)$ is a normal duoidal category.*

Proof. Since $J = \mathbf{Set}(1, \text{–}) \cdot 1$ and I is the inclusion functor $\mathbb{F} \to \mathbf{Set}$, the evaluation maps $\mathbf{Set}(1, X) \cdot 1 \to X$ provide an isomorphism $\upsilon \colon J \to I$ of unit objects; the other unit structure maps $\mu \colon I*I \to I$ and $\gamma \colon J \to J \circ J$ are determined uniquely from this, and it remains only to give the interchange maps $\xi \colon (F \circ G) * (H \circ K) \to (F * H) \circ (G * K)$.

We have already observed that the $\circ$-monoidal structure on $\mathcal{F}$ transports to give composition in $\mathbf{End}_\omega(\mathbf{Set})$; on the other hand, for any $F, G, H \in \mathbf{End}_\omega(\mathbf{Set})$, each natural family $FA \times GB \to H(A \times B)$ is uniquely determined by its components at finite cardinals A and B, and it follows that the $*$-tensor product on $\mathcal{F}$ transports to convolution in $\mathbf{End}_\omega(\mathbf{Set}) \subset [\mathbf{Set}, \mathbf{Set}]$. Consequently, to give the maps ξ it suffices to give natural families of maps $FG(A) \times HK(B) \to (F * H)(G * K)(A \times B)$ for all finitary endofunctors F, G, H, K of $\mathbf{Set}$. We obtain these as the composites

$$FG(A) \times HK(B) \xrightarrow{u_{GA,KB}} (F * H)(GA \times KB) \xrightarrow{(F*H)(u_{A,B})} (F * H)(G * K)(A \times B) \ .$$

The axioms for a duoidal category are now easily verified by exploiting the universal property of convolution. □

Applying our general framework to the normal duoidal $\mathcal{F}$ thus yields a theory of commutativity for algebraic theories (= $\circ$-monoids in $\mathcal{F}$); in particular, we have notions of commuting tensor product of theories, and of commutative algebraic theory. In order to identify these notions with those of Definition 34, we will first identify categories of models in our framework as certain *functor $\mathcal{F}$-categories*.

5.3. $\mathcal{F}$-enriched category theory

The theory of categories enriched over $(\mathcal{F}, \circ, I)$ was the object of study of [18]; one of its main results identifies categories $\mathcal{E}$ with finite powers as $\mathcal{F}$-enriched categories admitting certain enriched *absolute colimits* [47]. For our purposes, the salient points of this result are summarised by:

Proposition 37. *To each category $\mathcal{E}$ with finite powers we may associate an $\mathcal{F}$-category $\underline{\mathcal{E}}$ with the same objects as $\mathcal{E}$ and hom-objects given by:*

$$\underline{\mathcal{E}}(X,Y)(n) = \mathcal{E}(X^n, Y) \ .$$

The assignation $\mathcal{E} \mapsto \underline{\mathcal{E}}$ underlies a 2-fully faithful embedding $\mathbf{CAT}_{fp} \to \mathcal{F}\text{-}\mathbf{CAT}$ of the 2-category of categories with finite powers and power-preserving functors into the 2-category of $\mathcal{F}$-enriched categories.

Proof. The composition morphisms $\underline{\mathcal{E}}(Y,Z) \circ \underline{\mathcal{E}}(X,Y) \to \underline{\mathcal{E}}(X,Z)$ of $\underline{\mathcal{E}}$ are induced by the dinatural family of maps

$$\begin{aligned} \mathcal{E}(Y^m, Z) \times \mathcal{E}(X^n, Y)^m &\to \mathcal{E}(X^n, Z) \\ (f, g_1, \dots, g_m) &\mapsto f(g_1, \dots, g_m) \end{aligned}$$

while the n-component of the identities map $I \to \underline{\mathcal{E}}(X,X)$ is the map $n \to \mathcal{E}(X^n, X)$ picking out the n projection maps $\pi_0, \dots, \pi_{n-1}$. For the remaining details, we refer the reader to [18, Proposition 3.8]. □

The additional aspect of the theory enabled by our framework for commutativity is the existence of a symmetric monoidal closed structure on $\mathcal{F}$-**Cat**. Indeed, the normal duoidal $\mathcal{F}$ is $*$-symmetric and $*$-closed; moreover, its underlying category $\mathcal{F}$ is locally presentable (in particular complete) and the $\circ$-tensor product thereon is accessible, whence by Proposition 56, $\mathcal{F}$-**Cat** is cocomplete and free $\mathcal{F}$-categories exist. So Proposition 18 applies to show that $\mathcal{F}$-**Cat** admits a symmetric monoidal closed structure $(\odot, \mathcal{I}, [–,–])$ given by the commuting tensor product of $\mathcal{F}$-categories.

Proposition 38. *For any category $\mathcal{E}$ with finite powers and any algebraic theory T, we have an isomorphism of $\mathcal{F}$-categories*

$$\underline{\mathbf{Mod}(T; \mathcal{E})} \cong [\Sigma T, \underline{\mathcal{E}}] \ .$$

Proof. To give an $\mathcal{F}$-functor $F \colon \Sigma T \to \underline{\mathcal{E}}$ is equally to give an object $X \in \mathcal{E}$ together with a map of $\circ$-monoids $f \colon T \to \underline{\mathcal{E}}(X,X)$ in $\mathcal{F}$; now using the definition of $\underline{\mathcal{E}}$ in Proposition 37, we see that this is precisely to give a T-model in $\mathcal{E}$. Given another $\mathcal{F}$-functor $G \colon \Sigma T \to \underline{\mathcal{E}}$, corresponding to a $\circ$-monoid map $g \colon T \to \underline{\mathcal{E}}(Y,Y)$, say, we may trace through the construction of Proposition 15 to find that the hom-object $[\Sigma T, \underline{\mathcal{E}}](F,G)$ is obtained by forming the hexagon:

$$\begin{tikzcd}
 & \underline{\mathcal{E}}(X,Y) \circ T \arrow[r, "1 \circ f"] & \underline{\mathcal{E}}(X,Y) \circ \underline{\mathcal{E}}(X,X) \arrow[dr, "m"] & \\
\underline{\mathcal{E}}(X,Y) * T \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & & & \underline{\mathcal{E}}(X,Y) \ , \\
 & T \circ \underline{\mathcal{E}}(X,Y) \arrow[r, "g \circ 1"'] & \underline{\mathcal{E}}(Y,Y) \circ \underline{\mathcal{E}}(X,Y) \arrow[ur, "m"'] &
\end{tikzcd}$$

and then taking the equalizer of the transposed parallel pair $\underline{\mathcal{E}}(X,Y) \rightrightarrows [T, \underline{\mathcal{E}}(X,Y)]$. The two sides of the preceding hexagon send a pair $(f \in \mathcal{E}(X^n, Y), \alpha \in Tm)$ to the respective composites

$$X^{nm} \cong (X^m)^n \xrightarrow{[\![\alpha]\!]_X^n} X^n \xrightarrow{f} Y \qquad \text{and} \qquad X^{nm} \cong (X^n)^m \xrightarrow{f^m} Y^m \xrightarrow{[\![\alpha]\!]_Y} Y$$

from which it follows that $[\Sigma T, \underline{\mathcal{E}}](F,G)(n)$ is given by the set of all $f \in \mathcal{E}(X^n, Y)$ such that $f \,.\, [\![\alpha]\!]_X^n = [\![\alpha]\!]_Y \,.\, f^m$ for all $\alpha \in Tm$; that is, by the set of T-model homomorphisms from X^n to Y. It is easy to see that this bijection respects composition, and so we have $\underline{\mathbf{Mod}(T;\mathcal{E})} \cong [\Sigma T, \underline{\mathcal{E}}]$ as required. □

Using this result, we may identify the notions of our general theory with those given by Definition 34.

Corollary 39. *An algebraic theory U is the tensor product of theories S and T if and only if $\Sigma U \cong \Sigma S \odot \Sigma T$ as $\mathcal{F}$-categories.*

Proof. As is well-known [37, Thm III.1.1 & III.1.2], the functor $\mathbf{AlgTh}^{\mathrm{op}} \to \mathbf{CAT}/\mathbf{Set}$ sending each theory T to the forgetful functor $\mathbf{Mod}(T;\mathbf{Set}) \to \mathbf{Set}$ is fully faithful; and so $\Sigma U \cong \Sigma S \odot \Sigma T$ if and only if $\mathbf{Mod}(U;\mathbf{Set}) \cong \mathbf{Mod}(S \odot T;\mathbf{Set})$ over $\mathbf{Set}$. But by the preceding result, we have that

$$\begin{aligned}\underline{\mathbf{Mod}(S \odot T;\mathbf{Set})} &\cong [\Sigma S \odot \Sigma T, \underline{\mathbf{Set}}] \cong [\Sigma S, [\Sigma T, \underline{\mathbf{Set}}]] \cong [\Sigma S, \underline{\mathbf{Mod}(T;\mathbf{Set})}] \\ &\cong \underline{\mathbf{Mod}(S;\mathbf{Mod}(T;\mathbf{Set}))} = \underline{\mathbf{Mod}(S,T;\mathbf{Set})} \end{aligned} \tag{5.2}$$

as categories over $\mathbf{Set}$; whence $\Sigma U \cong \Sigma S \odot \Sigma T$ if and only if U is the tensor product of S and T in the sense of Definition 34. □

Corollary 40. *An algebraic theory T is commutative in the sense of Definition 34 just when it is a commutative $\circ$-monoid in the sense of Definition 29.*

Proof. T is commutative just when there is a factorisation as in the diagram

$$\begin{tikzcd} & \mathbf{Mod}(T;\mathbf{Set}) \arrow[d, dashed] \arrow[ddl, bend right, "1"'] \arrow[ddr, bend left, "1"] & \\ & \mathbf{Mod}(T,T;\mathbf{Set}) \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ \mathbf{Mod}(T;\mathbf{Set}) & & \mathbf{Mod}(T;\mathbf{Set}) \,. \end{tikzcd}$$

By the identification (5.2) of $\mathbf{Mod}(T,T;\mathbf{Set})$ with $\mathbf{Mod}(T \odot T;\mathbf{Set})$, together with the full fidelity of the model functor $\mathbf{AlgTh}^{\mathrm{op}} \to \mathbf{CAT}/\mathbf{Set}$, this is equally to ask that the identity cospan $T \to T \leftarrow T$ of $\circ$-monoids factor through the universal commuting cospan $T \to T \odot T \leftarrow T$; which is to ask that T be a commutative $\circ$-monoid in $\mathcal{F}$. □

Note that the hypotheses of Example 24 apply to $\mathcal{F}$, so that there is a duoidal adjunction $(-) \cdot J \dashv \mathcal{F}(J, -) \colon \mathcal{F} \to \mathbf{Set}$ inducing, by Proposition 25, a monoidal adjunction $\mathcal{F}\text{-}\mathbf{Cat} \leftrightarrows \mathbf{Cat}$. Restricting this to one-object categories gives a monoidal adjunction $(\mathbf{AlgTh}, \odot) \leftrightarrows (\mathbf{Mon}, \times)$ between the category of algebraic theories and the category of monoids, whose left adjoint views a monoid as an algebraic theory with only unary operations, and whose right adjoint sends an algebraic theory to its monoid of unary operations. The monoidality of this adjunction tells us, in particular, that a monoid is commutative just when the associated algebraic theory is commutative; that the monoid of unary operations of any commutative algebraic theory is commutative; and that the theory associated to a product monoid $M \times N$ is the tensor product of M and N *qua* theories.

5.4. Explicit formulae

There are well-known explicit formulae which give the tensor product of two algebraic theories, and which characterise when an algebraic theory is commutative; see [16], for example. We conclude this section by showing how these formulae can be reconstructed from our general framework. To do so, we first calculate the interchange map $\xi\colon (X \circ Y) * (Z \circ W) \to (X * Z) \circ (Y * W)$ of the duoidal structure on $\mathcal{F}$. Such a map classifies a natural family of maps $(X \circ Y)(n) \times (Z \circ W)(m) \to [(X * Z) \circ (Y * W)](nm)$, which, expanding out the definitions, is equally a family of maps

$$Xk \times (Yn)^k \times Z\ell \times (Wm)^\ell \to \int^k (X * Z)k \times (Y * W)(nm)^k$$

natural in n and m and dinatural in k and ℓ; which we calculate to be given by

$$(x, y_1, \ldots, y_k, z, w_1, \ldots, w_\ell) \mapsto (u(x, z), u(y_1, w_1), \ldots, u(y_k, w_k)) \tag{5.3}$$

(where $u\colon Xk \times Z\ell \to (X * Z)(k\ell)$ and $u\colon Yn \times Wm \to (Y * W)(nm)$ are the universal maps into the convolution tensor). It follows that the map $\sigma\colon X * Y \to X \circ Y$ associated to the duoidal structure on $\mathcal{F}$ classifies the natural family

$$\begin{aligned} Xn \times Ym \to (X \circ Y)(nm) &= \int^k Xk \times Y(nm)^k \\ (f, g) &\mapsto (f, Y\alpha_1(g), \ldots, Y\alpha_n(g)) \end{aligned}$$

where here $\alpha_j\colon n \to nm$ is the injection defined by $\alpha_j(i) = (i, j)$. Dually, the map $\tau\colon X * Y \to Y \circ X$ classifies the natural family

$$\begin{aligned} Xn \times Ym \to (X \circ Y)(nm) &= \int^k Xk \times Y(nm)^k \\ (f, g) &\mapsto (g, X\beta_1(f), \ldots, X\beta_m(f)) \end{aligned}$$

where now $\beta_i\colon m \to nm$ is given by $\beta_i(j) = (i, j)$. Given these calculations, we now see that, if T is an algebraic theory, then a cospan $f\colon A \to T \leftarrow B\colon g$ in $\mathcal{F}$ is commuting just when, for each $\varphi = fa \in Tn$ and each $\psi = gb \in Tm$, we have that

$$\psi(\varphi(\pi_{11}, \ldots, \pi_{n1}), \ldots, \varphi(\pi_{1m}, \ldots, \pi_{nm})) = \varphi(\psi(\pi_{11}, \ldots, \pi_{1m}), \ldots, \psi(\pi_{n1}, \ldots, \pi_{nm}))$$

as nm-ary operations. In particular, we see that an algebraic theory T is commutative just when this equality holds for all pairs of operations φ, ψ in T; furthermore, the commuting tensor product $S \odot T$ of theories S and T is the quotient of the coproduct theory $S + T$ which imposes these equalities for all $\varphi \in Sn$ and $\psi \in Tm$.

6. Example: symmetric operads

Symmetric operads were introduced by May in [42]; while his interest was in the case of *topological* operads, one can define symmetric operads over any braided monoidal base $\mathcal{V}$. We consider here the case $\mathcal{V} = \mathbf{Set}$ of symmetric plain operads; one way of understanding these is as special kinds of algebraic theory, whose equations between derived operations are generated by ones involving the same variables on each side of the equality without omission or repetition.

The category of symmetric plain operads admits a monoidal structure known as the *Boardman–Vogt* monoidal structure [6]; on viewing symmetric operads as algebraic theories, their Boardman–Vogt tensor product is precisely their tensor product as theories described in the previous section. The purpose of this section is to exhibit this fact as a consequence of our general framework for commutativity.

6.1. Species and symmetric operads

A *species* is a functor $X\colon \mathbb{P} \to \mathbf{Set}$, where $\mathbb{P}$ is the category of finite cardinals and bijective mappings between them; a symmetric operad is a monoid with respect to a suitable *substitution* tensor product on the category $\mathcal{S}p = [\mathbb{P}, \mathbf{Set}]$ of species.

To describe this, note first that sum and product of finite cardinals induce convolution monoidal structures on $\mathcal{S}p$, which we denote by $\oplus$ and $*$ respectively; the units of these monoidal structures are the respective representable functors $O = y_0$ and $J = y_1$. The sum monoidal structure on $\mathbb{P}$ in fact exhibits it as the free symmetric monoidal category on the object 1; it follows by [25, Theorem 5.1] that $(\mathcal{S}p, \oplus, O)$ is the *free symmetric monoidal closed cocomplete category* on the object y_1. Writing **CoctsStrMon** for the 2-category of symmetric monoidal closed cocomplete categories and cocontinuous symmetric strong monoidal functors, this is to say that, for any $\mathcal{V} \in \mathbf{CoctsStrMon}$, the functor $\mathbf{CoctsStrMon}(\mathcal{S}p, \mathcal{V}) \to \mathcal{V}$ which evaluates at the object y_1 is an equivalence of categories.

In particular, we have an equivalence $\mathbf{CoctsStrMon}(\mathcal{S}p, \mathcal{S}p) \simeq \mathcal{S}p$, under which the composition monoidal structure of the left-hand side transports to yield the *substitution* tensor product $\circ$ on $\mathcal{S}p = [\mathbb{P}, \mathbf{Set}]$. This has as unit the representable $I = y_1$, and binary tensor

$$X \circ Y = \int^{k \in \mathbb{P}} Xk \times Y^{\oplus k} \tag{6.1}$$

where here we write $Y^{\oplus k}$ for the k-fold tensor product $Y \oplus \cdots \oplus Y$. Observe that to give a natural transformation $\alpha\colon X \circ Y \to Z$ in $\mathcal{S}p$ is equally to give a family of maps

$$\tilde{\alpha}\colon Xk \times Ym_1 \times \cdots \times Ym_k \to Z(\Sigma_i m_i) \tag{6.2}$$

natural in the m_i's and dinatural in k.

Now a *symmetric operad* is a $\circ$-monoid (T, η, μ) in $\mathcal{S}p$. Like in the preceding section, we refer to elements $f \in Tn$ as *n-ary operations* of T; we write $\mathrm{id} \in T1$ for the element classified by $\eta\colon y_1 \to T$; and given $f \in Tn$ and $g_i \in Tm_i$ (for $i = 1, \ldots, n$), we write $f(g_1, \ldots, g_n) \in T(\Sigma_i m_i)$ for their image under $\tilde{\mu}$ as in (6.2).

6.2. Symmetric operads and theories

As indicated above, we wish to identify symmetric operads with certain kinds of algebraic theory. To this end, let us write $H\colon \mathbb{P} \to \mathbb{F}$ for the (non-full) inclusion functor, and write $\mathrm{Th}\colon \mathcal{S}p \to \mathcal{F}$ for the left Kan extension functor $\mathrm{Lan}_H\colon [\mathbb{P}, \mathbf{Set}] \to [\mathbb{F}, \mathbf{Set}]$. We now have:

Proposition 41. *The functor* $\mathrm{Th}\colon \mathcal{S}p \to \mathcal{F}$ *is faithful and full on isomorphisms, and is strong monoidal as a functor*

$$(\mathcal{S}p, \oplus) \to (\mathcal{F}, \times) \qquad \textit{and} \qquad (\mathcal{S}p, *) \to (\mathcal{F}, *) \qquad \textit{and} \qquad (\mathcal{S}p, \circ) \to (\mathcal{F}, \circ)\ ,$$

and symmetric in the first two cases. It follows that Th *exhibits $\mathcal{S}p$ as equivalent to a (non-full) subcategory of $\mathcal{F}$ closed under the $\circ$ and $*$ monoidal structures.*

Proof. For fidelity and fullness on isomorphisms, we refer to [28, Proposition 1] or [51, Proposition 10.9]. For strong symmetric monoidality as a functor $(\mathcal{S}p, \oplus) \to (\mathcal{F}, \times)$ and $(\mathcal{S}p, *) \to (\mathcal{F}, *)$, we observe that $H\colon \mathbb{P} \to \mathbb{F}$ is strong symmetric monoidal with respect to $+$ and $\times$; whence $\mathrm{Th} = \mathrm{Lan}_H\colon [\mathbb{P}, \mathbf{Set}] \to [\mathbb{F}, \mathbf{Set}]$ is strong symmetric monoidal with respect to the corresponding convolution tensor products, which are $\oplus$ and $*$ on $\mathcal{S}p$, and $\times$ and $*$ on $\mathcal{F}$; here we use [10, §5] to see that convolution with respect to coproduct on

$\mathbb{F}$ is in fact cartesian product in $\mathcal{F}$. For strong monoidality of Th as a functor $(\mathcal{S}p, \circ) \to (\mathcal{F}, \circ)$, we refer to [28, Section 2.1(iv)].

For the final clause of the proposition, let $\mathcal{F}' \subset \mathcal{F}$ be the subcategory whose objects are those isomorphic to ones in the image of Th, and whose morphisms $f: X \to Y$ are those for which there is an isomorphism $\mathrm{Th}(g) \cong f$ in $\mathcal{F}^{\mathbf{2}}$. It is easy to show using fidelity and fullness on isomorphisms that the corestriction $\mathcal{S}p \to \mathcal{F}'$ of Th is an equivalence of categories, as desired. $\square$

Using this proposition, we can identify symmetric operads with certain kinds of algebraic theory. In order to identify the Boardman–Vogt tensor product of symmetric operads with the tensor product of the associated theories, we will need to prove that $\mathcal{S}p$ is in fact equivalent to a sub-duoidal category of $\mathcal{F}$. To do so, it will be convenient to introduce the notion of a *rig category*.

Definition 42. A *rig category* [36] $(\mathcal{V}, \oplus, O, \otimes, I)$ is a category $\mathcal{V}$ equipped with a symmetric monoidal structure $(\oplus, O)$ and a monoidal structure $(\otimes, I)$, together with a lifting of the functor $\mathcal{V} \to [\mathcal{V}, \mathcal{V}]$ sending X to $X \otimes (-)$ to a functor $\mathcal{V} \to \mathbf{OpMon}_{\oplus}(\mathcal{V}, \mathcal{V})$ which is opmonoidal with respect to the $\oplus$-tensor product on $\mathcal{V}$ and the pointwise $\oplus$-tensor product on $\mathbf{OpMon}_{\oplus}(\mathcal{V}, \mathcal{V})$.

In more elementary terms, $\mathcal{V}$ is a rig category when its two monoidal structures $\oplus$ and $\otimes$ are related by nullary constraint morphisms $\alpha_\ell: O \otimes X \to O$ and $\alpha_r: X \otimes O \to O$ and binary constraint morphisms

$$(X \oplus Y) \otimes Z \xrightarrow{\nu_\ell} (X \otimes Z) \oplus (Y \otimes Z) \quad \text{and} \quad X \otimes (Y \oplus Z) \xrightarrow{\nu_r} (X \otimes Y) \oplus (X \otimes Z)$$

satisfying suitable axioms. In fact, both $\mathcal{S}p$ and $\mathcal{F}$ are rig categories as a consequence of the following result:

Proposition 43. *Let $\mathcal{A}$ be a small category and $\mathcal{V}$ a cartesian closed cocomplete one. If $(\mathcal{A}, \oplus, O, \otimes, I)$ is a rig category, then so is $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ equipped with the corresponding convolution monoidal structures.*

Proof. Writing $\oplus, O, \otimes, I$ also for the convolution tensors on $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, we see that to give the constraint maps $\alpha_\ell: O \otimes X \to O$ in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is equally to give maps $\tilde{\alpha}_\ell: XV \to \mathcal{V}(O \otimes V, O)$ in $\mathcal{V}$, natural in V; which we obtain as the composites

$$XV \xrightarrow{!} 1 \xrightarrow{(\alpha_\ell)_V} \mathcal{V}(O \otimes V, O)\ .$$

The constraint maps $\nu_\ell: X \otimes (Y \oplus Z) \to (X \otimes Y) \oplus (X \otimes Z)$ at $X, Y, Z \in [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ must by the universal property of convolution be induced by natural families

$$Xn \times Ym \times Zk \to [(X \otimes Y) \oplus (X \otimes Z)](n \otimes (m \oplus k))$$

in $\mathcal{V}$. We obtain these as the composites

$$\begin{aligned} Xn \times Ym \times Zk &\xrightarrow{(\pi_1,\pi_2,\pi_1,\pi_3)} Xn \times Ym \times Xn \times Zk \\ &\xrightarrow{u \times u} (X \otimes Y)(n \otimes m) \times (X \otimes Z)(n \otimes k) \\ &\xrightarrow{u} [(X \otimes Y) \oplus (X \otimes Z)]((n \otimes m) \oplus (n \otimes k)) \\ &\xrightarrow{[(X\otimes Y)\oplus(X\otimes Z)](\nu_\ell)} [(X \otimes Y) \oplus (X \otimes Z)](n \otimes (m \oplus k))\ . \end{aligned}$$

We define α_r and ν_r similarly; the coherence axioms for a rig now follow from those in $\mathcal{A}$ and the universal property of convolution. $\square$

Since both $\mathbb{P}^{\mathrm{op}}$ and $\mathbb{F}^{\mathrm{op}}$ are rig categories under disjoint union and product of finite cardinals, we conclude that $(\mathcal{S}p, \oplus, O, *, J)$ and $(\mathcal{F}, \times, 1, *, J)$ are both rig categories; since $H^{\mathrm{op}}\colon \mathbb{P}^{\mathrm{op}} \to \mathbb{F}^{\mathrm{op}}$ is a strong morphism of rig categories, so too is $\mathrm{Th}\colon \mathcal{S}p \to \mathcal{F}$.

Proposition 44. *$(\mathcal{S}p, *, J, \circ, I)$ is a normal duoidal category and $\mathrm{Th}\colon \mathcal{S}p \to \mathcal{F}$ a strong duoidal functor; whence Th exhibits $\mathcal{S}p$ as equivalent to a sub-duoidal category of $\mathcal{F}$.*

Proof. As in Proposition 36, the units J and I of the $*$ and $\circ$ monoidal structures on $\mathcal{S}p$ are both the representable y_1, and so the only difficulty lies in defining the interchange maps $\xi\colon (X \circ Y) * (W \circ Z) \to (X * W) \circ (Y * Z)$. Since $*$ is cocontinuous in each variable, and $\circ$ is cocontinuous in its first variable, it is enough to consider the case where X and W are representable; we must thus define maps $(y_n \circ Y) * (y_m \circ Z) \to (y_n * y_m) \circ (Y * Z) \cong y_{nm} \circ (Y * Z)$ natural in $n, m \in \mathbb{P}$. By definition of $\circ$, this is equally to give natural maps $Y^{\oplus n} * Z^{\oplus m} \to (Y * Z)^{\oplus nm}$. We obtain these using the rig structure on $\mathcal{S}p$, as the composites

$$Y^{\oplus n} * Z^{\oplus m} \xrightarrow{\nu_r} (Y^{\oplus n} * Z)^{\oplus m} \xrightarrow{(\nu_\ell)^{\oplus m}} ((Y * Z)^{\oplus n})^{\oplus m} = (Y * Z)^{\oplus nm} \tag{6.3}$$

where the two non-identity maps are built from repeated applications of the opmonoidal constraints ν_r and ν_ℓ respectively.

Leaving aside the duoidal coherence axioms in $\mathcal{S}p$ for the moment, we next show strong duoidality of $\mathrm{Th}\colon \mathcal{S}p \to \mathcal{F}$. Preservation of unit coherences is straightforward, so it suffices to show that each square of the form:

$$\begin{tikzcd}
(\mathrm{Th}(X) \circ \mathrm{Th}(Y)) * (\mathrm{Th}(W) \circ \mathrm{Th}(Z)) \arrow[r, "\xi"] \arrow[d, "\cong"'] & (\mathrm{Th}(X) * \mathrm{Th}(W)) \circ (\mathrm{Th}(Y) * \mathrm{Th}(Z)) \\
\mathrm{Th}((X \circ Y) * (W \circ Z)) \arrow[r, "\mathrm{Th}(\xi)"] & \mathrm{Th}((X * W) \circ (Y * Z)) \arrow[u, "\cong"']
\end{tikzcd}$$

commutes in $\mathcal{F}$. Note that, since Th is faithful, once we have this, we are done, since we may deduce the duoidal coherence axioms in $\mathcal{S}p$ from those in $\mathcal{F}$. To show the required commutativity, observe that all vertices in the preceding diagram are cocontinuous in the variables X and W; so it is enough to consider the case where $X = y_n$ and $W = y_m$. Using again the definitions (5.1) and (6.1) of the $\circ$ tensor products, this means showing that each square

$$\begin{tikzcd}
\mathrm{Th}(Y)^n * \mathrm{Th}(Z)^m \arrow[r, "\xi"] \arrow[d, "\cong"'] & (\mathrm{Th}(Y) * \mathrm{Th}(Z))^{nm} \\
\mathrm{Th}(Y^{\oplus n} * Z^{\oplus m}) \arrow[r, "\mathrm{Th}(\xi)"] & \mathrm{Th}((Y * Z)^{\oplus nm}) \arrow[u, "\cong"']
\end{tikzcd}$$

is commutative. Since Th is a strong morphism of rig categories, the lower composite is the analogue of the map (6.3) for the rig $(\mathcal{F}, \times, *)$; as such, it is the map whose (i,j)th projection is $\pi_i * \pi_j\colon \mathrm{Th}(Y)^n * \mathrm{Th}(Z)^m \to \mathrm{Th}(Y) * \mathrm{Th}(Z)$; which, comparing with the formula (5.3) for interchange in $\mathcal{F}$, is precisely the upper map. $\square$

Remark 45. The normal duoidal structure on $\mathcal{S}p = [\mathbb{P}, \mathbf{Set}]$ described by this proposition can in fact be constructed on $[\mathbb{P}, \mathcal{V}]$ for any cocomplete cartesian closed $\mathcal{V}$; for the case of $\mathcal{V}$ being the category of simplicial sets, the interchange maps of this duoidal structure were described in [12, Proposition 1.20]. Note that if $\mathcal{V}$ is a non-cartesian symmetric monoidal category, then $[\mathbb{P}, \mathcal{V}]$ bears the two monoidal structures $(\circ, I)$ and $(*, J)$ but is *not* duoidal; the essential problem is that Proposition 43 fails in this case, which obstructs the construction of the interchange maps.

6.3. The Boardman–Vogt tensor product

The normal duoidal structure on the locally presentable $\mathcal{S}p$ just described is $*$-symmetric and $*$-closed and has a $\circ$-accessible tensor product, whence by Propositions 56 and 18 as before, $\mathcal{S}p$-**Cat** admits the commuting monoidal structure $(\odot, \mathcal{I})$. In particular, restricting to the one-object case, we obtain a tensor product of symmetric operads on **Set**, called the *Boardman–Vogt* tensor product. By transcribing the explicit calculations of Section 5.4 above, we re-find the well-known formula describing this tensor product: given operads $\mathcal{O}$ and $\mathcal{P}$, their tensor is obtained from the operadic coproduct $\mathcal{O} + \mathcal{P}$ by quotienting out by the equalities

$$\psi(\varphi, \dots, \varphi) = \varphi(\psi, \dots, \psi) \cdot \sigma$$

for each $\psi \in \mathcal{O}(n) \subset (\mathcal{O} + \mathcal{P})(n)$ and $\varphi \in \mathcal{P}(m) \subset (\mathcal{O} + \mathcal{P})(m)$; here $(-) \cdot \sigma$ indicates the action on $(\mathcal{O} + \mathcal{P})(nm)$ of the symmetry $\sigma \colon nm \to mn$.

The strong duoidal $\mathrm{Th} \colon \mathcal{S}p \to \mathcal{F}$ of the preceding proposition has right adjoint given by restriction along $H \colon \mathbb{P} \to \mathbb{F}$, and so is the left adjoint of a duoidal adjunction $\mathrm{Th} \dashv [H, 1] \colon \mathcal{F} \to \mathcal{S}p$; whence by Proposition 25, $\mathrm{Th}_* \colon (\mathcal{S}p\text{-}\mathbf{Cat}, \odot) \to (\mathcal{F}\text{-}\mathbf{Cat}, \odot)$ is a strong monoidal 2-functor—and so sends the Boardman–Vogt tensor product of symmetric operads to the tensor product of theories, as claimed above.

7. Normalizing duoidal categories

Before giving our remaining examples, we break off briefly in order to describe a construction which will be useful in producing them. This construction assigns to any reasonably well-behaved duoidal category with non-isomorphic units I and J a *normalization* in which I and J are forced to coincide. It involves taking bimodules over the bimonoid I, and is in fact an instance of a more general bimodule construction, which we now explain.

7.1. Bimodules over a bimonoid

Suppose that the maps $e \colon J \to M \leftarrow M * M \colon m$ and $u \colon I \leftarrow M \to M \circ M \colon d$ exhibit M as a bimonoid in the duoidal $\mathcal{V}$. We write $\mathbf{Bimod}_M$ for the category of algebras for the monad $M * (-) * M$ on $\mathcal{V}$, and call its objects *M-bimodules*. By exploiting the $\circ$-comonoid structure on M, we may lift the $\circ$-monoidal structure on $\mathcal{V}$ to one on $\mathbf{Bimod}_M$, whose unit is I with the action

$$M * I * M \xrightarrow{u*I*u} I * I * I \xrightarrow{\mu \cdot (I*\mu)} I$$

and whose binary tensor of $a \colon M * A * M \to A$ and $b \colon M * B * M \to B$ is $A \circ B$ equipped with the following action—wherein we temporarily write $\circ$ as juxtaposition:

$$M * AB * M \xrightarrow{d*AB*d} MM * AB * MM \xrightarrow{\xi \cdot (\xi * I)} (M * A * M)(M * B * M) \xrightarrow{a*b} AB \ .$$

Suppose now that $\mathcal{V}$ admits reflexive coequalizers which are preserved by $*$ in each variable. Under these circumstances, $\mathbf{Bimod}_M$ also admits the tensor product $*_M$, whose unit is M itself with the regular action, and whose binary tensor, constructed by a reflexive coequalizer $A * M * B \rightrightarrows A * B \twoheadrightarrow A *_M B$, classifies M-bilinear maps. We claim that, in fact, $(\mathbf{Bimod}_M, *_M, M, \circ, I)$ is a duoidal category.

The constraint maps in $\mathbf{Bimod}_M$ are obtained as follows. We take υ and γ to be $u \colon M \to I$ and $d \colon M \to M \circ M$ respectively; these are easily seen to be maps of M-bimodules. To define μ, we observe that $\mu \colon I * I \to I$ in $\mathcal{V}$ is I-bilinear, and so by restriction along the $*$-monoid map $u \colon M \to I$ also M-bilinear; it thus descends to the required $\mu\colon I *_M I \to I$ in $\mathbf{Bimod}_M$. Finally, we must construct the interchange maps; so let $A, B, C, D \in \mathbf{Bimod}_M$, and consider the diagram

$$\begin{tikzcd}
(A \circ B) * (C \circ D) \arrow[r, "\xi"] \arrow[d, "q"'] & (A * C) \circ (B * D) \arrow[d, "q \circ q"] \\
(A \circ B) *_M (C \circ D) \arrow[r, dashed] & (A *_M C) \circ (B *_M D)
\end{tikzcd}$$

in $\mathcal{V}$, where the q's to the left and right are the universal M-bilinear maps. A straightforward calculation from the duoidal axioms shows that the upper composite is also a M-bilinear map; whence there is a unique induced map in $\mathbf{Bimod}_M$ as displayed, which we take to be the component of ξ for $\mathbf{Bimod}_M$. Now each of the duoidal axioms in $\mathbf{Bimod}_M$ is either exactly the corresponding axiom in $\mathcal{V}$, or else a direct consequence of it after descending along the regular epimorphisms q.

7.2. Normalization

If we specialize to the case of the preceding construction where M is the unit object I made into a bimonoid via the structure maps $v\colon J \to I \leftarrow I * I\colon \mu$ and $1_I\colon I \leftarrow I \to I \circ I\colon \lambda_I$, we see that the induced duoidal structure on $\mathbf{Bimod}_I$ has its structure cell v an *identity*; as such, it is normal, and so we can give:

Definition 46. If $\mathcal{V}$ is duoidal with reflexive coequalizers preserved by $*$ in each variable, then its *normalization* is the normal duoidal category $\mathrm{N}(\mathcal{V}) = (\mathbf{Bimod}_I, *_I, I, \circ, I)$.

If $\mathcal{V}$ is $*$-biclosed with equalizers, then the $*_M$-monoidal structure on $\mathbf{Bimod}_M$ is also biclosed (see [3] for example); in particular, this means that the $*$-biclosedness of a duoidal structure is preserved under normalization. On the other hand, the $*$-braidedness need *not* be preserved; for this we need to modify the construction.

Suppose that $\mathcal{V}$ is a $*$-braided monoidal category, and that M is a bimonoid in $\mathcal{V}$ with commutative underlying $*$-monoid. There is a full embedding $\mathbf{Mod}_M \to \mathbf{Bimod}_M$ of the category of left M-modules into the category of M-bimodules which sends $\ell\colon M * A \to A$ to the bimodule $\ell\colon M * A \to A \leftarrow A * M\colon \ell c$. The image of $\mathbf{Mod}_M$ under this embedding is closed under both the tensor product $*_M$ and also, by (2.1), under the lifted $\circ$-monoidal structure; whence $\mathbf{Mod}_M$ acquires a duoidal structure $(*_M, \circ)$. Moreover, for any A and B in $\mathbf{Mod}_M$, the braidings $c\colon A * B \to B * A$ descend to maps $c\colon A *_M B \to B *_M A$ in $\mathbf{Mod}_M$ which witness the duoidal structure as $*$-braided. As the terminal $*$-monoid I in M is commutative, we may define:

Definition 47. Let $\mathcal{V}$ be $*$-braided duoidal with reflexive coequalizers preserved by $*$ in each variable. The *braided normalization* $\mathrm{N}_c(\mathcal{V})$ is the $*$-braided normal duoidal category $(\mathbf{Mod}_I, *_I, I, \circ, I)$.

7.3. Normalization and commutativity

With an eye to our applications, it will be convenient to have a description of commutativity in $\mathrm{N}(\mathcal{V})$ and $\mathrm{N}_c(\mathcal{V})$ in terms of data in $\mathcal{V}$. In fact it suffices to consider only $\mathrm{N}(\mathcal{V})$ as $\mathrm{N}_c(\mathcal{V})$ sits inside this as a full sub-duoidal category.

Lemma 48. *Let $\mathcal{V}$ be duoidal with normalization $\mathrm{N}(\mathcal{V})$, and let C be a $\circ$-monoid in $\mathrm{N}(\mathcal{V})$. A cospan $f\colon A \to C \leftarrow B\colon g$ in $\mathrm{N}(\mathcal{V})$ is commuting just when the diagram*

$$\begin{tikzcd}[column sep=small]
 & A \circ B \arrow[r, "f \circ g"] & C \circ C \arrow[dr, "m"] & \\
A * B \arrow[ur, "\bar{\sigma}"] \arrow[dr, "\bar{\tau}"'] & & & C \\
 & B \circ A \arrow[r, "g \circ f"] & C \circ C \arrow[ur, "m"'] &
\end{tikzcd} \tag{7.1}$$

commutes in $\mathcal{V}$, where $\bar{\sigma}$ and $\bar{\tau}$ are the composites

$$A * B \cong (A \circ I) * (I \circ B) \xrightarrow{\xi} (A * I) \circ (I * B) \xrightarrow{r \circ \ell} A \circ B$$

$$A * B \cong (I \circ A) * (B \circ I) \xrightarrow{\xi} (I * B) \circ (A * I) \xrightarrow{\ell \circ r} A \circ B$$

constructed from interchange in $\mathcal{V}$ and the I-bimodule structures on A and B.

Proof. Since the forgetful $\mathbf{Bimod}_I \to \mathcal{V}$ is faithful, to ask that (4.4) commutes in $\mathbf{Bimod}_I$ is equally to ask that its precomposition with the epimorphism $A * B \twoheadrightarrow A *_I B$ commutes in $\mathcal{V}$; which is exactly to ask that (7.1) commutes. □

Corollary 49. *Let $\mathcal{V}$ be duoidal with normalization $\mathrm{N}(\mathcal{V})$. The forgetful $U \colon \mathrm{N}(\mathcal{V}) \to \mathcal{V}$ induces an isomorphism between the category of commutative $\circ$-monoids in $\mathrm{N}(\mathcal{V})$ and the category of duoids in $\mathcal{V}$.*

Proof. By Proposition 30, it suffices to show that U induces an isomorphism between the categories of duoids in $\mathrm{N}(\mathcal{V})$ and in $\mathcal{V}$. To give a $*_I$-monoid in $\mathrm{N}(\mathcal{V})$ is, by a well-known calculation, to give an $*$-monoid C in $\mathcal{V}$ together with an $*$-monoid morphism $I \to C$. It follows that $\circ$-monoids in $\mathbf{Mon}_{*_I}(\mathrm{N}(\mathcal{V}))$ may be identified via U with $\circ$-monoids in $\mathbf{Mon}_*(\mathcal{V})$. □

8. Example: strong and commutative monads

Another way of viewing the algebraic theories of Section 5 is as finitary monads on **Set**. All aspects of our framework for commutativity may be re-expressed in this context in a purely monad-theoretic form; this then allows them to be generalised further to the context of *strong monads* on any monoidal category $\mathcal{V}$. In this monad-theoretic setting, the theory of commutativity is due to [34]; in this section, we show how to reconstruct it from our general framework.

8.1. The duoidal category of κ-accessible endofunctors

In this section, we suppose that $\mathcal{V}$ is a monoidal biclosed category which is *locally κ-presentable as a closed category* [32]; this means that the category $\mathcal{V}$ is locally κ-presentable, with the κ-presentable objects being closed under nullary and binary tensor. We will establish a framework for commutativity for monads on $\mathcal{V}$ which are *κ-accessible*—meaning that their underlying endofunctor preserves κ-filtered colimits. This covers most cases of practical interest; in Section 9.4, we will see another approach which can deal with non-accessible monads on non-locally presentable categories.

We let $\mathbf{End}_\kappa(\mathcal{V})$ denote the category of κ-accessible endofunctors of $\mathcal{V}$. If $\mathcal{V}_\kappa$ is a small skeleton of the subcategory of κ-presentable objects in $\mathcal{V}$, then restriction and left Kan extension along the inclusion $\mathcal{V}_\kappa \to \mathcal{V}$ establishes an equivalence between $\mathbf{End}_\kappa(\mathcal{V})$ and $[\mathcal{V}_\kappa, \mathcal{V}]$. Since $\mathcal{V}_\kappa$ is closed under the monoidal structure of $\mathcal{V}$, Day convolution gives a biclosed monoidal structure on $[\mathcal{V}_\kappa, \mathcal{V}]$ and so, by transporting across the equivalence, a biclosed monoidal structure $(*, J)$ on $\mathbf{End}_\kappa(\mathcal{V})$. Since κ-accessible endofunctors compose, $\mathbf{End}_\kappa(\mathcal{V})$ also has its composition monoidal structure $(\circ, I)$; now generalising Proposition 36, we have:

Proposition 50. $(\mathbf{End}_\kappa(\mathcal{V}), *, J, \circ, I)$ *is a duoidal category.*

Proof. If $F, G, H \in \mathbf{End}_\kappa(\mathcal{V})$, then any natural family $FA \otimes GB \to H(A \otimes B)$ is uniquely determined by its components at κ-presentable A and B; it follows that the $*$-tensor product on $\mathbf{End}_\kappa(\mathcal{V})$ is in fact convolution in $[\mathcal{V}, \mathcal{V}]$. We now use this fact in constructing the duoidal structure maps.

First, by the universal property of convolution, $*$-monoids in $\mathbf{End}_\kappa(\mathcal{V})$ correspond to lax monoidal functors; so in particular, the strict monoidal identity functor is an $*$-monoid with structure maps $\upsilon\colon J \to I$ and $\mu\colon I * I \to I$. Next, to give $\gamma\colon J \to J \circ J$ is equally well, by the universal property of J, to give a map $I \to JJI$ in $\mathcal{V}$; which we take to be the composite $Jj \cdot j\colon I \to JI \to JJI$. It remains to give the interchange maps $\xi\colon (F \circ G) * (H \circ K) \to (F * H) \circ (G * K)$, which are equally well specified by giving natural families of maps $FG(A) \otimes HK(B) \to (F * H)(G * K)(A \otimes B)$. We obtain these as the composites

$$FG(A) \otimes HK(B) \xrightarrow{u_{GA,KB}} (F * H)(GA \otimes KB) \xrightarrow{(F*H)(u_{A,B})} (F * H)(G * K)(A \otimes B)\ .$$

The axioms for a duoidal category are now easily verified by exploiting the universal property of convolution. $\square$

8.2. Bistrong endofunctors

When $\mathcal{V} = \mathbf{Set}$ and $\kappa = \omega$, the duoidal structure induced by the preceding result on $\mathbf{End}_\omega(\mathbf{Set})$ corresponds under the equivalence with $[\mathbb{F}, \mathbf{Set}]$ to the duoidal structure of Proposition 36; in particular, it is *normal*. We may explicitly calculate the isomorphisms $I * F \to F$ and $F * I \to F$ as corresponding to the natural families

$$\begin{array}{ccc} t\colon X \times FY \to F(X \times Y) & \quad\text{and} & \bar{t}\colon FX \times Y \to F(X \times Y) \\ (x, y) \mapsto F(\lambda z.\,(x, z))(y) & & (x, y) \mapsto F(\lambda z.\,(z, y))(x) \end{array}$$

expressing the fact that any finitary endofunctor of **Set** has a canonical *strength* and *costrength* [34]. The same argument pertains when $\mathcal{V} = \mathbf{Set}$ and κ is any infinite regular cardinal, and also in the (uninteresting) cases where $\mathcal{V} = \mathbf{2}$ or $\mathcal{V} = \mathbf{1}$; but for any other $\mathcal{V}$, the map $\upsilon\colon J \to I$ of the preceding proposition is not an isomorphism, and so we must apply the normalization of Section 7. In this case, passing to the category of bimodules over the $*$-monoid I means explicitly equipping $F \in \mathbf{End}_\kappa(\mathcal{V})$ with strength and costrength maps

$$t\colon A \otimes FB \to F(A \otimes B) \qquad \text{and} \qquad \bar{t}\colon FA \otimes B \to F(A \otimes B)\ ,$$

natural in A and B and satisfying evident associativity and unit axioms. If we call such an F *bistrong*, then the normalization of the duoidal $\mathbf{End}_\kappa(\mathcal{V})$ is given by the category $\mathbf{BiStr}_\kappa(\mathcal{V})$ of κ-accessible bistrong endofunctors and bistrength-preserving natural transformations, with the two monoidal structures being composition $(\circ, I)$ (where the composition of strengths is the obvious one) and the quotiented convolution $(*_I, I)$, whose value $F *_I G$ is obtained by identifying in $F * G$ the costrength of F and the strength of G.

Now a $\circ$-monoid (T, η, μ) in $\mathbf{BiStr}_\kappa(\mathcal{V})$ is a bistrong κ-accessible monad on $\mathcal{V}$; and by the explicit description of ξ in $\mathbf{End}_\kappa(\mathcal{V})$ given above, together with Lemma 48, we see that a cospan $f\colon M \to T \leftarrow N\colon g$ in $\mathbf{BiStr}_\kappa(\mathcal{V})$ is commuting just when the diagram

$$\begin{tikzcd}
& M(X \otimes NY) \arrow[r, "Mt"] & MN(X \otimes Y) \arrow[r, "fg_{X \otimes Y}"] & TT(X \otimes Y) \arrow[dr, "\mu"] & \\
MX \otimes NY \arrow[ur, "\bar{t}"] \arrow[dr, "t"'] & & & & T(X \otimes Y) \\
& N(MX \otimes Y) \arrow[r, "N\bar{t}"] & NM(X \otimes Y) \arrow[r, "gf_{X \otimes Y}"] & TT(X \otimes Y) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

commutes for all $X, Y \in \mathcal{V}$, while T itself is commutative just when

$$\begin{tikzcd}
& T(X \otimes TY) \arrow[r, "Tt"] & TT(X \otimes Y) \arrow[dr, "\mu"] & \\
TX \otimes TY \arrow[ur, "\bar{t}"] \arrow[dr, "t"'] & & & T(X \otimes Y) \\
& T(TX \otimes Y) \arrow[r, "T\bar{t}"] & TT(X \otimes Y) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

commutes for all $X, Y \in \mathcal{V}$. We have thus reconstructed the classical notion of *commutative monad* [34] on a monoidal category $\mathcal{V}$.

If the monoidal $\mathcal{V}$ is braided, then the duoidal $\mathbf{End}_\kappa(\mathcal{V})$ is $*$-braided, and so we have the option of taking the *braided* normalization of $\mathbf{End}_\kappa(\mathcal{V})$. This is the $*$-braided normal duoidal category of strong κ-accessible endofunctors of $\mathcal{V}$, which by [34, Section 1] and [32, Remark 7.7] is equally the category $\mathcal{V}$-$\mathbf{End}_\kappa(\mathcal{V})$ of κ-accessible $\mathcal{V}$-enriched endofunctors of $\mathcal{V}$. The two tensor products giving the duoidal structure now admit direct description as composition $(\circ, I)$ and $\mathcal{V}$*-enriched* convolution $(*_\mathcal{V}, I)$. A $\circ$-monoid is thus a κ-accessible $\mathcal{V}$-monad on $\mathcal{V}$, while the notions of commuting map and commutative $\mathcal{V}$-monad are exactly as above, where the strength t and costrength $\bar{t}$ are derived from the $\mathcal{V}$-enrichment via the formulae of [34, Section 1].

8.3. Commuting tensor product

Since $\mathbf{End}_\kappa(\mathcal{V})$ is equivalent to $[\mathcal{V}_\kappa, \mathcal{V}]$, it is locally presentable; since $\mathbf{BiStr}_\kappa(\mathcal{V})$ is the category of algebras for a cocontinuous monad on $\mathbf{End}_\kappa(\mathcal{V})$ it is thus also locally presentable, and in particular complete. The duoidal structure thereon is $*$-biclosed, while the $\circ$-tensor product preserves κ-filtered colimits in each variable; whence $\mathbf{End}_\kappa(\mathcal{V})$-$\mathbf{Cat}$ is cocomplete and free $\mathbf{End}_\kappa(\mathcal{V})$-categories exist by Proposition 56. So by Proposition 18, $\mathbf{End}_\kappa(\mathcal{V})$-$\mathbf{Cat}$ admits the commuting tensor product $\odot$. In particular, we have a tensor product of κ-accessible monads on $\mathcal{V}$; the universal property of this tensor product was first described in [23, Definition 2].

Note that the restriction to κ-accessible monads is essential to ensure existence of the tensor product; for *unbounded* monads—ones which, like the power-set monad on **Set**, are not κ-accessible for *any* κ—it may be that their commuting tensor product fails to exist entirely; see [19, Theorem 22].

Finally, let us note the force of Proposition 30 in the context of this example. Together with Lemma 48, it tells us that commutative $\circ$-monoids in $\mathbf{BiStr}_\kappa(\mathcal{V})$ are the same thing as duoids in $\mathbf{End}_\kappa(\mathcal{V})$; while in the $*$-braided case, it tells us that commutative $\circ$-monoids in $\mathcal{V}$-$\mathbf{End}_\kappa(\mathcal{V})$ are the same as $*$-commutative monoids in $\mathbf{End}_\kappa(\mathcal{V})$. This latter case reconstructs the main Theorem 3.2 of [34]: that a strong κ-accessible monad on a braided monoidal category is commutative just when it is a monoidal monad, which is then automatically a braided monoidal monad. The non-braided case tells us similarly that a bistrong κ-accessible monad on a monoidal category is commutative just when it is a monoidal monad.

9. Example: Freyd-categories

Strong and commutative monads play a prominent role in computer science, in particular in work on computational side-effects growing out of Moggi's [44]. In this context, a useful generalisation of strong monads are the *arrows* of [22], which find categorical expression as the *Freyd*-categories of [38]. The notion of *Freyd*-category is a slightly delicate one; the purpose of this section is to show how it fits naturally into our framework.

9.1. Duoidal categories from monoidales

In order to define the duoidal categories giving rise to Freyd-categories, we will appeal to the following general construction. Let $(\mathcal{M}, \otimes, I)$ be a monoidal bicategory [20], and (A, j, p) a monoidale (= pseudomonoid) in $\mathcal{M}$ whose unit $j\colon I \to A$ and multiplication $p\colon A \otimes A \to A$ have right adjoints $j^*\colon A \to I$ and $p^*\colon A \to A \otimes A$; since left adjoint morphisms in a bicategory are often called *maps*, we call A a *map monoidale* in $\mathcal{M}$.

In this situation, the category $\mathcal{M}(A, A)$ bears two monoidal structures; the first $(\circ, I)$ is the composition of the bicategory $\mathcal{M}$, while the second $(*, J)$ is convolution with respect to the monoidale (A, j, p) and comonoidale (A, j^*, p^*). The unit J is thus the composite $jj^*\colon A \to I \to A$, the tensor $f * g$ of $f, g\colon A \to A$ is the composite

$$A \xrightarrow{p^*} A \otimes A \xrightarrow{f \otimes g} A \otimes A \xrightarrow{p} A \ ,$$

and the coherence constraints are derived from the constraint 2-cells of the monoidale A and their mates; see [11, Proposition 4].

Proposition 51. *If A is a map monoidale in the monoidal bicategory $\mathcal{M}$, then $(\mathcal{M}(A, A), *, J, \circ, I)$ is a duoidal category. Moreover, if $\mathcal{M}$ is a braided monoidal bicategory [43, Appendix A], and A a braided monoidale [43, Section 3], then this duoidal structure is $*$-braided; and if $\mathcal{M}$ is a biclosed monoidal bicategory which admits all right liftings and extensions, then the duoidal structure is $*$-biclosed and $\circ$-biclosed.*

In the final clause, to say that $\mathcal{M}$ is *biclosed* is to say that the homomorphisms $A \otimes (-)$ and $(-) \otimes A\colon \mathcal{M} \to \mathcal{M}$ have right biadjoints $[A, -]_\ell$ and $[A, -]_r$; while to ask for *right liftings* and *right extensions* is to ask that, for each object C and 1-cell $f\colon A \to B$ in $\mathcal{M}$, the functors $f \circ (-)\colon \mathcal{M}(C, A) \to M(C, B)$ and $(-) \circ f\colon M(B, C) \to \mathcal{M}(A, C)$ have right adjoints.

Proof. The constraint cells for the duoidal structure on $\mathcal{M}(A, A)$ are obtained from the constraints of the monoidal bicategory $\mathcal{M}$ and the units and counits of the adjunctions $j \dashv j^*$ and $p \dashv p^*$:

$$\mu\colon p(1 \otimes 1)p^* \xrightarrow{\cong} pp^* \xrightarrow{\epsilon} 1 \qquad \upsilon\colon jj^* \xrightarrow{\epsilon} 1 \qquad \gamma\colon jj^* \xrightarrow{j\eta j^*} jj^*jj^*$$

$$\xi\colon p(fh \otimes gk)p^* \xrightarrow{\cong} p(f \otimes g)(h \otimes k)p^* \xrightarrow{p(f \otimes g)\eta(h \otimes k)p^*} p(f \otimes g)p^*p(h \otimes k)p^* \ .$$

The duoidal axioms are straightforward consequences of the monoidal bicategory axioms and the triangle identities for an adjunction.

When $\mathcal{M}$ is braided and A is a braided monoidale, the convolution monoidal structure $*$ on $\mathcal{M}(A, A)$ itself acquires a braiding, and it is now easy to verify the coherence making $\mathcal{M}(A, A)$ into a $*$-braided duoidal category. Finally, suppose that $\mathcal{M}$ is biclosed with all right liftings and extensions. This immediately implies $\circ$-biclosedness of $\mathcal{M}(A, A)$. To show $*$-biclosedness, we first claim that the functors

$$\otimes\colon \mathcal{M}(A,B)\times\mathcal{M}(C,D)\to\mathcal{M}(A\otimes C,B\otimes D)$$

have right adjoints in each variable. To see this, suppose given $f\in\mathcal{M}(A,B)$; then for each $C,D\in\mathcal{M}$ we have the composite functor

$$\mathcal{M}(C,D)\xrightarrow{f\otimes(-)}\mathcal{M}(A\otimes C,B\otimes D)\xrightarrow{\overline{(-)}}\mathcal{M}(C,[A,B\otimes D])\ .$$

Since these functors are clearly pseudonatural in C, they must be given to within equivalence by composition with a 1-cell $D\to[A,B\otimes D]$, and so—using the right liftings—must have right adjoints. Since $\overline{(-)}$ is an equivalence, we conclude that each $f\otimes(-)$ has a right adjoint $[f,-]_r$, and dually each $(-)\otimes g$ has a right adjoint $[g,-]_\ell$. Under these assumptions, it follows that the duoidal structure on $\mathcal{M}(A,A)$ is $*$-biclosed: given $g,h\colon A\to A$, the two internal homs are given by $[g,p^*hp]_\ell\colon A\to A$ and $[g,p^*hp]_r\colon A\to A$ respectively. $\square$

9.2. Bistrong profunctors

We now apply the construction of the previous section to the bicategory **Prof** of *profunctors*, whose objects are small categories and whose 1- and 2-cells $\mathcal{A}\nrightarrow\mathcal{B}$ are cocontinuous functors and transformations $[\mathcal{A}^{\mathrm{op}},\mathbf{Set}]\to[\mathcal{B}^{\mathrm{op}},\mathbf{Set}]$. In practice, we prefer the equivalent formulation which views these 1- and 2-cells as functors and transformations $\mathcal{B}^{\mathrm{op}}\times\mathcal{A}\to\mathbf{Set}$, with composition given by coend. **Prof** is in fact a monoidal bicategory, with $\mathcal{A}\otimes\mathcal{B}:=\mathcal{A}\times\mathcal{B}$ and with $(N\colon\mathcal{A}\nrightarrow\mathcal{B})\otimes(M\colon\mathcal{C}\nrightarrow\mathcal{D})$ defined by $(N\otimes M)\big((b,d),(a,c)\big)=N(b,a)\times M(d,c)$.

There is an identity-on-objects strong monoidal homomorphism $(-)_*\colon\mathbf{Cat}\to\mathbf{Prof}$ which sends $F\colon\mathcal{A}\to\mathcal{B}$ to the profunctor $F_*\colon\mathcal{A}\nrightarrow\mathcal{B}$ with $F_*(b,a)=\mathcal{B}(b,Fa)$. Note that F_* has a right adjoint $F^*\colon\mathcal{B}\nrightarrow\mathcal{A}$ with $F^*(a,b)=\mathcal{B}(Fa,b)$; it follows that each small monoidal category $\mathcal{A}$ (= monoidale in **Cat**) is sent by $(-)_*$ to a map monoidale in **Prof**. Applying the construction of the preceding section, we conclude that if $(\mathcal{A},\otimes,i)$ is a small monoidal category, then the functor category $\mathbf{Prof}(\mathcal{A},\mathcal{A})=[\mathcal{A}^{\mathrm{op}}\times\mathcal{A},\mathbf{Set}]$ has a duoidal structure $(*,J,\circ,I)$ whose tensor products have the following characterisations:

- Maps $M\circ N\to L$ classify families $M(c,b)\times N(b,a)\to L(c,a)$ that are bilinear in the sense of Definition 13;
- Maps $I\to M$ classify extranatural families $I\to M(u,u)$;
- Maps $M*N\to L$ classify natural families $M(a,b)\times N(c,d)\to L(a\otimes c,b\otimes d)$;
- Maps $J\to M$ classify morphisms $I\to M(i,i)$.

The exchange maps ξ of this duoidal structure are given as follows. For $M,N,P,Q\in\mathbf{Prof}(\mathcal{A},\mathcal{A})$, the universal natural families for $M*P$ and $N*Q$ yield an arrow

$$M(a,b)\times N(b,c)\times P(d,e)\times Q(e,f)\to(M*P)(a\otimes d,b\otimes e)\times(N*Q)(b\otimes e,c\otimes f)\ ;$$

now postcomposing with the universal dinatural family for $(M*P)\circ(N*Q)$ yields

$$M(a,b)\times N(b,c)\times P(d,e)\times Q(e,f)\to(M*P)\circ(N*Q)(a\otimes d,c\otimes f)\ ,$$

a family natural in a,c,d and f and bilinear in b and e. Using the classifying property of $\circ$, this corresponds to a family

$$(M\circ N)(a,c)\times(P\circ Q)(d,f)\to(M*P)\circ(N*Q)(a\otimes d,c\otimes f)$$

natural in all variables; and so, by the classifying property of $*$, to the required morphism $(M\circ N)*(P\circ Q)\to(M*P)\circ(N*Q)$. The other duoidal constraint maps are derived in a similar fashion.

Unless $\mathcal{A} \simeq 1$, the duoidal structure on $\mathbf{Prof}(\mathcal{A}, \mathcal{A})$ will not be normal, and so we must apply the normalization of Section 7; we now describe the effect this has. To equip $M \in \mathbf{Prof}(\mathcal{A}, \mathcal{A})$ with a left action $I * M \to M$ is to give a natural family of maps $I(a, b) \times M(c, d) \to M(a \otimes c, b \otimes d)$, or equally, by the classifying property of I, a family of maps

$$t_{ucd} \colon M(c, d) \to M(u \otimes c, u \otimes d) \tag{9.1}$$

natural in c and d and dinatural in u, and satisfying the associativity and unit axioms:

$$\begin{tikzcd}
M(c,d) \arrow[r, "t"] \arrow[d, "t"'] & M((u\otimes v)\otimes c, (u\otimes v)\otimes d) \arrow[d, "M(\alpha^{-1},\alpha)"] \\
M(v\otimes c, v\otimes d) \arrow[r, "t"] & M(u\otimes(v\otimes c), u\otimes(v\otimes d))
\end{tikzcd}
\qquad
\begin{tikzcd}
M(c,d) \arrow[rr, equal] \arrow[dr, "t"'] & & M(c,d) \\
& M(i\otimes c, i\otimes d) \arrow[ur, "M(\lambda^{-1},\lambda)"'] &
\end{tikzcd} .$$

Similarly, a right action $M * I \to M$ involves a family of maps

$$\bar{t}_{cdv} \colon M(c, d) \to M(c \otimes v, d \otimes v) \tag{9.2}$$

satisfying dual associativity and unit axioms; the further axiom required for (9.1) and (9.2) to comprise an I-bimodule structure on M is that

$$\begin{tikzcd}
& M(u\otimes c, u\otimes d) \arrow[r, "\bar{t}"] & M((u\otimes c)\otimes v, (u\otimes d)\otimes v) \arrow[dd, "M(\alpha^{-1},\alpha)"] \\
M(c,d) \arrow[ur, "t"] \arrow[dr, "\bar{t}"'] & & \\
& M(c\otimes v, d\otimes v) \arrow[r, "t"] & M(u\otimes(c\otimes v), u\otimes(d\otimes v))
\end{tikzcd}$$

should commute. Note that when $M = F_*$ for some $\mathcal{V}$-functor F, giving an I-bimodule structure reduces by the Yoneda lemma to giving a compatible strength and costrength on F, in the sense of Section 8.2 above; it thus seems reasonable for a general M to call this structure a *bistrong profunctor* (note that in [45], the name *Tambara module* was used for the same structure).

The normalization of the duoidal $\mathbf{Prof}(\mathcal{A}, \mathcal{A})$ is thus the normal duoidal category $\mathbf{BiStrProf}(\mathcal{A}, \mathcal{A})$ of bistrong profunctors and bistrength-preserving transformations, equipped with the two monoidal structures $(*_I, I)$ given by the I-bilinear quotient of the convolution $*$, and $(\circ, I)$ given by composition of profunctors with the induced bistrength. Note that, by the observations of the preceding paragraph, the embedding $(-)_* \colon \mathbf{End}(\mathcal{A}) \to \mathbf{Prof}(\mathcal{A}, \mathcal{A})$ lifts to an embedding

$$(-)_* \colon \mathbf{BiStr}(\mathcal{A}, \mathcal{A}) \to \mathbf{BiStrProf}(\mathcal{A}, \mathcal{A}) \tag{9.3}$$

which is strong monoidal for the respective composition monoidal structures.

9.3. Bistrong promonads

A $\circ$-monoid in the normal duoidal $\mathbf{BiStrProf}(\mathcal{A}, \mathcal{A})$ is a $\circ$-monoid in $\mathbf{Prof}(\mathcal{A}, \mathcal{A})$—thus a monad on $\mathcal{A}$ in $\mathbf{Prof}$—equipped with a bistrength compatible with the monad structure. We call this structure a *bistrong promonad*. Note that, by strong monoidality of (9.3), any bistrong monad on $\mathcal{A}$ can be regarded as a bistrong promonad whose underlying profunctor is representable; in the next section, we consider how our commutativity notions specialise to this case.

As is well-known, a monad on $\mathcal{A}$ in **Prof** is the same thing as an identity-on-objects functor $F\colon \mathcal{A} \to \mathcal{M}$: given F, the corresponding monad is $F^*F_*\colon \mathcal{A} \nrightarrow \mathcal{A}$; while given a promonad $M\colon \mathcal{A} \nrightarrow \mathcal{A}$, the corresponding $\mathcal{M}$ has the same objects as $\mathcal{A}$ and hom-sets $\mathcal{M}(a,b) = M(a,b)$, with the monad multiplication giving composition in $\mathcal{M}$ and the monad unit giving the action of $F\colon \mathcal{A} \to \mathcal{M}$ on homs. In these terms, giving a bistrength for M amounts to giving natural families $\mathcal{M}(a,b) \to \mathcal{M}(u \otimes a, u \otimes b)$ and $\mathcal{M}(a,b) \to \mathcal{M}(a \otimes v, b \otimes v)$. Compatibility with the monad structure of M says that these functions provide the actions on homs of functors $u \otimes (-)$ and $(-) \otimes v\colon \mathcal{M} \to \mathcal{M}$ fitting into commuting diagrams as on the left and right in

$$\begin{tikzcd} \mathcal{A} \arrow[r, "u\otimes(-)"] \arrow[d, "F"'] & \mathcal{A} \arrow[d, "F"] \\ \mathcal{M} \arrow[r, "u\otimes(-)"'] & \mathcal{M} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, "(-)\otimes v"] \arrow[d, "F"'] & \mathcal{A} \arrow[d, "F"] \\ \mathcal{M} \arrow[r, "(-)\otimes v"'] & \mathcal{M} \end{tikzcd} . \tag{9.4}$$

The unit axioms for the strength and costrength express that the maps $F\lambda_a\colon i \otimes a \to a$ and $F\rho_a\colon a \otimes i \to a$ are components of natural transformations $i \otimes (-) \Rightarrow 1$ and $(-) \otimes i \Rightarrow 1\colon \mathcal{M} \to \mathcal{M}$; the associativity and bimodule axioms express that the maps $F\alpha_{abc}\colon (a \otimes b) \otimes c \to a \otimes (b \otimes c)$ are components of natural transformations $(a \otimes b) \otimes - \Rightarrow a \otimes (b \otimes -)$ and $(a \otimes -) \otimes b \Rightarrow a \otimes (- \otimes c)$ and $(- \otimes b) \otimes c \Rightarrow - \otimes (b \otimes c)\colon \mathcal{M} \to \mathcal{M}$. The naturality of the strength and costrength maps in a and b is automatic from the structure defined so far, while the extranaturality of t in u and of $\bar{t}$ in v express that, for each $f\colon u \to v$ in $\mathcal{A}$ and $g\colon c \to d$ in $\mathcal{M}$, the following diagrams commute:

$$\begin{tikzcd} u \otimes c \arrow[r, "u\otimes g"] \arrow[d, "Ff\otimes c"'] & u \otimes d \arrow[d, "Ff\otimes d"] \\ v \otimes c \arrow[r, "v\otimes g"'] & v \otimes d \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} c \otimes u \arrow[r, "g\otimes u"] \arrow[d, "c\otimes Ff"'] & d \otimes u \arrow[d, "d\otimes Ff"] \\ c \otimes v \arrow[r, "g\otimes v"'] & d \otimes v \end{tikzcd} . \tag{9.5}$$

In the case where the monoidal structure on $\mathcal{A}$ is cartesian, and the symmetry isomorphisms descend to $\mathcal{M}$ in an obvious sense, the above structure was shown in [26, Theorem 6.1] to be equivalent to that of a *Freyd*-category in the sense of [38, Definition 4.1]. We now explain how to generalise this equivalence to the non-cartesian situation; we begin by recalling the necessary definitions.

Definition 52. A *premonoidal* structure on a category $\mathcal{M}$ is given by the following data: (i) a unit object $i \in \mathcal{M}$; (ii) for each $u \in \mathcal{M}$ a functor $u \otimes (-)\colon \mathcal{M} \to \mathcal{M}$ and for each $v \in \mathcal{M}$ a functor $(-) \otimes v\colon \mathcal{M} \to \mathcal{M}$ such that the assignation on objects $u, v \mapsto u \otimes v$ on objects is unambiguously defined; and (iii), families of maps

$$\lambda_a\colon i \otimes a \to a \qquad \rho_a\colon a \otimes i \to a \qquad \alpha_{abc}\colon (a \otimes b) \otimes c \to a \otimes (b \otimes c)$$

where the λ's and ρ's are natural, and the α's are natural in each variable separately. The data in (iii) must satisfy the usual triangle and pentagon axioms for a monoidal category; moreover, we require each map λ_a, ρ_a and α_{abc} to be central. Here, a map $f\colon u \to v$ in $\mathcal{M}$ is called *central* if, for every $g\colon c \to d$ in $\mathcal{M}$, the two squares

$$\begin{tikzcd} u \otimes c \arrow[r, "u\otimes g"] \arrow[d, "f\otimes c"'] & u \otimes d \arrow[d, "f\otimes d"] \\ v \otimes c \arrow[r, "v\otimes g"'] & v \otimes d \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} c \otimes u \arrow[r, "g\otimes u"] \arrow[d, "c\otimes f"'] & d \otimes u \arrow[d, "d\otimes f"] \\ c \otimes v \arrow[r, "g\otimes v"'] & d \otimes v \end{tikzcd}$$

commute. Note that monoidal categories are the same thing as premonoidal categories in which every map is central. If $\mathcal{M}$ and $\mathcal{N}$ are premonoidal categories, then a *strict premonoidal functor* $\mathcal{M} \to \mathcal{N}$ is one which preserves the unit, tensors and constraint morphisms on the nose, and which moreover sends central maps to central maps.

Definition 53. A generalised *Freyd*-category is a triple $(\mathcal{A}, \mathcal{M}, F)$ where $\mathcal{A}$ is a small monoidal category, $\mathcal{M}$ is a small premonoidal category, and $F\colon \mathcal{A} \to \mathcal{M}$ is a bijective-on-objects strict premonoidal functor.

As indicated above, this generalises the definition of *Freyd*-category in [38] in two ways: we do not require $\mathcal{A}$ be *cartesian* monoidal, and do not assume that $\mathcal{M}$ and F are *symmetric* premonoidal in the obvious sense. The following result correspondingly generalises [26, Theorem 6.1].

Proposition 54. *Let $\mathcal{A}$ be a small monoidal category. To give a bistrong promonad on $\mathcal{A}$ is equally to give a generalised Freyd-category of the form $(\mathcal{A}, \mathcal{M}, F)$.*

Proof. Let M be a promonad on $\mathcal{A}$, and let $F\colon \mathcal{A} \to \mathcal{M}$ be the corresponding identity-on-objects functor; we must show that a compatible bistrength is precisely what is needed to make F into a generalised *Freyd*-category. The premonoidal structure on $\mathcal{M}$ will have unit object i, with the data of (9.4) providing the functors $u \otimes (-)$ and $(-) \otimes v$ in $\mathcal{M}$, and the unit and associativity axioms for the strength and costrength yielding the required natural families of constraint maps $F\lambda_a$, $F\rho_a$ and $F\alpha_{abc}$. Now commutativity in (9.5) is the assertion that F sends central maps (= all maps) in $\mathcal{A}$ to central maps in $\mathcal{M}$. In particular, each $F\lambda_a$, $F\rho_a$ and $F\alpha_{abc}$ is central in $\mathcal{M}$, whence $\mathcal{M}$ is premonoidal; it is now clear from the definitions that $F\colon \mathcal{A} \to \mathcal{M}$ is strict premonoidal. $\square$

Note that every premonoidal $\mathcal{M}$ arises in this way for a suitable $\mathcal{A}$; for example, we may take $\mathcal{A}$ to be the *centre* $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$, comprising the same objects but only the central maps. The centre of a premonoidal category is a monoidal category, and the identity-on-objects inclusion $\mathcal{Z}(\mathcal{M}) \to \mathcal{M}$ is a strict premonoidal functor.

We now interpret our basic commutativity notions in light of this proposition. Let M be a bistrong promonad, corresponding to the generalised *Freyd*-category $F\colon \mathcal{A} \to \mathcal{M}$, and let $h\colon H \to M \leftarrow K\colon k$ be a cospan in $\mathbf{BiStrProf}(\mathcal{A}, \mathcal{A})$. By Lemma 48 and the explicit description of the tensor products in $\mathbf{Prof}(\mathcal{A}, \mathcal{A})$, this cospan is commuting just when the diagram

$$\begin{tikzcd}
 & \mathcal{M}(a,b) \times \mathcal{M}(c,d) \arrow[r, "(-\otimes c)\times(b\otimes -)"] & \mathcal{M}(a\otimes c, b\otimes c) \times \mathcal{M}(b\otimes c, b\otimes d) \arrow[dr, "\circ"] & \\
H(a,b) \times K(c,d) \arrow[ur, "h\times k"] \arrow[dr, "h\times k"'] & & & \mathcal{M}(a\otimes c, b\otimes d) \\
 & \mathcal{M}(a,b) \times \mathcal{M}(c,d) \arrow[r, "(a\otimes -)\times(-\otimes d)"] & \mathcal{M}(a\otimes c, a\otimes d) \times \mathcal{M}(a\otimes d, b\otimes d) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

commutes for all $a, b, c, d \in \mathcal{A}$. This is equally to ask that, for each $x \in H(a,b)$ and each $y \in K(c,d)$, the square

$$\begin{tikzcd} a \otimes c \arrow[r, "h(x)\otimes c"] \arrow[d, "a\otimes k(y)"'] & b \otimes c \arrow[d, "b\otimes k(y)"] \\ a \otimes d \arrow[r, "h(x)\otimes d"'] & b \otimes d \end{tikzcd} \tag{9.6}$$

should commute in $\mathcal{M}$. In particular, M is commutative just when $F\colon \mathcal{A} \to \mathcal{M}$ is a strict monoidal functor between monoidal categories. Moreover, the generalised *Freyd*-category $\mathcal{A} \to \mathcal{Z}(\mathcal{M})$ given by factorising F through the centre of $\mathcal{A}$ is precisely the intersection of the left and right centres (in the sense of Definition 27) of the $\circ$-monoid M.

Finally, we observe that **Prof** is a symmetric monoidal bicategory; thus, by Proposition 51, if $\mathcal{A}$ is a symmetric monoidal category, then the resultant duoidal structure on $\mathbf{Prof}(\mathcal{A}, \mathcal{A})$ is naturally $*$-braided. We thus have the option of taking its braided normalization, the category of *strong profunctors* $\mathbf{StrProf}(\mathcal{A}, \mathcal{A})$. In this circumstance, to give a $\circ$-monoid is to give a generalised *Freyd*-category which is symmetric in the obvious sense.

9.4. Commutative monads

As observed in Section 9.3 above, the category of bistrong monad on a small monoidal category $\mathcal{A}$ can be identified with the full subcategory of bistrong promonads on $\mathcal{A}$ whose underlying endofunctor is representable. It is thus natural to ask when the bistrong promonad $M = T_*$ induced by a bistrong monad T is commutative. In this circumstance, the profunctor $M\colon \mathcal{A} \nrightarrow \mathcal{A}$ is given by $M(a, b) = \mathcal{A}(a, Tb)$, and so the generalised *Freyd*-category corresponding to M under Proposition 54 is of the form $(\mathcal{A}, \mathcal{A}_T, F)$, where $F\colon \mathcal{A} \to \mathcal{A}_T$ is the free functor into the Kleisli category of the monad T. The commutativity of a cospan of profunctors $h\colon H \to M \leftarrow K\colon k$, described in the previous section as the commutativity of (9.6), now becomes the commutativity of the diagram:

$$\begin{tikzcd} a \otimes c \arrow[r, "h(x)\otimes c"] \arrow[d, "a\otimes k(y)"'] & Tb \otimes c \arrow[r, "\bar{t}_{bc}"] & T(b\otimes c) \arrow[r, "T(b\otimes k(y))"] & T(b \otimes Td) \arrow[d, "Tt_{bd}"] \\ a \otimes Td \arrow[d, "t_{ad}"'] & & & T^2(b\otimes d) \arrow[d, "\mu"] \\ T(a\otimes d) \arrow[r, "T(h(x)\otimes d)"] & T(Tb\otimes d) \arrow[r, "T\bar{t}_{ad}"] & T^2(b\otimes d) \arrow[r, "\mu"] & T(b\otimes d) \end{tikzcd}$$

for all $x \in H(a, b)$ and $y \in K(c, d)$. When $H = L_*$ and $K = R_*$ for endofunctors L, R of $\mathcal{A}$, the morphisms h and k are necessarily induced by natural transformations $f\colon L \Rightarrow T \Leftarrow R\colon g$, and an application of Yoneda lemma shows that the commutativity of the cospan (h, k) in this circumstance reduces to the commutativity of the following diagrams in $\mathcal{A}$:

$$\begin{tikzcd} Lb \otimes Rd \arrow[r, "f_b\otimes Rd"] \arrow[d, "Lb\otimes g_d"'] & Tb \otimes Rd \arrow[r, "\bar{t}_{bRd}"] & T(b\otimes Rd) \arrow[r, "T(b\otimes g_d)"] & T(b \otimes Td) \arrow[d, "Tt_{bd}"] \\ Lb \otimes Td \arrow[d, "t_{Lbd}"'] & & & T^2(b\otimes d) \arrow[d, "\mu"] \\ T(Lb\otimes d) \arrow[r, "T(f_b\otimes d)"] & T(Tb\otimes d) \arrow[r, "T\bar{t}_{bd}"] & T^2(b\otimes d) \arrow[r, "\mu"] & T(b\otimes d)\,. \end{tikzcd}$$

We have thus proved:

Proposition 55. *If T is a bistrong monad on $\mathcal{A}$, then a cospan $L \Rightarrow T \Leftarrow R$ in* $\mathbf{End}(\mathcal{A})$ *commutes in the sense of [23] precisely when the induced cospan $L_* \Rightarrow T_* \Leftarrow R_*$ commutes in the normal duoidal* $\mathbf{BiStrProf}(\mathcal{A}, \mathcal{A})$*. A bistrong monad T on $\mathcal{A}$ is commutative in the sense of [34] if and only it is so as a bistrong promonad T_* on $\mathcal{A}$.*

As in Section 8.3, we may reconstruct from this the classical result that a bistrong monad T is commutative precisely when it is a monoidal monad. Indeed, a duoid structure on an object $T_* \in \mathbf{Prof}(\mathcal{A}, \mathcal{A})$ is easily seen to be the same as a monoidal monad structure on T; on the other hand, a commutative monoid structure on T_* in the normalization $\mathbf{BiStProf}(\mathcal{A}, \mathcal{A})$ is, by the proposition above, a commutative monad on $\mathcal{A}$.

Finally, in this section, we observe that $\mathbf{Prof}$ is biclosed (even autonomous) as a monoidal bicategory, and admits all right extensions and right liftings; so by Proposition 51, the duoidal structure on $\mathbf{Prof}(\mathcal{A}, \mathcal{A})$ is $*$- and $\circ$-biclosed. It follows that $\mathbf{BiStProf}(\mathcal{A}, \mathcal{A})$ is also $*$ and $\circ$-biclosed; moreover, being the category of algebras for a cocontinuous monad on $\mathbf{Prof}(\mathcal{A}, \mathcal{A}) = [\mathcal{A}^{\mathrm{op}} \times \mathcal{A}, \mathbf{Set}]$, it is locally presentable (indeed, a presheaf category). Applying Proposition 56 now shows that the hypotheses of Proposition 18 are satisfied by $\mathbf{BiStProf}(\mathcal{A}, \mathcal{A})$, so that we have a good notion of commuting tensor product for bistrong promonads, or equally, for generalised *Freyd*-categories on $\mathcal{A}$. In the functional programming literature, strong monads encode notions of computational side-effect, and an important role is played by the tensorial combination of two such monads. *Freyd*-categories were introduced as a generalisation of strong monads (as implied by Proposition 55) and so this tensor product of generalised *Freyd*-categories is both extremely natural and of potential interest to computer scientists.

10. Example: sesquicategories

Our final example deals with the *sesquicategories* of [48], which, as explained in the introduction, are "2-categories without middle-four interchange"; we will exhibit sesquicategories with fixed underlying category as $\circ$-monoids in a suitable duoidal category, which is obtained by normalizing the duoidal category of *derivation schemes* defined in [4, Section 6].

10.1. Derivation schemes

Suppose that $(\mathcal{E}, \otimes, I)$ is a monoidal category with pullbacks, for which the functor $\otimes\colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ preserves pullbacks. In this circumstance, the bicategory $\mathbf{Span}(\mathcal{E})$ of spans in $\mathcal{E}$ becomes a monoidal bicategory, whose tensor product is that of $\mathcal{E}$ on objects, and on 1-cells is given by pointwise tensor product of spans; moreover, the identity-on-objects inclusion $(-)_*\colon \mathcal{E} \to \mathbf{Span}(\mathcal{E})$ sending $f\colon X \to Y$ to $1\colon X \leftarrow X \to Y\colon f$ then becomes a strong monoidal homomorphism. Each morphism f_* has a right adjoint $f^* = (f\colon Y \leftarrow X \to X\colon 1)$ in $\mathbf{Span}(\mathcal{E})$, and so each monoid (A, j, p) in $\mathcal{E}$ is sent by $(-)_*$ to a map monoidale in $\mathbf{Span}(\mathcal{E})$. It follows by the construction of Section 9.1 that, for each monoid (A, j, p) in $\mathcal{E}$, the category $\mathbf{Span}(\mathcal{E})(A, A) = \mathcal{E}/A \times A$ acquires a duoidal structure.

Now fix a set X_0; we apply the preceding general considerations to the monoidal category $\mathcal{E} = \mathbf{Set}/X_0 \times X_0$ with (pullback-preserving) tensor product given by composition of spans. To give a monoid in $\mathcal{E}$ is to give a small category $\mathbb{X} = X_1 \rightrightarrows X_0$, and the above construction now derives from this a duoidal structure on the category $\mathbf{Span}(\mathbf{Set}/X_0 \times X_0)(X_1, X_1) = \mathbf{Set}/X_1 \times_{X_0 \times X_0} X_1$. In the terminology of [4], this duoidal category would be called $\mathbf{Sp}_2(\mathbb{X}, \mathbf{Set})$; however, we follow [48,49] in referring to it as the duoidal category $\mathbf{Ds}(\mathbb{X})$ of *derivation schemes* on $\mathbb{X}$.

Explicitly, an object of $A \in \mathbf{Ds}(\mathbb{X})$ amounts to a function assigning to each parallel pair of morphisms $f, g\colon x \rightrightarrows y$ in $\mathbb{X}$ a set of “2-cells” $A_{f,g}$, while a morphism $\alpha\colon A \to B$ in $\mathbf{Ds}(\mathbb{X})$ is a collection of functions $\alpha_{f,g}\colon A_{f,g} \to B_{f,g}$. As for the duoidal structure $(*, J, \circ, I)$ on $\mathbf{Ds}(\mathbb{X})$, the unit objects are given by

$$J_{f,g} = \begin{cases} 1 & \text{when } f = g = 1_a; \\ \emptyset & \text{otherwise,} \end{cases} \qquad I_{f,g} = \begin{cases} 1 & \text{when } f = g; \\ \emptyset & \text{otherwise,} \end{cases} \tag{10.1}$$

while the binary tensors $\circ$ and $*$ are characterised by the properties that:

- Maps $A * B \to C$ classify families of functions $A_{f_1,g_1} \times B_{f_2,g_2} \to C_{f_1f_2,g_1g_2}$ indexed by all $f_1, f_2\colon x \to y$ and $g_1, g_2\colon y \to z$ in $\mathbb{X}$; and
- Maps $A \circ B \to C$ classify families of functions $A_{g,h} \times B_{f,g} \to C_{f,h}$ indexed by all $f, g, h\colon x \to y$ in $\mathbb{X}$.

It follows that $*$-monoid and $\circ$-monoid structures on a derivation scheme A endow its 2-cells with horizontal and vertical composition operations respectively, and that a duoid structure—involving compatible vertical and horizontal composition—is an enrichment of $\mathbb{X}$ to a 2-category with 2-cells given by A.

10.2. Whiskering schemes and sesquicategories

As is clear from (10.1), the duoidal category $\mathbf{Ds}(\mathbb{X})$ is not normal, so that in order for our theory of commutativity to be applicable we must first pass to its normalization as in Section 7. From the above classification of the tensor products on $\mathbf{Ds}(\mathbb{X})$, we see that to equip a derivation scheme A with an $I * (-) * I$-algebra structure is to give functions $h \cdot (-)\colon A_{f,g} \to A_{hf,hg}$ and $(-) \cdot k\colon A_{f,g} \to A_{fk,gk}$ for all $k\colon x \to y$ and $f, g\colon y \to z$ and $h\colon z \to w$ in $\mathbb{X}$, subject to the evident associativity and unitality axioms; in other words, to endow the 2-cells in A with a notion of whiskering. It seems reasonable to call this structure a *whiskering scheme*; note that it amounts to a lifting of $\mathbf{Hom}_{\mathbb{X}}\colon \mathbb{X}^{\mathrm{op}} \times \mathbb{X} \to \mathbf{Set}$ through the set-of-objects functor $\mathbf{Gph} \to \mathbf{Set}$.

The category $\mathbf{Ws}(\mathbb{X})$ of whiskering schemes is thus the normalization of the duoidal $\mathbf{Ds}(\mathbb{X})$ when equipped with the lifted monoidal structure $(\circ, I)$ and the bilinear quotient $(*_I, I)$ of $(*, J)$. To give a $\circ$-monoid structure on a whiskering scheme is to endow it with a vertical composition of 2-cells which is compatible with whiskering, or in other words, to make it into a sesquicategory; we have thus shown that the category of $\circ$-monoids in $\mathbf{Ws}(\mathbb{X})$ is the category $\mathbf{Sesq}(\mathbb{X})$ of sesquicategory structures on $\mathbb{X}$.

10.3. Commutativity in sesquicategories

We now interpret the basic notions of our general theory in the context of the normal duoidal $\mathbf{Ws}(\mathbb{X})$. Let C be a $\circ$-monoid in $\mathbf{Ws}(\mathbb{X})$, corresponding to the sesquicategory $\mathbb{C}$ extending $\mathbb{X}$, and let $\alpha\colon A \to C \leftarrow B\colon \beta$ be a cospan in $\mathbf{Ws}(\mathbb{X})$. By Lemma 48 and our explicit description of the tensor products in $\mathbf{Ds}(\mathbb{X})$, we see that the cospan (f, g) is commuting just when, for every $f, g\colon x \to y$ and $h, k\colon y \to z$ in $\mathbb{X}$, the diagram

$$\begin{tikzcd} & C_{h,k} \times C_{f,g} \arrow[r, "(-\cdot g)\times(h\cdot -)"] & C_{hg,kg} \times C_{hf,hg} \arrow[dr, "\circ"] & \\ A_{h,k} \times B_{f,g} \arrow[ur, "\alpha\times\beta"] \arrow[dr, "(\beta\times\alpha).c"'] & & & C_{hf,kg} \\ & C_{f,g} \times C_{h,k} \arrow[r, "(k\cdot -)\times(-\cdot f)"] & C_{kf,kg} \times C_{hf,kf} \arrow[ur, "\circ"'] & \end{tikzcd}$$

commutes; which is equally to ask that, for each $x \in A_{h,k}$ and $y \in B_{f,g}$, the interchange axiom holds for the composable pair of 2-cells

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha(x)"] & y \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta(y)"] & z \end{tikzcd}$$

in the sesquicategory $\mathbb{C}$. In particular, we conclude that C is a commutative $\circ$-monoid just when the sesquicategory $\mathbb{C}$ is in fact a 2-category.

Appendix A. $\mathcal{V}$-Cat is locally presentable when $\mathcal{V}$ is so

In [33, Theorem 4.5], Lack and Kelly show that if the locally presentable category $\mathcal{V}$ bears a monoidal biclosed structure, then $\mathcal{V}$-**Cat** is locally presentable and moreover free $\mathcal{V}$-categories exist. In this appendix, we generalise this result by weakening the assumption of closedness.

Proposition 56. *If $\mathcal{V}$ is a locally presentable category equipped with a monoidal structure $(\circ, I)$ for which each functor $A \circ (–)$ and $(–) \circ A$ is accessible, then $\mathcal{V}$-**Cat** is locally presentable and free $\mathcal{V}$-categories exist.*

Proof. Fixing a set X, write $\mathcal{V}\text{-}\mathbf{Cat}_X$ for the fibre of $\mathrm{ob}\colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathbf{Set}$ over X, and similarly for $\mathcal{V}\text{-}\mathbf{Gph}_X$. We first show that each $U_X\colon \mathcal{V}\text{-}\mathbf{Cat}_X \to \mathcal{V}\text{-}\mathbf{Gph}_X$ is an accessible functor between accessible categories. To this end, define $P^0 \in \mathcal{V}\text{-}\mathbf{Gph}_X$ by $P^0(x,x) = I$ and $P^0(x,y) = 0$ for $x \neq y$, let $P^2\colon (\mathcal{V}\text{-}\mathbf{Gph}_X)^2 \to \mathcal{V}\text{-}\mathbf{Gph}_X$ and $P^3\colon (\mathcal{V}\text{-}\mathbf{Gph}_X)^3 \to \mathcal{V}\text{-}\mathbf{Gph}_X$ be defined by

$$\begin{aligned} P^2(A,B)(x,y) &= \sum_{z \in X} A(z,y) \circ B(x,z) \\ P^3(A,B,C)(x,y) &= \sum_{z,w \in X} A(w,y) \circ B(z,w) \circ C(x,z) \ , \end{aligned}$$

and for any $A \in \mathcal{V}\text{-}\mathbf{Gph}_X$, let

$$\begin{aligned} &\varphi^\ell\colon P^3(A,A,A) \to P^2(P^2(A,A),A) \qquad && \varphi^r\colon P^3(A,A,A) \to P^2(A,P^2(A,A)) \\ &\psi^\ell\colon A \to P^2(P^0,A) && \psi^r\colon A \to P^2(A,P^0) \end{aligned}$$

denote the canonical comparison maps induced by the universal property of coproduct. In these terms, to endow $A \in \mathcal{V}\text{-}\mathbf{Gph}_X$ with $\mathcal{V}$-category structure is to give maps $e\colon P^0 \to A$ and $m\colon P^2(A,A) \to A$ rendering commutative the diagrams:

$$\begin{tikzcd} & P^2(P^2(A,A),A) \arrow[r, "P^2(m{,}1)"] & P^2(A,A) \arrow[dr, "m"] & \\ P^3(A,A,A) \arrow[ur, "\varphi^\ell"] \arrow[dr, "\varphi^r"'] & & & A \\ & P^2(A,P^2(A,A)) \arrow[r, "P^2(1{,}m)"'] & P^2(A,A) \arrow[ur, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} P^2(P^0,A) \arrow[d, "P^2(e{,}1)"] & A \arrow[l, "\psi^\ell"'] \arrow[r, "\psi^r"] \arrow[d, equal] & P^2(A,P^0) \arrow[d, "P^2(1{,}e)"'] \\ P^2(A,A) \arrow[r, "m"'] & A & P^2(A,A) \arrow[l, "m"] \end{tikzcd} .$$

Consequently $\mathcal{V}\text{-}\mathbf{Cat}_X$ can be constructed from $\mathcal{V}\text{-}\mathbf{Gph}_X$ using bilimits in **CAT**: first, one takes the inserter $J\colon \mathcal{E} \to \mathcal{V}\text{-}\mathbf{Gph}_X$ of the endofunctors $A \mapsto P^2(A,A)$ and $A \mapsto A$ of $\mathcal{V}\text{-}\mathbf{Gph}_X$, and then the inserter $K\colon \mathcal{G} \to \mathcal{E}$ of the functors ΔP^0 and $J\colon \mathcal{E} \to \mathcal{V}\text{-}\mathbf{Gph}_X$. An object of $\mathcal{G}$ comprises $A \in \mathcal{V}\text{-}\mathbf{Gph}_X$ equipped with maps $e\colon P^0 \to A$ and $m\colon P^2(A,A) \to A$, and now the inclusion $L\colon \mathcal{V}\text{-}\mathbf{Cat}_X \to \mathcal{G}$ exhibits $\mathcal{V}\text{-}\mathbf{Cat}_X$ as the joint inserter of the three parallel pairs of 2-cells in $\mathbf{CAT}(\mathcal{G}, \mathcal{V}\text{-}\mathbf{Gph}_X)$ corresponding to the three axioms displayed above. Now, $\mathcal{V}\text{-}\mathbf{Gph}_X = \mathcal{V}^{X\times X}$ is accessible since $\mathcal{V}$ is, while P^0, P^2, P^3 are accessible since the tensor product of $\mathcal{V}$ is so; since by [40, Theorem 5.1.6], the 2-category **ACC** of accessible categories and accessible functors is closed under bilimits in **CAT**, we conclude that $U_X\colon \mathcal{V}\text{-}\mathbf{Cat}_X \to \mathcal{V}\text{-}\mathbf{Gph}_X$ is an accessible functor between accessible categories as required.

Now for any map $f\colon X \to Y$ in **Set**, the reindexing functor $\mathcal{V}^{f\times f}\colon \mathcal{V}^{Y\times Y} \to \mathcal{V}^{X\times X}$ is easily seen to lift to a functor

$$\begin{tikzcd} \mathcal{V}\text{-}\mathbf{Cat}_Y \arrow[r, dashed, "f^*"] \arrow[d, "U_Y"'] & \mathcal{V}\text{-}\mathbf{Cat}_X \arrow[d, "U_X"] \\ \mathcal{V}\text{-}\mathbf{Gph}_Y \arrow[r, "\mathcal{V}^{f\times f}"'] & \mathcal{V}\text{-}\mathbf{Gph}_X \end{tikzcd}\ .$$

Since $\mathcal{V}^{f\times f}$ has a right adjoint (given by right Kan extension), it preserves all colimits and is in particular accessible. On the other hand, composing f^* with the bilimiting cone that defines $\mathcal{V}\text{-}\mathbf{Cat}_X$ over $\mathcal{V}\text{-}\mathbf{Gph}_X$ gives a cone of accessible categories and functors; whence by [40, Theorem 5.1.6] f^* is also accessible. Consequently, the indexed categories $\mathcal{V}\text{-}\mathbf{Cat}_{(-)}$ and $\mathcal{V}\text{-}\mathbf{Gph}_{(-)}\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{CAT}$ and the indexed transformation U between them all take values in **ACC**, so that by [40, Theorem 5.4], the induced functor $U\colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Gph}$ between total categories is an accessible functor between accessible categories. Now since $\mathcal{V}\text{-}\mathbf{Cat}$ and $\mathcal{V}\text{-}\mathbf{Gph}$ are also complete, they must be locally presentable; since U is also continuous, it must by [17, Satz 14.6] be a right adjoint as required. $\square$

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