Double clubs

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Abstract. Nous développons une théorie des *double clubs* qui étend la théorie de Kelly des clubs aux pseudo-double catégories de Paré et Grandis. Nous montrons alors que le club pour les catégories monoïdales symétriques strictes sur **Cat** s'étend en un double club sur la pseudo-double catégorie $\mathbb{C}at$ des 'catégories, foncteurs, profoncteurs et transformations'.

1 Introduction

Kelly's theory of *clubs* [9, 10, 12, 13] captures an important intuition, that of adding structure in a 'generic way'. In the case of **Cat**, it tells us that, given a description of this added structure at the terminal category 1, we should be able to derive it at an arbitrary category **C** by 'labelling with objects and maps of **C**'.

The genesis of this paper was an attempt to do something similar for **Mod**, the bicategory of categories and profunctors. As it stands, the theory of clubs is inadequate: it deals with categories with pullbacks, whilst **Mod** is neither a category nor has pullbacks. Therefore, we must look for a suitable generalisation of the theory of clubs which is amenable to application in **Mod**.

Now, taking pullbacks is fundamental to the theory of clubs, so we are led to question whether or not **Mod** is the correct place to work; ideally, we should like to replace it with something where we *can* take lots of pullbacks. Now, observe that **Mod** has certain peculiar properties: it has all lax colimits, but these lax colimits have a universal property up to *isomorphism* rather than up to *equivalence*; unfortunately, the language of bicategories cannot express what this universal property is. Similarly, the operation given on objects by cartesian product of categories induces a structure of monoidal bicategory on **Mod**; again, this structure ought to be associative up to *isomorphism* rather than *equivalence*, and again, the language of bicategories is simply unable to express this.

Inspired by this, we are led to consider the *pseudo double categories* of [7] and [8] (and also considered briefly by [14]). These are a weakening of Ehresmann's notion

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of *double category* [3, 4], and have two directions, one 'category-like' and the other 'bicategory-like'. The presence of a 'category-like' direction allows us to express 'up-to-isomorphism' as well as 'up-to-equivalence' notions, and more saliently, to take lots of pullbacks. Indeed, in our case, we can generalise **Mod** to the pseudo double category $\mathbb{C}at$ of 'categories, functors, profunctors and transformations' which in an appropriate sense, has *all* pullbacks.

The main thrust of this paper, then, is to develop a suitable generalisation of the theory of clubs from plain categories to pseudo double categories. Concurrently, we generalise the leading example of a club on **Cat**, the club for symmetric strict monoidal categories, to such a 'double club' on the pseudo double category $\mathbb{C}at$.

This paper is not mere theory for theory's sake: it has been developed very much with an application in mind. In [5], we make extensive use of these results to get a handle on the "higher-dimensional bookwork" involved in the construction of a *pseudo-distributive law* [18] on **Mod**. An examination of [5], therefore, may give the reader a better feel for the motivation behind the present work.

Structure of the paper. In Section 2, we summarise the basic concepts and definitions of pseudo double categories, and prove some new results about double functor categories. In Section 3, we recap the theory of plain clubs, before, in Section 4, starting our generalisation of this theory to the setting of double categories. First, we explore some necessary further aspects of the theory of pseudo double categories, considering slice double categories, equivalences of double categories and cartesian maps in double categories, and then prove a key equivalence of double categories.

In Section 5, we develop the theory of 'monoidal double categories': with this in place, we are ready, in Section 6, to give two definitions of 'double club', one more abstract, the other more tractable. Finally, in Section 7, we show that we can extend the club S for symmetric strict monoidal categories on **Cat** to a double club on $\mathbb{C}at$.

Two Appendices gives a result on equivalences in double categories (Appendix A), and a technical result on 'whiskering' which is of some use in applying the theory of double clubs (Appendix B).

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2 Pseudo double categories

We begin by recapping some of the theory of *pseudo double categories*. Since the full details of this can be found in [7, 8], we shall merely set out our notation and give a few examples.

2.1 Basic theory

Definition 1. A **pseudo double category** $\mathbb{K}$ consists of:

- A diagram of categories $K_1 \overset{s}{\underset{t}{\rightrightarrows}} K_0$. We write X for a typical object and f for a typical arrow of K_0, and call them **objects** and **vertical maps** of $\mathbb{K}$;

similarly, we write $\mathbf{X}$ for a typical object and $\mathbf{f}$ for a typical arrow of K_1, and call them **horizontal maps** and **cells** of $\mathbb{K}$. We call s and t the **source** and **target** functors of $\mathbb{K}$, and write X_s and X_t for $s(\mathbf{X})$ and $t(\mathbf{X})$; similarly, we write f_s and f_t for $s(\mathbf{f})$ and $t(\mathbf{f})$.

- A **horizontal units** functor $\mathbf{I}\colon K_0 \to K_1$.
- A **horizontal composition** functor $\otimes\colon K_1 {}_s\!\times_t K_1 \to K_1$, where $K_1 {}_s\!\times_t K_1$ is the evident pullback.
- Special isomorphisms

$$\begin{aligned} \mathfrak{l}_{\mathbf{X}}\colon \mathbf{X} \to \mathbf{I}_{X_t} \otimes \mathbf{X}, \quad \mathfrak{r}_{\mathbf{X}}\colon \mathbf{X} \to \mathbf{X} \otimes \mathbf{I}_{X_s}, \\ \text{and} \quad \mathfrak{a}_{\mathbf{XYZ}}\colon \mathbf{X} \otimes (\mathbf{Y} \otimes \mathbf{Z}) \to (\mathbf{X} \otimes \mathbf{Y}) \otimes \mathbf{Z}, \end{aligned}$$

  in K_1, natural in all variables. Note that we say a map $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ in K_1 is **special** if $X_s = Y_s$, $X_t = Y_t$, $f_s = \mathrm{id}_{X_s}$ and $f_t = \mathrm{id}_{X_t}$.

These data are required to satisfy five straightforward axioms.

We write a typical object of $\mathbb{K}$ as X, a typical vertical arrow as $f\colon X \to Y$, a typical horizontal arrow as $\mathbf{X}\colon X_s \to\!\!\!\!\!+ \; X_t$ and a typical cell as

$$\begin{array}{ccc} X_s & \xrightarrow{\mathbf{X}} & X_t \\ {\scriptstyle f_s}\downarrow & \Downarrow\mathbf{f} & \downarrow{\scriptstyle f_t} \\ Y_s & \xrightarrow[\mathbf{Y}]{} & Y_t, \end{array}$$

which we may abbreviate to $\mathbf{f}\colon \mathbf{X} \Rightarrow \mathbf{Y}$. We observe that any pseudo double category $\mathbb{K}$ contains a bicategory $\mathcal{B}\mathbb{K}$, with objects the objects of K_0, 1-cells the objects of K_1 and 2-cells the special maps in K_1. Therefore, given $\mathbf{X}\colon A \to\!\!\!\!\!+ \; B$ and $\mathbf{Y}\colon B \to\!\!\!\!\!+ \; C$ in K_1, we may notate $\mathbf{Y} \otimes \mathbf{X}$ as

$$\mathbf{Y} \otimes \mathbf{X}\colon A \xrightarrow{\mathbf{X}} B \xrightarrow{\mathbf{Y}} C,$$

and can extend this notation to horizontal composition of cells. As for bicategories, this notation is ambiguous for chains of three or more such composites: any such will need a choice of bracketing in order to specify a composite horizontal arrow of $\mathbb{K}$. However, as for bicategories, we may use pasting diagrams to specify composites of special maps in K_1: it follows from the bicategorical pasting theorem [17, 19] that such diagrams uniquely specify a special map in K_1 once a bracketing for the start and end edge has been chosen.

Example 2. The pseudo double category $\mathbb{C}at$ is given as follows:

- **Objects** are small categories $X, Y, \ldots$;
- **Vertical maps** are functors $F\colon X \to Y$;
- **Horizontal maps** $\mathbf{X}\colon X_s \to\!\!\!\!\!+ \; X_t$ are profunctors from X_s to X_t; i.e., functors $\mathbf{X}\colon X_t^{\mathrm{op}} \times X_s \to \mathbf{Set}$. We shall specify such by giving:

– The **proarrows** $g\colon x_t \relbar\joinrel\mapstochar\joinrel\rightarrow x_s$, for $x_t \in X_t$ and $x_s \in X_s$: in other words, the elements of $\mathbf{X}(x_t; x_s)$;

– The **actions** by maps of X_t and X_s; so for $h\colon x_s \to x_s'$ in X_s and $f\colon x_t' \to x_t$ in X_t, we give the functors

$$h \bullet (-) = \mathbf{X}(\mathrm{id}_{x_t}; h)\colon \mathbf{X}(x_t; x_s) \to \mathbf{X}(x_t; x_s')$$

and

$$(-) \bullet f = \mathbf{X}(f; \mathrm{id}_{x_s})\colon \mathbf{X}(x_t; x_s) \to \mathbf{X}(x_t'; x_s).$$

Given a proarrow $g\colon x_t \relbar\joinrel\mapstochar\joinrel\rightarrow x_s$, we write the elements $h \bullet g$ and $g \bullet f$ as

$$x_t \overset{g}{\relbar\joinrel\mapstochar\joinrel\rightarrow} x_s \xrightarrow{h} x_s' \quad \text{and} \quad x_t' \xrightarrow{f} x_t \overset{g}{\relbar\joinrel\mapstochar\joinrel\rightarrow} x_s$$

respectively. By analogy with categorical composition, we'll tend to drop the '$\bullet$' symbol where convenient, and denote these actions simply by juxtaposition;

- **Cells** $\mathbf{F}\colon \mathbf{X} \Rightarrow \mathbf{Y}$ are natural transformations

$$\begin{array}{ccc} X_t^{\mathrm{op}} \times X_s & \xrightarrow{F_t^{\mathrm{op}} \times F_s} & Y_t^{\mathrm{op}} \times Y_s \\ & {\scriptstyle \mathbf{X}}\searrow \quad \overset{\mathbf{F}}{\Rightarrow} \quad \swarrow{\scriptstyle \mathbf{Y}} & \\ & \mathbf{Set}. & \end{array}$$

We shall specify a cell by giving its action on proarrows of $\mathbf{X}$; in other words, by giving the components

$$\mathbf{F}_{x_t, x_s}\colon \mathbf{X}(x_t; x_s) \to \mathbf{Y}\big(F_t(x_t); F_s(x_s)\big).$$

In practice, we drop the suffices and refer to all of these maps simply as '$\mathbf{F}$'. Note that naturality of $\mathbf{F}$ amounts to verifying the equivariance formulae

$$\mathbf{F}(h \bullet g) = F_s(h) \bullet \mathbf{F}(g) \quad \text{and} \quad \mathbf{F}(g \bullet f) = \mathbf{F}(g) \bullet F_t(f).$$

Vertical composition is given as in **Cat**, whilst horizontal composition $\otimes$, horizontal units $\mathbf{I}$, associativity $\mathfrak{a}$ and unitality $\mathfrak{l}$, $\mathfrak{r}$ are given as in **Mod**, the bicategory of categories and profunctors. In particular, we notate the proarrows of $\mathbf{I}_X$ (the identity at X) by

$$\mathbf{I}_f\colon x \relbar\joinrel\mapstochar\joinrel\rightarrow y \quad \text{where} \quad f \in \mathbf{X}(x, y)$$

and the proarrows of $\mathbf{Y} \otimes \mathbf{X}\colon A \overset{\mathbf{X}}{\relbar\joinrel\mapstochar\joinrel\rightarrow} B \overset{\mathbf{Y}}{\relbar\joinrel\mapstochar\joinrel\rightarrow} C$ by

$$k \otimes g\colon c \relbar\joinrel\mapstochar\joinrel\rightarrow a \quad \text{where} \quad k \in \mathbf{Y}(c; b), \quad g \in \mathbf{X}(b; a).$$

Note that in the latter case, the 'proarrows' are subject to the equivalence relations $gf \otimes k \simeq g \otimes fk$ for suitable $f \in B(b, b')$; as usual we shall conflate $k \otimes g$ with its image under this equivalence relation.

From this example, we can derive several more useful examples: we can restrict our attention to the *discrete* categories, to get the pseudo double category of sets, maps and spans; we can replace categories with $\mathcal{V}$-categories (for some suitable base for enrichment $\mathcal{V}$) to produce the pseudo double category $\mathcal{V}$-$\mathbb{C}at$; and we can restrict this last to *one-object* $\mathcal{V}$-categories, thereby producing the pseudo double category of monoids, monoid maps and modules in $\mathcal{V}$. In particular, setting $\mathcal{V} = \mathbf{Ab}$, the category of abelian groups, we get the pseudo double category of rings, ring homomorphisms and bimodules.

Definition 3. A **morphism of pseudo double categories** (or **double morphism** for short) $F \colon \mathbb{K} \to \mathbb{L}$ consists of functors $F_0 \colon K_0 \to L_0$ and $F_1 \colon K_1 \to L_1$ – and to ease notation we write 'F' interchangeably for both – together with special maps $\mathfrak{e}_X \colon \mathbf{I}_{FX} \to F\mathbf{I}_X$ and $\mathfrak{m}_{\mathbf{X},\mathbf{Y}} \colon F\mathbf{X} \otimes F\mathbf{Y} \to F(\mathbf{X} \otimes \mathbf{Y})$, natural in all variables, all satisfying five evident axioms.

Pseudo double categories and the morphisms between them form themselves into a category **DblCat**. Similarly, we may define the category $\mathbf{DblCat}_o$ of 'pseudo double categories and double opmorphisms' and $\mathbf{DblCat}_\psi$ of 'pseudo double categories and homomorphisms': for an opmorphism, $\mathfrak{e}_X$ and $\mathfrak{m}_{\mathbf{X},\mathbf{Y}}$ point in the opposite direction, whilst for a homomorphism, $\mathfrak{e}_X$ and $\mathfrak{m}_{\mathbf{X},\mathbf{Y}}$ are invertible.

Example 4. We give an example of a homomorphism on the pseudo double category $\mathbb{C}at$ of Example 2. This homomorphism $S \colon \mathbb{C}at \to \mathbb{C}at$ extends the 'free symmetric strict monoidal category 2-functor' on **Cat**, and given as follows:

- **On objects**: Given a small category X, the category SX has:
  - **Objects** being pairs $(n, \langle x_i \rangle)$, where $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathrm{ob}\, X$;
  - **Arrows** being
  $$(\sigma, \langle g_i \rangle) \colon (n, \langle x_i \rangle) \to (n, \langle y_i \rangle),$$
  where $\sigma \in S_n$ and $g_i \colon x_i \to y_{\sigma(i)}$ in X (note that there are no maps from $(n, \langle x_i \rangle)$ to $(m, \langle y_j \rangle)$ for $n \neq m$).

  Composition and identities in SX are given in the evident way; namely,
  $$\mathrm{id}_{(n,\langle x_i \rangle)} = (\mathrm{id}_n, \langle \mathrm{id}_{x_i} \rangle)$$
  $$\text{and } (\tau, \langle g_i \rangle) \circ (\sigma, \langle f_i \rangle) = (\tau\sigma, \big\langle g_{\sigma(i)} \circ f_i \big\rangle).$$

- **On vertical maps**: Given a functor $F \colon X \to Y$, we give $SF \colon SX \to SY$ by
  $$SF(n, \langle x_i \rangle) = (n, \langle Fx_i \rangle)$$
  $$SF(\sigma, \langle g_i \rangle) = (\sigma, \langle Fg_i \rangle).$$

- **On horizontal maps**: Given a profunctor $\mathbf{X} \colon X \to\!\!\!\!\!\!+\;\, Y$, we give the profunctor $S\mathbf{X} \colon SX \to\!\!\!\!\!\!+\;\, SY$ as follows:
  - **Proarrows** are
  $$(\sigma, \langle g_i \rangle) \colon (n, \langle y_i \rangle) \to\!\!\!\!\!\!+\;\, (n, \langle x_i \rangle),$$
  where $\sigma \in S_n$ and $g_i \colon y_i \to\!\!\!\!\!\!+\;\, x_{\sigma(i)}$ in $\mathbf{X}$ (no proarrows exist from $(n, \langle y_i \rangle)$ to $(m, \langle x_j \rangle)$ for $n \neq m$);

– **Actions** by maps of SX and SY are given in the obvious way, i.e.,

$$(\tau, \langle h_i \rangle) \bullet (\sigma, \langle g_i \rangle) = (\tau\sigma, \big\langle g_{\sigma(i)} \bullet f_i \big\rangle)$$

for the left action by SX, and similarly for the right action by SY.

- **On cells**: Given a cell $\mathbf{F}\colon \mathbf{X} \Rightarrow \mathbf{Y}$, the cell $S\mathbf{F}\colon S\mathbf{X} \Rightarrow S\mathbf{Y}$ is given by

$$S\mathbf{F}\big((\sigma, \langle g_i \rangle)\big) = (\sigma, \langle \mathbf{F}(g_i) \rangle).$$

Vertical functoriality is immediate, whilst horizontal pseudo-functoriality is easily defined and checked to be coherent. There are straightforward variations on the above theme; we can construct homomorphisms T and $P\colon \mathbb{C}at \to \mathbb{C}at$ which lift, respectively, the 2-functors on **Cat** for the 'free (non-symmetric) strict monoidal category' and the 'free category with finite products'.

There are general principles at work here: in all three cases, we have a 2-functor F on **Cat** which *lifts* to a homomorphism $\hat{F}$ on **Mod** in the sense of [18]. Any such lifting will give rise to a double homomorphism on $\mathbb{C}at$ which 'looks like' F in the vertical direction and 'looks like' $\hat{F}$ in the horizontal direction.

Definition 5. Given morphisms $F, G\colon \mathbb{K} \to \mathbb{L}$ of pseudo double categories, a **vertical transformation** $\alpha\colon F \Rightarrow G$ consists of natural transformations $\alpha_0\colon F_0 \Rightarrow G_0$ and $\alpha_1\colon F_1 \Rightarrow G_1$ (and again, we shall use 'α' indifferently for α_0 and α_1), subject to four straightforward axioms.

Given pseudo double categories $\mathbb{K}$ and $\mathbb{L}$, the double morphisms $\mathbb{K} \to \mathbb{L}$ and vertical transformations between them form a category $[\mathbb{K}, \mathbb{L}]_v$. These categories provide us with hom-categories enriching **DblCat** to a 2-category. Horizontal composition of 2-cells is given by the horizontal composition in **Cat** of the underlying natural transformations.

$[\mathbb{K}, \mathbb{L}]_v$ has a full subcategory $[\mathbb{K}, \mathbb{L}]_{v\psi}$ given by restricting to the double *homomorphisms.* Since double homomorphisms are closed under horizontal composition, these fit together to give the locally full sub-2-category $\mathbf{DblCat}_\psi$ of **DblCat**, consisting of pseudo double categories, double homomorphisms and vertical transformations.

Example 6. Following on from Examples 2 and 4, we give a vertical transformation $\eta\colon \mathrm{id}_{\mathbb{C}at} \Rightarrow S$ as follows. Its component at an object X of $\mathbb{C}at_0$ is the functor $\eta_X\colon X \to SX$ given by

$$\eta_X(x) = (1, \langle x \rangle) \quad \text{and} \quad \eta_X(f) = (\mathrm{id}_1, \langle f \rangle),$$

whilst its component at an object $\mathbf{X}$ of $\mathbb{C}at_1$ is the cell $\eta_{\mathbf{X}}\colon \mathbf{X} \Rightarrow S\mathbf{X}$ given by

$$\eta_{\mathbf{X}}(g) = (\mathrm{id}_1, \langle g \rangle).$$

Likewise, we can give a vertical transformation $\mu\colon SS \Rightarrow S$ which 'flattens lists of lists' by removing the inner sets of brackets. It's easy to check that η and μ as defined above obey the monad laws

$$\mu \circ \eta S = \mathrm{id}_S = \mu \circ S\eta \quad \text{and} \quad \mu \circ \mu S = \mu \circ S\mu$$

and thus describe a monad on the object $\mathbb{C}at$ in the 2-category $\mathbf{DblCat}_\psi$, one which lifts the 2-monad for symmetric strict monoidal categories on $\mathbf{Cat}$.

We can repeat the above exercise for the 2-monads for (non-symmetric) strict monoidal categories and categories with finite products, lifting them to double monads on $\mathbb{C}at$. Again, there are general principles at work: we are utilising a *pseudo-distributive law* in the sense of [15, 18], which allows us to lift our 2-monad on $\mathbf{Cat}$ to a pseudomonad on $\mathbf{Mod}$. From this, we can deduce the existence of a double monad on $\mathbb{C}at$ combining the two.

In general, we shall call a monad in $\mathbf{DblCat}_\psi$ a *double monad*: Grandis and Paré consider such double monads and their more general cousins, monads in $\mathbf{DblCat}$, in [8].

Definition 7. Given double morphisms $A_s, A_t \colon \mathbb{K} \to \mathbb{L}$, a **horizontal transformation** $\mathbf{A} \colon A_s \Rrightarrow A_t$ consists of a **components functor** $A_c \colon K_0 \to L_1$ (and to simplify notation, we shall write $\mathbf{A}X$ for A_cX and $\mathbf{A}f$ for A_cf) together with special invertible maps $A_{\mathbf{X}} \colon A_t\mathbf{X} \otimes \mathbf{A}X_s \to \mathbf{A}X_t \otimes A_s\mathbf{X}$ natural in $\mathbf{X}$, which we call the *pseudonaturality* of $\mathbf{A}$; in pasting notation

$$\begin{array}{ccc} A_sX_s & \xrightarrow{A_s\mathbf{X}} & A_sX_t \\ {\scriptstyle \mathbf{A}X_s}\big\downarrow & \Uparrow A_{\mathbf{X}} & \big\downarrow{\scriptstyle \mathbf{A}X_t} \\ A_tX_s & \xrightarrow[A_t\mathbf{X}]{} & A_tX_t. \end{array}$$

These data satisfy four evident axioms.

Example 8. The vertical transformations $\eta \colon \mathrm{id}_{\mathbb{C}at} \Rightarrow S$ and $\mu \colon SS \Rightarrow S$ of Example 6 have horizontal counterparts $(\eta)_* \colon \mathrm{id}_{\mathbb{C}at} \Rrightarrow S$ and $(\mu)_* \colon SS \Rrightarrow S$, with components at $X \in (\mathbb{C}at)_0$ given by

$$(\eta_*)_X = (\eta_X)_* \colon X \to SX \quad \text{and} \quad (\mu_*)_X = (\mu_X)_* \colon SSX \to SX,$$

where $(\ )_*$ is the usual embedding homomorphism $\mathbf{Cat} \to \mathbf{Mod}$. We leave the remaining details to the reader.

Definition 9. Given horizontal transformations $\mathbf{A} \colon A_s \Rrightarrow A_t$ and $\mathbf{B} \colon B_s \Rrightarrow B_t$, a **modification** $\boldsymbol{\gamma} \colon \mathbf{A} \Rrightarrow \mathbf{B}$ consists of a pair of vertical transformations $\gamma_s \colon A_s \Rightarrow B_s$ (the 'vertical source') and $\gamma_t \colon A_t \Rightarrow B_t$ (the 'vertical target'); together with a natural transformation $\gamma_c \colon A_c \Rightarrow B_c$ (the 'central natural transformation'). To simplify notation, we shall refer to the components of γ_c as 'the components of $\boldsymbol{\gamma}$', and write a typical such component as $\boldsymbol{\gamma}_X$. This data must satisfy three evident axioms. We shall notate such a modification as:

$$\begin{array}{ccc} A_s & \xRightarrow{\mathbf{A}} & A_t \\ {\scriptstyle \gamma_s}\Big\Downarrow & \Downarrow\boldsymbol{\gamma} & \Big\Downarrow{\scriptstyle \gamma_t} \\ B_s & \xRightarrow[\mathbf{B}]{} & B_t. \end{array}$$

Given two pseudo double categories $\mathbb{K}$ and $\mathbb{L}$, the horizontal transformations and modifications between them form a category $[\mathbb{K},\mathbb{L}]_h$; further, there are two evident projections $s,t\colon [\mathbb{K},\mathbb{L}]_h \to [\mathbb{K},\mathbb{L}]_v$ which provide data for a **functor double category** $[\mathbb{K},\mathbb{L}]$ as follows:

- The horizontal composite $(\mathbf{C}\colon C_s \nRightarrow C_t) \otimes (\mathbf{A}\colon A_s \nRightarrow C_s)$ has components functor $C_c(-) \otimes A_c(-)$ and pseudonaturality maps

$$(C \otimes A)_{\mathbf{X}}\colon C_t\mathbf{X} \otimes (\mathbf{C}X_s \otimes \mathbf{A}X_s) \to (\mathbf{C}X_t \otimes \mathbf{A}X_t) \otimes A_s\mathbf{X}$$

given by the pasting

$$\begin{array}{ccc} A_sX_s & \xrightarrow{A_s\mathbf{X}} & A_sX_t \\ {\scriptstyle \mathbf{A}X_s}\downarrow & \Uparrow A_{\mathbf{X}} & \downarrow{\scriptstyle \mathbf{A}X_t} \\ C_sX_s & \xrightarrow{C_s\mathbf{X}} & C_sX_t \\ {\scriptstyle \mathbf{C}X_s}\downarrow & \Uparrow C_{\mathbf{X}} & \downarrow{\scriptstyle \mathbf{C}X_t} \\ C_tX_s & \xrightarrow[C_t\mathbf{X}]{} & C_tX_t. \end{array}$$

Given modifications

$$\begin{array}{ccc} A_s & \xRightarrow{\mathbf{A}} & C_s \\ {\scriptstyle \gamma_s}\Downarrow & \Downarrow\boldsymbol{\gamma} & \Downarrow{\scriptstyle \delta_s} \\ B_s & \xRightarrow[\mathbf{B}]{} & D_s \end{array} \quad \text{and} \quad \begin{array}{ccc} C_s & \xRightarrow{\mathbf{C}} & C_t \\ \Downarrow{\scriptstyle \delta_s} & \Downarrow\boldsymbol{\delta} & \Downarrow{\scriptstyle \delta_t} \\ D_s & \xRightarrow[\mathbf{D}]{} & D_t, \end{array}$$

the composite modification $\boldsymbol{\delta}\otimes\boldsymbol{\gamma}$ has $(\delta\otimes\gamma)_s = \gamma_s$, $(\delta\otimes\gamma)_t = \delta_t$ and component at X given by $\boldsymbol{\delta}_X \otimes \boldsymbol{\gamma}_X\colon \mathbf{C}X \otimes \mathbf{A}X \to \mathbf{D}X \otimes \mathbf{B}X$.

- The horizontal unit $\mathbf{I}_F\colon F \nRightarrow F$ at F has components functor $\mathbf{I}_{F(-)}$, and pseudonaturality maps $(I_F)_{\mathbf{X}}$ given by

$$(I_F)_{\mathbf{X}} = F\mathbf{X} \otimes \mathbf{I}_{FX_s} \xrightarrow{\mathfrak{r}^{-1}_{F\mathbf{X}}} F\mathbf{X} \xrightarrow{\mathfrak{l}_{F\mathbf{X}}} \mathbf{I}_{FX_t} \otimes F\mathbf{X}.$$

Given a vertical transformation $\alpha\colon F \Rightarrow G$, the modification $\mathbf{I}_\alpha$ has $(\mathbf{I}_\alpha)_s = \alpha = (\mathbf{I}_\alpha)_t$, and component at X given by $\mathbf{I}_{\alpha_X}\colon \mathbf{I}_{FX} \to \mathbf{I}_{GX}$.

- Unit and associativity constraints $\mathfrak{l}$, $\mathfrak{r}$ and $\mathfrak{a}$ for $[\mathbb{K},\mathbb{L}]$ are given 'componentwise' from those in $\mathbb{L}$.

There is a sub-pseudo double category $[\mathbb{K},\mathbb{L}]_\psi$, given by restricting to *homo*morphisms as objects, and taking all vertical and horizontal transformations and modifications between them.

2.2 Whiskering of homomorphisms

Given a double morphism $G\colon \mathbb{L} \to \mathbb{M}$, we know by virtue of the 2-category structure of **DblCat** that we can 'whisker' F on either side; that is, given vertical transformations

$$\alpha\colon F_1 \Rightarrow F_2\colon \mathbb{K} \to \mathbb{L} \quad \text{and} \quad \beta\colon H_1 \Rightarrow H_2\colon \mathbb{M} \to \mathbb{N}$$

we can form vertical transformations

$$G\alpha\colon GF_1 \Rightarrow GF_2\colon \mathbb{K} \to \mathbb{M} \quad \text{and} \quad \beta G\colon H_1G \Rightarrow H_2G\colon \mathbb{L} \to \mathbb{N}.$$

What we shall do in this section is to produce a similar whiskering operation on *horizontal* transformations, and show that it is compatible with the vertical whiskering:

Proposition 10. *Let $G\colon \mathbb{L} \to \mathbb{M}$ be a double morphism. Then 'precomposition with G' extends to a strict double homomorphism*

$$(-)G\colon [\mathbb{M},\mathbb{N}] \to [\mathbb{L},\mathbb{N}].$$

Proof. We give $(-)G$ as follows:

- $\big((-)G\big)_0\colon [\mathbb{M},\mathbb{N}]_v \to [\mathbb{L},\mathbb{N}]_v$ is given by the whiskering operation in the 2-category **DblCat**. Thus we take the double morphism $H\colon \mathbb{M} \to \mathbb{N}$ to the double morphism $HG\colon \mathbb{L} \to \mathbb{N}$ and the vertical transformation $\alpha\colon H \Rightarrow H'$ to the vertical transformation $\alpha G\colon HG \Rightarrow H'G$.
- $\big((-)G\big)_1\colon [\mathbb{M},\mathbb{N}]_h \to [\mathbb{L},\mathbb{N}]_h$ is given as follows. Given a horizontal transformation $\mathbf{A}\colon A_s \Rrightarrow A_t$, the horizontal transformation $\mathbf{A}G\colon A_sG \Rrightarrow A_tG$ has components functor A_cG_0 (and therefore component at X given by $\mathbf{A}GX\colon A_sGX \twoheadrightarrow A_tGX$) and pseudonaturality maps given by

  $$(\mathbf{A}G)_{\mathbf{X}} = A_tG\mathbf{X} \otimes \mathbf{A}GX_s \xrightarrow{A_{G\mathbf{X}}} \mathbf{A}GX_t \otimes A_sG\mathbf{X}.$$

  Given a modification $\boldsymbol{\gamma}\colon \mathbf{A} \Rrightarrow \mathbf{B}$, the modification $\boldsymbol{\gamma}G$ has $(\boldsymbol{\gamma}G)_s = \gamma_sG$, $(\boldsymbol{\gamma}G)_t = \gamma_tG$, and $(\boldsymbol{\gamma}G)_c = \gamma_cG_0$, and therefore component at X given by:

  $$(\boldsymbol{\gamma}G)_X = \boldsymbol{\gamma}_{GX}\colon \mathbf{A}GX \to \mathbf{B}GX.$$

Visibly, $\big((-)G\big)_1$ and $\big((-)G\big)_0$ are compatible with source and target, and we observe that $(\mathbf{A}\otimes\mathbf{B})G = \mathbf{A}G\otimes\mathbf{B}G$ and $\mathbf{I}_HG = \mathbf{I}_{HG}$, so that $(-)G$ is a *strict* homomorphism. □

We now move on to whiskerings on the left. As for bicategories, we cannot in general whisker *morphisms* with horizontal transformations on the left; we must instead restrict to *homo*morphisms.

Proposition 11. *Let $G\colon \mathbb{L} \to \mathbb{M}$ be a double homomorphism. Then 'postcomposition with G' induces a double homomorphism*

$$G(-)\colon [\mathbb{K},\mathbb{L}] \to [\mathbb{K},\mathbb{M}].$$

Proof. We give $G(-)$ as follows:

- $\big(G(-)\big)_0\colon [\mathbb{K},\mathbb{L}]_v \to [\mathbb{K},\mathbb{M}]_v$ is given by the whiskering operation in the 2-category **DblCat**. Thus we take the double morphism $F\colon \mathbb{K} \to \mathbb{L}$ to the double morphism $GF\colon \mathbb{K} \to \mathbb{M}$ and the vertical transformation $\alpha\colon F \Rightarrow F'$ to the vertical transformation $G\alpha\colon GF \Rightarrow GF'$.

- $\big(G(\text{–})\big)_1 \colon [\mathbb{K},\mathbb{L}]_h \to [\mathbb{K},\mathbb{M}]_h$ is given as follows. Given a horizontal transformation $\mathbf{A}\colon A_s \Rrightarrow A_t$, the horizontal transformation $G\mathbf{A}\colon GA_s \Rrightarrow GA_t$ has components functor G_1A_c (and therefore component at X given by $G\mathbf{A}X\colon GA_sX \nrightarrow GA_tX$) and pseudonaturality maps $(G\mathbf{A})_{\mathbf{X}}$ given by

$$GA_t\mathbf{X}\otimes G\mathbf{A}X_s \xrightarrow{\mathfrak{m}_{A_t\mathbf{X},\mathbf{A}X_s}} G(A_t\mathbf{X}\otimes \mathbf{A}X_s) \xrightarrow{GA_{\mathbf{X}}} G(\mathbf{A}X_t\otimes A_s\mathbf{X}) \xrightarrow{\mathfrak{m}^{-1}_{\mathbf{A}X_t,A_s\mathbf{X}}} G\mathbf{A}X_t\otimes GA_s\mathbf{X}.$$

Given a modification $\boldsymbol{\gamma}\colon \mathbf{A} \Rrightarrow \mathbf{B}$, the modification $G\boldsymbol{\gamma}$ has $(G\boldsymbol{\gamma})_s = G\gamma_s$, $(G\boldsymbol{\gamma})_t = G\gamma_t$ and $(G\boldsymbol{\gamma})_c = G_1\gamma_c$, and therefore component at X given by

$$(G\boldsymbol{\gamma})_X = G\boldsymbol{\gamma}_X \colon G\mathbf{A}X \to G\mathbf{B}X.$$

Again, it's clear that these functors are compatible with source and target. It remains to give $\mathfrak{m}$ and $\mathfrak{e}$, so we take the special invertible modification $\mathfrak{e}_A\colon \mathbf{I}_{GA} \Rrightarrow G\mathbf{I}_A$ to have components

$$(\mathfrak{e}_A)_X = \mathfrak{e}_{AX}\colon \mathbf{I}_{GAX} \to G\mathbf{I}_{AX}.$$

and the special invertible modification $\mathfrak{m}_{\mathbf{A},\mathbf{B}}\colon G\mathbf{A}\otimes G\mathbf{B} \Rrightarrow G(\mathbf{A}\otimes\mathbf{B})$ to have components

$$(\mathfrak{m}_{\mathbf{A},\mathbf{B}})_X = \mathfrak{m}_{\mathbf{A}X,\mathbf{B}X}\colon G\mathbf{A}X\otimes G\mathbf{B}X \to G(\mathbf{A}X\otimes\mathbf{B}X).$$

Checking naturality and coherence is routine. □

Observe also that $G(\text{–})$ and $(\text{–})G$ restrict to respective homomorphisms

$$(\text{–})G\colon [\mathbb{M},\mathbb{N}]_\psi \to [\mathbb{L},\mathbb{N}]_\psi \quad \text{and} \quad G(\text{–})\colon [\mathbb{K},\mathbb{L}]_\psi \to [\mathbb{K},\mathbb{M}]_\psi.$$

These propositions give us an 'action' of homomorphisms on functor pseudo double categories (we shall see below the precise sense in which this *is* an action), which can be extended from homomorphisms to the vertical transformations between them. We begin with whiskerings on the right.

Proposition 12. *Let G and $G'\colon \mathbb{L}\to\mathbb{M}$ be double morphisms, and let $\alpha\colon G \Rightarrow G'$ be a vertical transformation. Then precomposition with α induces a vertical transformation*

$$(\text{–})\alpha\colon (\text{–})G \Rightarrow (\text{–})G'\colon [\mathbb{M},\mathbb{N}] \to [\mathbb{L},\mathbb{N}].$$

Proof. We give $(\text{–})\alpha$ as follows:

- $\big((\text{–})\alpha\big)_0$ has component at $H \in [\mathbb{M},\mathbb{N}]_v$ given by the map $H\alpha\colon HG \Rightarrow HG'$ in $[\mathbb{L},\mathbb{N}]_v$. The naturality of these components in H is the equality $\beta G' \circ H\alpha = H'\alpha \circ \beta G$ in **DblCat**;

- $\big((\text{–})\alpha\big)_1$ is given as follows. Its component at $\mathbf{A} \in [\mathbb{M},\mathbb{N}]_h$ is the modification $\mathbf{A}\alpha\colon \mathbf{A}G \Rrightarrow \mathbf{A}G'$ whose central natural transformation is $A_c\alpha_0$. The naturality of these components in $\mathbf{A}$ follows from the equality $\beta_cG'_0 \circ A_c\alpha_0 = A'_c\alpha_0 \circ \beta_cG_0$ in **Cat**.

These natural transformations are compatible with source and target, and checking the vertical transformation axioms is routine. ☐

Proposition 13. *Let G and $G'\colon \mathbb{L} \to \mathbb{M}$ be double homomorphisms, and let $\alpha\colon G \Rightarrow G'$ be a vertical transformation. Then postcomposition with α induces a vertical transformation*

$$\alpha(-)\colon G(-) \Rightarrow G'(-)\colon [\mathbb{K},\mathbb{L}] \to [\mathbb{K},\mathbb{M}].$$

Proof. We give the vertical transformation $\alpha(-)$ as follows:

- $\big(\alpha(-)\big)_0$ has component at $F \in [\mathbb{K},\mathbb{L}]_v$ given by the map $\alpha F\colon GF \Rightarrow G'F$ in $[\mathbb{K},\mathbb{M}]_v$. The naturality of these components in F is the equality $G'\beta \circ \alpha F = \alpha F' \circ G\beta$ in **DblCat**.
- $\big(\alpha(-)\big)_1$ has component at $\mathbf{A} \in [\mathbb{K},\mathbb{L}]_h$ given by the modification $\alpha\mathbf{A}\colon G\mathbf{A} \Rrightarrow G'\mathbf{A}$ whose central natural transformation is $\alpha_1 A_c$. The naturality of these components in $\mathbf{A}$ is the equality $G'_1\beta_c \circ \alpha_1 A_c = \alpha_1 A'_c \circ G_1\beta_c$ in **Cat**.

These natural transformations are compatible with source and target, and checking coherence is routine. ☐

Observe that $\alpha(-)$ and $(-)\alpha$ restrict to respective vertical transformations

$$(-)\alpha\colon (-)G \Rightarrow (-)G'\colon [\mathbb{M},\mathbb{N}]_\psi \to [\mathbb{L},\mathbb{N}]_\psi$$
$$\alpha(-)\colon G(-) \Rightarrow G'(-)\colon [\mathbb{K},\mathbb{L}]_\psi \to [\mathbb{K},\mathbb{M}]_\psi.$$

We make one final remark: given a vertical transformation $\alpha\colon G \Rightarrow G'$ in $[\mathbb{L},\mathbb{M}]_\psi$ and a modification $\boldsymbol{\gamma}\colon \mathbf{A} \Rrightarrow \mathbf{B}$ in $[\mathbb{M},\mathbb{N}]$, the two modifications $\mathbf{B}\alpha\circ\boldsymbol{\gamma}G$ and $\boldsymbol{\gamma}G'\circ\mathbf{A}\alpha$ are the same, by naturality of $\big((-)\alpha\big)_1$. Thus we shall write this common value as $\boldsymbol{\gamma}\alpha$. Similarly, if we have $\boldsymbol{\gamma}\colon \mathbf{A} \Rrightarrow \mathbf{B}$ now in $[\mathbb{K},\mathbb{L}]$ we write $\alpha\boldsymbol{\gamma}$ for the modification $\alpha\mathbf{B} \circ G\boldsymbol{\gamma} = G'\boldsymbol{\gamma} \circ \alpha\mathbf{A}$ in $[\mathbb{K},\mathbb{M}]$.

2.3 The hom 2-functor on $\mathbf{DblCat}_\psi$

It's not hard to see that the operations of the previous section are functorial with respect to vertical transformations. To be more precise, given double categories $\mathbb{K}$, $\mathbb{L}$, $\mathbb{M}$ and $\mathbb{N}$, the above operations induce functors

$$[\mathbb{K},-]\colon [\mathbb{L},\mathbb{M}]_{v\psi} \to \big[\,[\mathbb{K},\mathbb{L}],[\mathbb{K},\mathbb{M}]\,\big]_{v\psi}$$
$$\text{and } [-,\mathbb{N}]\colon [\mathbb{L},\mathbb{M}]_{v\psi} \to \big[\,[\mathbb{M},\mathbb{N}],[\mathbb{L},\mathbb{N}]\,\big]_{v\psi},$$

along with their 'pseudo' restrictions

$$[\mathbb{K},-]_\psi\colon [\mathbb{L},\mathbb{M}]_{v\psi} \to \big[\,[\mathbb{K},\mathbb{L}]_\psi,[\mathbb{K},\mathbb{M}]_\psi\,\big]_{v\psi}$$
$$\text{and } [-,\mathbb{N}]_\psi\colon [\mathbb{L},\mathbb{M}]_{v\psi} \to \big[\,[\mathbb{M},\mathbb{N}]_\psi,[\mathbb{L},\mathbb{N}]_\psi\,\big]_{v\psi}.$$

Moreover, it's straightforward to check that the following equalities hold:

$$\big((-)G_1\big)G_2 = (-)(G_1G_2), \qquad \big((-)\alpha_1\big)\alpha_2 = (-)(\alpha_1\alpha_2),$$
$$G_1\big(G_2(-)\big) = (G_1G_2)(-), \qquad \alpha_1\big(\alpha_2(-)\big) = (\alpha_1\alpha_2)(-),$$
$$\big(G_1(-)\big)G_2 = G_1\big((-)G_2\big), \quad \text{and} \quad \big(\alpha_1(-)\big)\alpha_2 = \alpha_1\big((-)\alpha_2\big).$$

which can be more succinctly stated as follows:

Proposition 14. *The functors* $[\mathbb{K}, \text{–}]$ *and* $[\text{–}, \mathbb{N}]$ *defined above provide data for 2-functors*

$$[\mathbb{K}, \text{–}]\colon \mathbf{DblCat}_\psi \to \mathbf{DblCat}_\psi \quad \textit{and} \quad [\text{–}, \mathbb{N}]\colon \mathbf{DblCat}_\psi^{\mathrm{op}} \to \mathbf{DblCat}_\psi$$

which are compatible in the sense that they provide data for a 2-functor

$$[\,\text{–}, ?\,]\colon \mathbf{DblCat}_\psi^{\mathrm{op}} \times \mathbf{DblCat}_\psi \to \mathbf{DblCat}_\psi.$$

Similarly, the functors $[\mathbb{K}, \text{–}]_\psi$ *and* $[\text{–}, \mathbb{N}]_\psi$ *defined above provide data for 2-functors*

$$[\mathbb{K}, \text{–}]_\psi\colon \mathbf{DblCat}_\psi \to \mathbf{DblCat}_\psi \quad \textit{and} \quad [\text{–}, \mathbb{N}]_\psi\colon \mathbf{DblCat}_\psi^{\mathrm{op}} \to \mathbf{DblCat}_\psi$$

which are compatible in the sense that they provide data for a 2-functor

$$[\,\text{–}, ?\,]_\psi\colon \mathbf{DblCat}_\psi^{\mathrm{op}} \times \mathbf{DblCat}_\psi \to \mathbf{DblCat}_\psi.$$

Now, what *are* these 2-functors? Does either of the bivariant 2-functors provide an 'internal hom' for $\mathbf{DblCat}_\psi$? Let us make this question precise: observe that $\mathbf{DblCat}_\psi$ has all finite products, and thus can be viewed as a monoidal bicategory, with the tensor product given by cartesian product. Then by an 'internal hom' for $\mathbf{DblCat}_\psi$, we mean a homomorphism of bicategories

$$\langle -, ?\,\rangle\colon \mathbf{DblCat}_\psi^{\mathrm{op}} \times \mathbf{DblCat}_\psi \to \mathbf{DblCat}_\psi$$

such that for all pseudo double categories $\mathbb{K}$, we have a biadjunction $(-) \times \mathbb{K} \dashv \langle \mathbb{K}, - \rangle$. In other words, $\langle -, ?\,\rangle$, if it exists, exhibits $\mathbf{DblCat}_\psi$ as a *biclosed* monoidal bicategory in the sense of [2].

Now, there is *no* good biadjunction for the 'lax hom' 2-functor $[-, ?\,]$, for the same reason as there is no good whiskering on the left by morphisms: at some point, we have to produce pseudo-naturality data for a horizontal transformation, and, due to the laxity of the morphisms involved, no choice of such data exists. However, it *is* the case that the 'pseudo hom' 2-functor $[-, ?\,]_\psi$ provides an internal hom in the above described sense. We don't intend to work through the rather messy details here, but we do note that although both $(-) \times \mathbb{K}$ and $[\,\mathbb{K}, -]$ are 2-functors, the adjunction between them is still only a *bi*adjunction rather than an honest 2-adjunction.

3 Clubs

We now recall some of the basic definitions and results of the theory of clubs. A rather more detailed account of this material can be found in [13] or [20].

Definition 15. A natural transformation $\alpha\colon A \Rightarrow S\colon \mathbf{C} \to \mathbf{D}$ is called a **cartesian natural transformation** if all its naturality squares are pullbacks.

The following is immediate by elementary properties of pullback:

Proposition 16. *Suppose that* $\mathbf{C}$ *has a terminal object* 1*. Then a natural transformation* $\alpha\colon A \Rightarrow S\colon \mathbf{C} \to \mathbf{D}$ *is cartesian if and only if every naturality square of the form*

$$\begin{array}{ccc} AX & \xrightarrow{A!} & A1 \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ SX & \xrightarrow[S!]{} & S1 \end{array}$$

is a pullback.

Thus, if we are given S, the cartesian natural transformations into it are determined up to isomorphism by their component $\alpha_1\colon A1 \to S1$. We can make this statement precise as follows. Given a category $\mathbf{C}$ and an object $X \in \mathbf{C}$, the slice category $\mathbf{C}/X$ has:

- **Objects** being pairs (U, f) where $U \in \mathbf{C}$ and $f\colon U \to X$;
- **Maps** $j\colon (U, f) \to (V, g)$ being maps $j\colon U \to V$ in $\mathbf{C}$ with $gj = f$.

In particular, given a functor $S\colon \mathbf{C} \to \mathbf{D}$, we form the slice category $[\mathbf{C}, \mathbf{D}]/S$; consider now the full subcategory of this given by the objects (A, α) where α is a *cartesian* natural transformations into S. We write $\mathit{Coll}(S)$ for this subcategory and call it the **category of collections** over S. We have a functor $F\colon \mathit{Coll}(S) \to \mathbf{D}/S1$ which evaluates at 1:

$$\begin{aligned} F\colon \mathit{Coll}(S) &\to \mathbf{D}/S1 \\ (A, \alpha) &\mapsto (A1, \alpha_1) \\ \gamma &\mapsto \gamma_1, \end{aligned}$$

and our above statement now becomes:

Proposition 17. *[13] Suppose* $\mathbf{D}$ *has all pullbacks; then evaluation at* 1 *induces an equivalence of categories* $\mathit{Coll}(S) \simeq \mathbf{D}/S1$*.*

Now suppose we are given a category $\mathbf{C}$ together with a monad (S, η, μ) on $\mathbf{C}$. As above, we can form the slice category $[\mathbf{C}, \mathbf{C}]/S$, but now we can go further; indeed, $[\mathbf{C}, \mathbf{C}]$ is a (strict) monoidal category and (S, η, μ) is a monoid in it. Thus the slice category $[\mathbf{C}, \mathbf{C}]/S$ acquires a canonical monoidal structure, given by

$$\mathbf{I} = (\mathrm{id}_{\mathbf{C}} \xRightarrow{\eta} S) \quad \text{and} \quad (A, \alpha) \otimes (B, \beta) = (AB \xRightarrow{\alpha\beta} SS \xRightarrow{\mu} S).$$

This structure is 'canonical' in the following sense: giving a monoid S in $[\mathbf{C}, \mathbf{C}]$ is equivalent to giving a lax monoidal functor $\ulcorner S \urcorner : 1 \to [\mathbf{C}, \mathbf{C}]$, and $[\mathbf{C}, \mathbf{C}]/S$ equipped with the above monoidal structure is a *lax limit* for this arrow in the 2-category of monoidal categories, lax monoidal functors and lax monoidal transformations.

Now, we may naturally ask whether the subcategory $\mathit{Coll}(S)$ of $[\mathbf{C}, \mathbf{C}]/S$ is closed under the above monoidal structure. Explicitly:

Definition 18. We say that a subcategory $\mathbf{D}$ of a monoidal category $\mathbf{C}$ is a **monoidal subcategory** if $\mathbf{D}$ can be made into a monoidal category such that the inclusion $\mathbf{D} \hookrightarrow \mathbf{C}$ is a strict monoidal functor.

Definition 19. We say that a monad (S, η, μ) is a **club** on $\mathbf{C}$ if $Coll(S)$ is a monoidal subcategory of $[\mathbf{C}, \mathbf{C}]/S$.

Given a club (S, η, μ), we can exploit the equivalence of categories $Coll(S) \simeq \mathbf{C}/S1$ to transport the monoidal structure on $Coll(S)$ to a monoidal structure on $\mathbf{C}/S1$. Explicitly, this monoidal structure has unit given by $\mathbf{I} = \eta_1 \colon 1 \to S1$, and tensor product $(a, \theta) \otimes (b, \phi)$ given by the left-hand composite in the following diagram:

$$\begin{array}{ccc} a \otimes b & \longrightarrow & a \\ \downarrow & & \downarrow{\scriptstyle \theta} \\ Sb & \xrightarrow[S!]{} & S1 \\ {\scriptstyle S\phi}\downarrow & & \\ SS1 & & \\ {\scriptstyle \mu_1}\downarrow & & \\ S1. & & \end{array}$$

Now, the above definition of club is not easy to work with in practice, so the following alternative description is often useful:

Proposition 20. *[13] A monad (S, η, μ) is a club on $\mathbf{C}$ if and only if:*

1. *η is a cartesian natural transformation;*
2. *μ is a cartesian natural transformation;*
3. *S preserves cartesian natural transformations into S: that is, whenever $\alpha \colon A \Rightarrow S$ is cartesian, so is $S\alpha \colon SA \Rightarrow SS$.*

Example 21. Straightforward calculation using the previous proposition shows all of the following to be clubs on **Cat**:

- The 'free symmetric strict monoidal category' monad S;
- The 'free (non-symmetric) strict monoidal category' monad T;
- The 'free category with finite products' monad P.

In Example 6 of the previous section, we saw that S, T and P extend from 2-monads on **Cat** to double monads on $\mathbb{C}at$. What we are going to show is that S, T and P also extend from *clubs* on **Cat** to *double clubs* on $\mathbb{C}at$. To do this, we first need to know what we *mean* by a double club, and this is the objective of the next three sections.

4 Double clubs I

We shall assume without further mention that $\mathbb{K}$ and $\mathbb{L}$ are pseudo double categories such that:

- $\mathbb{K}$ has a **double terminal object**; that is, an object $1 \in K_0$ such that 1 is terminal in K_0 and $\mathbf{I}_1$ is terminal in K_1;

- L_1 and L_0 have all pullbacks and are equipped with a choice of such; and furthermore, s and t preserve these choices *strictly*.

In the terminology of [7], this latter condition amounts to a lax functorial choice of double pullbacks. In fact, we can rephrase much of the work of this section *globally*, in terms of double pullbacks in double functor categories. However, by doing so we would lose sight of why we have to impose technical conditions such as property (hps) below. Therefore we shall work at the *local* level of components and leave it to the reader to translate into a global view.

Example 22. The pseudo double category $\mathbb{C}at$ satisfies both the above criteria. The terminal category 1 provides a double terminal object. For the lax functorial choice of double pullbacks, we observe that $\mathbb{C}at_0 = \mathbf{Cat}$ certainly has all pullbacks, whilst $\mathbb{C}at_1$ is isomorphic to the category $\mathbf{Cat}/\mathbf{2}$ (where $\mathbf{2}$ is the arrow category $0 \to 1$), and hence also has all pullbacks. Further, given a choice of pullbacks in $\mathbb{C}at_0$, we can choose pullbacks in $\mathbb{C}at_1$ such that s and t strictly preserve them.

4.1 Slice double categories

We begin by extending the notion of slice category from plain categories to double categories. The details of this construction are already known, and can be found (along with a discussion of the more general 'comma double categories') in [8]. Thus we shall merely recap the details.

Definition 23. A **monad** in the pseudo double category $\mathbb{K}$ consists of:

- An object X in K_0;
- An object $\mathbf{X}\colon X \twoheadrightarrow X$ in K_1;
- Special maps $\mathfrak{m}\colon \mathbf{X} \otimes \mathbf{X} \to \mathbf{X}$ and $\mathfrak{e}\colon \mathbf{I}_X \to \mathbf{X}$ subject to the commutativity of the usual unitality and associativity diagrams.

Equivalently, this is to give a double morphism $\ulcorner \mathbf{X} \urcorner : 1 \to \mathbb{K}$. So, given a pseudo double category $\mathbb{K}$ together with a monad $(\mathbf{X}, \mathfrak{m}, \mathfrak{e})$ in $\mathbb{K}$, we form the **slice double category** $\mathbb{K}/\mathbf{X}$ as follows: $(\mathbb{K}/\mathbf{X})_1 = K_1/\mathbf{X}$ and $(\mathbb{K}/\mathbf{X})_0 = K_0/X$, whilst s and t are given by

$$s(\mathbf{U} \xrightarrow{\mathbf{f}} \mathbf{X}) = (U_s \xrightarrow{f_s} X), \qquad s(\mathbf{j}) = j_s,$$
$$t(\mathbf{U} \xrightarrow{\mathbf{f}} \mathbf{X}) = (U_t \xrightarrow{f_t} X) \qquad \text{and } t(\mathbf{j}) = j_t.$$

$\mathbf{I}$ and $\otimes$ are given on objects by

$$\mathbf{I}_{(U,f)} = (\mathbf{I}_U \xrightarrow{\mathbf{I}_f} \mathbf{I}_X \xrightarrow{\mathfrak{e}} \mathbf{X}) \quad \text{and} \quad (\mathbf{U}, \mathbf{f}) \otimes (\mathbf{V}, \mathbf{g}) = (\mathbf{U} \otimes \mathbf{V} \xrightarrow{\mathbf{f} \otimes \mathbf{g}} \mathbf{X} \otimes \mathbf{X} \xrightarrow{\mathfrak{m}} \mathbf{X})$$

and inherit their action on maps from $\mathbb{K}$, whilst the natural transformations $\mathfrak{l}$, $\mathfrak{r}$ and $\mathfrak{a}$ have components inherited from $\mathbb{K}$; that is,

$$\mathfrak{l}_{(\mathbf{U},\mathbf{f})} = \mathfrak{l}_{\mathbf{U}}, \quad \mathfrak{r}_{(\mathbf{U},\mathbf{f})} = \mathfrak{r}_{\mathbf{U}} \quad \text{and} \quad \mathfrak{a}_{(\mathbf{U},\mathbf{f}),(\mathbf{V},\mathbf{g}),(\mathbf{W},\mathbf{h})} = \mathfrak{a}_{\mathbf{U},\mathbf{V},\mathbf{W}}.$$

The remaining details are easily checked. We now describe the slice double categories we shall need for the theory of double clubs.

Proposition 24. *Given a pseudo double category* $\mathbb{K}$ *and an object* $X \in K_0$*, the functor* $\ulcorner X \urcorner : 1 \to K_0$ *extends to a double homomorphism* $\ulcorner \mathbf{I}_X \urcorner : 1 \to \mathbb{K}$*.*

Proof. To give $\ulcorner \mathbf{I}_X \urcorner$ is to give an 'iso-monad' in $\mathbb{K}$ whose multiplication and unit are invertible; for this we take $\mathbf{I}_X \colon X \nrightarrow X$, with multiplication and unit given by

$$\mathfrak{m} = \mathfrak{l}_{\mathbf{I}_X}^{-1} = \mathfrak{r}_{\mathbf{I}_X}^{-1} \colon \mathbf{I}_X \otimes \mathbf{I}_X \to \mathbf{I}_X \quad \text{and} \quad \mathfrak{e} = \mathrm{id}_{\mathbf{I}_X} \colon \mathbf{I}_X \to \mathbf{I}_X. \qquad \square$$

In particular, given a double homomorphism $S \colon \mathbb{K} \to \mathbb{L}$, we have the object $\mathrm{id}_{\mathbb{K}} \in [\mathbb{K}, \mathbb{K}]_\psi$, and thus the double homomorphism

$$1 \xrightarrow{\ulcorner \mathbf{I}_{\mathrm{id}_{\mathbb{K}}} \urcorner} [\mathbb{K}, \mathbb{K}]_\psi \xrightarrow{S(-)} [\mathbb{K}, \mathbb{L}]_\psi.$$

Writing $S\mathbf{I}$ for the corresponding monad in $[\mathbb{K}, \mathbb{L}]_\psi$, we can form the slice double category $[\mathbb{K}, \mathbb{L}]_\psi / S\mathbf{I}$. Similarly, we have the monad $S\mathbf{I}_1$ given by

$$1 \xrightarrow{\ulcorner \mathbf{I}_1 \urcorner} \mathbb{K} \xrightarrow{S} \mathbb{L},$$

and so can form the slice double category $\mathbb{L}/S\mathbf{I}_1$.

Example 25. Consider once more the double homomorphism $S \colon \mathbb{C}at \to \mathbb{C}at$ of Example 4. For this, the pseudo double category $\mathbb{C}at/S\mathbf{I}_1$ has:

- **Objects** (X, F) given by a category X together with a functor $F \colon X \to S1$. We observe that we can identify $S1$ with (a skeleton of) the category of *finite sets and bijections.*
- **Vertical maps** $H \colon (X, F) \to (Y, G)$ given by commutative triangles

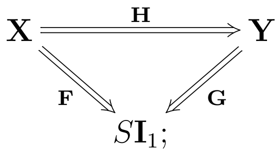
$$\begin{array}{ccc} X & \xrightarrow{H} & Y \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle G} & \\ & S1. & \end{array}$$

- **Horizontal maps** $(\mathbf{X}, \mathbf{F}) \colon (X_s, F_s) \nrightarrow (X_t, F_t)$ given by a profunctor $\mathbf{X} \colon X_s \nrightarrow X_t$ together with a cell

$$\begin{array}{ccc} X_s & \overset{\mathbf{X}}{\nrightarrow} & X_t \\ {\scriptstyle F_s}\downarrow & \Downarrow \mathbf{F} & \downarrow{\scriptstyle F_t} \\ S1 & \underset{S\mathbf{I}_1}{\nrightarrow} & S1. \end{array}$$

  We identify the profunctor $S\mathbf{I}_1 \colon S1 \nrightarrow S1$ with the hom functor on $S1$; thus to give a horizontal map $(\mathbf{X}, \mathbf{F})$ is to give a profunctor $\mathbf{X}$ together with an assignation to each proarrow f of $\mathbf{X}$ an arrow $\mathbf{F}f$ of $S1$, compatible with F_s and F_t.

- **Cells H**: $(\mathbf{X},\mathbf{F}) \Rightarrow (\mathbf{Y},\mathbf{G})$ are given by commutative triangles of cells in $\mathbb{C}at$



  thus to each proarrow f of $\mathbf{X}$, we assign a compatible proarrow $\mathbf{H}f$ of $\mathbf{Y}$ such that $\mathbf{GH}f = \mathbf{F}f$.

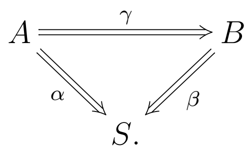
- **Horizontal identity** is given on objects (X, F) by $(\mathbf{I}_X, \hat{\mathbf{I}}_F)$, where $\mathbf{I}_X$ is the identity profunctor on X and $\hat{\mathbf{I}}_F$ is given by $\hat{\mathbf{I}}_F(\mathbf{I}_f) = Ff$, for f an arrow of X.

- **Horizontal composition** is given by $(\mathbf{X},\mathbf{F}) \otimes (\mathbf{X}',\mathbf{F}') = (\mathbf{X} \otimes \mathbf{X}', \mathbf{F}\hat{\otimes}\mathbf{F}')$, where $\mathbf{X} \otimes \mathbf{X}'$ is usual profunctor composition, and where $(\mathbf{F}\hat{\otimes}\mathbf{F}')(f \otimes f') = \mathbf{F}f \circ \mathbf{F}'(f')$.

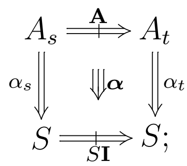
The pseudo double category $[\mathbb{C}at, \mathbb{C}at]_\psi/S\mathbf{I}$ has:

- **Objects** (A, α) given by a double homomorphism $A\colon \mathbb{C}at \to \mathbb{C}at$ together with a vertical transformation $\alpha\colon A \Rightarrow S$.

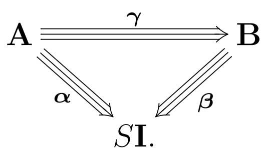
- **Vertical maps** $\gamma\colon (A,\alpha) \to (B,\beta)$ given by commutative triangles



- **Horizontal maps** $(\mathbf{A},\boldsymbol{\alpha})\colon (A_s,\alpha_s) \to\!\!\!\!\!+ (A_t,\alpha_t)$ given by pairs $(\mathbf{A},\boldsymbol{\alpha})$ where $\mathbf{A}$ is a horizontal transformation and $\boldsymbol{\alpha}$ a modification as follows:



- **Cells** $\boldsymbol{\gamma}\colon (\mathbf{A},\boldsymbol{\alpha}) \Rightarrow (\mathbf{B},\boldsymbol{\beta})$ given by commutative triangles



- **Horizontal identities** given on objects (A,α) by

$$\mathbf{I}_{(A,\alpha)} = \mathbf{I}_A \xRightarrow{\mathbf{I}_\alpha} \mathbf{I}_S \xRightarrow{\mathfrak{e}} S\mathbf{I}$$

(where $\mathfrak{e}$ is the unit of the monad $S\mathbf{I}$, with components $\mathfrak{e}_X \colon \mathbf{I}_{SX} \to S\mathbf{I}_X$), and on maps $\gamma \colon (A, \alpha) \to (B, \beta)$ by

$$\begin{array}{ccc} \mathbf{I}_A & \xRightarrow{\ \mathbf{I}_\gamma\ } & \mathbf{B} \\ & {\scriptstyle \mathfrak{e}\circ\mathbf{I}_\alpha} \searrow \quad \swarrow {\scriptstyle \mathfrak{e}\circ\mathbf{I}_\beta} & \\ & S\mathbf{I}; & \end{array}$$

- **Horizontal composition** given on objects by

$$(\mathbf{A}, \boldsymbol{\alpha}) \otimes (\mathbf{A}', \boldsymbol{\alpha}') = \big(\mathbf{A} \otimes \mathbf{A}' \xRightarrow{\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}'} S\mathbf{I} \otimes S\mathbf{I} \xRightarrow{\ \mathfrak{m}\ } S\mathbf{I}\big)$$

(where $\mathfrak{m}$ is the multiplication of the monad $S\mathbf{I}$, with components

$$\mathfrak{m}_X = S\mathbf{I}_X \otimes S\mathbf{I}_X \xrightarrow{\mathfrak{m}_{\mathbf{I}_X,\mathbf{I}_X}} S(\mathbf{I}_X \otimes \mathbf{I}_X) \xrightarrow{S\mathfrak{l}^{-1}_{\mathbf{I}_X}} S\mathbf{I}_X\ ),$$

and on maps by

$$\begin{array}{ccc} \mathbf{A} \otimes \mathbf{A}' & \xRightarrow{\ \boldsymbol{\gamma}\otimes\boldsymbol{\gamma}'\ } & \mathbf{B} \otimes \mathbf{B}' \\ & {\scriptstyle \mathfrak{m}\circ(\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}')} \searrow \quad \swarrow {\scriptstyle \mathfrak{m}\circ(\boldsymbol{\beta}\otimes\boldsymbol{\beta}')} & \\ & S\mathbf{I}. & \end{array}$$

4.2 The double category of collections

We return now to our general theory. We should like to restrict from the full double slice category $[\mathbb{K}, \mathbb{L}]_\psi / S\mathbf{I}$ to something mimicking the category of collections. To do this, we need a double category analogue of Definition 15's 'cartesian natural transformation':

Definition 26.

- A vertical transformation $\alpha \colon F \Rightarrow G \colon \mathbb{K} \to \mathbb{L}$ is called a **cartesian vertical transformation** if the natural transformations $\alpha_1 \colon F_1 \Rightarrow G_1$ and $\alpha_0 \colon F_0 \Rightarrow G_0$ are cartesian;

- A modification $\boldsymbol{\gamma} \colon \mathbf{A} \Rrightarrow \mathbf{B}$ is called a **cartesian modification** if γ_s and γ_t are cartesian vertical transformations and the natural transformation $\gamma_c \colon A_c \Rightarrow B_c$ is cartesian.

We should like the double category of collections $\mathbb{C}oll(S)$ to have:

- $\mathbb{C}oll(S)_0$ being the full subcategory of $\big([\mathbb{K}, \mathbb{L}]_\psi / S\mathbf{I}\big)_0$ whose objects are the cartesian vertical transformations into S;

- $\mathbb{C}oll(S)_1$ being the full subcategory of $\big([\mathbb{K}, \mathbb{L}]_\psi / S\mathbf{I}\big)_1$ whose objects are the cartesian modifications into $S\mathbf{I}$,

with the remaining data inherited from the double category $[\mathbb{K}, \mathbb{L}]_\psi / S\mathbf{I}$. In order for this to make sense, we need $\mathbb{C}oll(S)$ to be closed under the horizontal units and composition of $[\mathbb{K}, \mathbb{L}]_\psi / S\mathbf{I}$, for which we require S to have the following property.

Definition 27. Let $S\colon \mathbb{K} \to \mathbb{L}$ be a double homomorphism; we say that S has **property (hps)** (horizontal pullback stability) if it satisfies:

- **Property (hps1)**: given horizontally composable pullbacks

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\mathbf{p}_1} & \mathbf{B} \\ {\scriptstyle \mathbf{p}_2}\downarrow & & \downarrow{\scriptstyle \mathbf{f}} \\ S\mathbf{C} & \xrightarrow[S!]{} & S\mathbf{I}_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{A}' & \xrightarrow{\mathbf{p}'_1} & \mathbf{B}' \\ {\scriptstyle \mathbf{p}'_2}\downarrow & & \downarrow{\scriptstyle \mathbf{f}'} \\ S\mathbf{C}' & \xrightarrow[S!]{} & S\mathbf{I}_1, \end{array}$$

  in L_1, the diagram

$$\begin{array}{ccc} \mathbf{A}' \otimes \mathbf{A} & \xrightarrow{\mathbf{p}'_1 \otimes \mathbf{p}_1} & \mathbf{B}' \otimes \mathbf{B} \\ {\scriptstyle \mathbf{p}'_2 \otimes \mathbf{p}_2}\downarrow & & \downarrow{\scriptstyle \mathbf{f}' \otimes \mathbf{f}} \\ S\mathbf{C}' \otimes S\mathbf{C} & \xrightarrow[S! \otimes S!]{} & S\mathbf{I}_1 \otimes S\mathbf{I}_1 \end{array}$$

  is a pullback in L_1; and

- **Property (hps2)**: given a pullback

$$\begin{array}{ccc} A & \xrightarrow{p_1} & B \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ SC & \xrightarrow[S!]{} & S1 \end{array}$$

  in L_0, the diagram

$$\begin{array}{ccc} \mathbf{I}_A & \xrightarrow{\mathbf{I}_{p_1}} & \mathbf{I}_B \\ {\scriptstyle \mathbf{I}_{p_2}}\downarrow & & \downarrow{\scriptstyle \mathbf{I}_f} \\ \mathbf{I}_{SC} & \xrightarrow[\mathbf{I}_{S!}]{} & \mathbf{I}_{S1} \end{array}$$

  is a pullback in L_1.

Proposition 28. *Given a homomorphism $S\colon \mathbb{K} \to \mathbb{L}$ with property (hps), the categories $\mathbb{C}oll(S)_0$ and $\mathbb{C}oll(S)_1$ provide data for a pseudo double category whose remaining data is inherited from $[\mathbb{K}, \mathbb{L}]_\psi / S\mathbf{I}$.*

Proof. We must check that the horizontal units of $[\mathbb{K}, \mathbb{L}]_\psi / S\mathbf{I}$ are cartesian modifications, and that the horizontal composition of two cartesian modifications is another cartesian modification. For the first of these, given $(A, \alpha) \in \mathbb{C}oll(S)_0$, we have $\mathbf{I}_{(A,\alpha)}$ given by the modification

$$\mathbf{I}_{(A,\alpha)} = \mathbf{I}_A \overset{\mathbf{I}_\alpha}{\Longrightarrow} \mathbf{I}_S \overset{\mathfrak{e}}{\Longrightarrow} S\mathbf{I};$$

so consider the diagram

$$\begin{array}{ccc} \mathbf{I}_{AX} & \xrightarrow{\mathbf{I}_{A!}} & \mathbf{I}_{A1} \\ {\scriptstyle \mathbf{I}_{\alpha_X}}\downarrow & & \downarrow{\scriptstyle \mathbf{I}_{\alpha_1}} \\ \mathbf{I}_{SX} & \xrightarrow{\mathbf{I}_{S!}} & \mathbf{I}_{S1} \\ {\scriptstyle \mathfrak{e}_X}\downarrow & & \downarrow{\scriptstyle \mathfrak{e}_1} \\ S\mathbf{I}_X & \xrightarrow[S\mathbf{I}_!]{} & S\mathbf{I}_1. \end{array}$$

It follows from property (hps2) and the cartesianness of α that the top square is a pullback; and the lower square commutes, and so is a pullback since both vertical arrows are isomorphisms. Thus the outer edge is again a pullback, and so $\mathbf{I}_{(A,\alpha)}$ is cartesian as required.

For the second, suppose we are given horizontally composable objects $(\mathbf{A}, \boldsymbol{\alpha})$ and $(\mathbf{B}, \boldsymbol{\beta})$ of $\mathbb{C}oll(S)_1$; we must show that the modification

$$\mathbf{A} \otimes \mathbf{B} \xRightarrow{\boldsymbol{\alpha} \otimes \boldsymbol{\beta}} S\mathbf{I} \otimes S\mathbf{I} \xRightarrow{\mathfrak{m}} S\mathbf{I}$$

is also cartesian. So consider the diagram:

$$\begin{array}{ccc} \mathbf{A}X \otimes \mathbf{B}X & \xrightarrow{\mathbf{A}! \otimes \mathbf{B}!} & \mathbf{A}1 \otimes \mathbf{B}1 \\ {\scriptstyle \boldsymbol{\alpha}_X \otimes \boldsymbol{\beta}_X} \downarrow & & \downarrow {\scriptstyle \boldsymbol{\alpha}_1 \otimes \boldsymbol{\beta}_1} \\ S\mathbf{I}_X \otimes S\mathbf{I}_X & \xrightarrow[S\mathbf{I}_! \otimes S\mathbf{I}_!]{} & S\mathbf{I}_1 \otimes S\mathbf{I}_1 \\ {\scriptstyle \mathfrak{m}_X} \downarrow & & \downarrow {\scriptstyle \mathfrak{m}_1} \\ S\mathbf{I}_X & \xrightarrow[S\mathbf{I}_!]{} & S\mathbf{I}_1. \end{array}$$

The upper square is a pullback by property (hps2) and the cartesianness of α and β; the lower square commutes and has isomorphisms down the sides, and hence is a pullback. So the outer edge is also a pullback as required. ☐

4.3 Evaluation at 1 in $\mathbb{C}oll(S)$

In order to see that our definition of $\mathbb{C}oll(S)$ is the correct one, we need to show that there is a suitable analogue at the pseudo double category level of the equivalence of categories $Coll(S) \simeq \mathbf{D}/S1$ exhibited in Proposition 17.

For this, we need a suitable notion of 'equivalence of double categories'. There is an obvious candidate for this, namely equivalence in the 2-category $\mathbf{DblCat}_\psi$, and the following proposition gives us an elementary characterisation of such equivalences.

Proposition 29. *Suppose we are given double categories $\mathbb{K}$ and $\mathbb{L}$, and:*

- *A double homomorphism $F \colon \mathbb{K} \to \mathbb{L}$;*
- *Functors $G_1 \colon L_1 \to K_1$ and $G_0 \colon L_0 \to K_0$;*
- *Natural isomorphisms $\eta_i \colon \mathrm{id}_{K_i} \cong G_i F_i$ and $\epsilon_i \colon F_i G_i \cong \mathrm{id}_{K_i}$ $(i = 0, 1)$;*

such that $sG_1 = G_0 s$, $tG_1 = G_0 t$, $s\epsilon_1 = \epsilon_0 s$, $t\epsilon_1 = \epsilon_0 t$, $s\eta_1 = \eta_0 s$ and $t\eta_1 = \eta_0 t$. Then $\mathbb{K}$ and $\mathbb{L}$ are equivalent in $\mathbf{DblCat}_\psi$.

Proof. See Appendix A, Corollary 54. ☐

Now let $S \colon \mathbb{K} \to \mathbb{L}$ be a double homomorphism with property (hps), and consider the double category of collections $\mathbb{C}oll(S)$. We have a strict homomorphism

$F\colon \mathbb{C}oll(S) \to \mathbb{L}/S\mathbf{I}_1$ which 'evaluates at 1':

$$\begin{aligned} F_0\colon \mathbb{C}oll(S)_0 &\to L_0/S1 \\ (A,\alpha) &\mapsto (A1,\alpha_1) \\ \gamma &\mapsto \gamma_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} F_1\colon \mathbb{C}oll(S)_1 &\to L_1/S\mathbf{I}_1 \\ (\mathbf{A},\boldsymbol{\alpha}) &\mapsto (\mathbf{A}1,\boldsymbol{\alpha}_1) \\ \boldsymbol{\gamma} &\mapsto \boldsymbol{\gamma}_1. \end{aligned}$$

Using this, we can prove the following analogue of Proposition 17.

Proposition 30. *Let S be a homomorphism $\mathbb{K} \to \mathbb{L}$ satisfying property (hps). Then evaluation at 1 induces an equivalence of double categories $\mathbb{C}oll(S) \simeq \mathbb{L}/S\mathbf{I}_1$.*

Proof. We exhibit all the data required for Proposition 29. We have the strict homomorphism $F\colon \mathbb{C}oll(S) \to \mathbb{L}/S\mathbf{I}_1$ as above; in the opposite direction, we must exhibit functors $G_i\colon (\mathbb{L}/S\mathbf{I}_1)_i \to \mathbb{C}oll(S)_i$. We can form categories of collections $Coll(S_0)$ and $Coll(S_1)$, and by Proposition 17 we have equivalences of categories

$$Coll(S_0) \simeq L_0/S1 \quad \text{and} \quad Coll(S_1) \simeq L_1/S\mathbf{I}_1$$

where the rightward direction of these equivalences is given by evaluation at 1 and $\mathbf{I}_1$ respectively. We are now ready to give G_0:

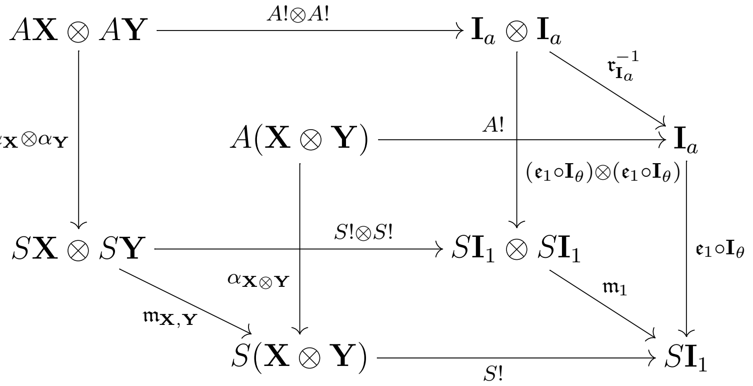
- **On objects**: given an object $(a,\theta) \in L_0/S1$, under the first equivalence we produce an object $(A_0,\alpha_0) \in Coll(S_0)$. We can also form the object $\mathbf{I}_{(a,\theta)} \in L_1/S\mathbf{I}_1$: under the second equivalence this produces an object $(A_1,\alpha_1) \in Coll(S_1)$. Explicitly, A_0, α_0, A_1 and α_1 are the specified objects and maps in the following pullback diagrams:

$$\begin{array}{ccc} A_0X & \xrightarrow{A_0!} & a \\ {\scriptstyle (\alpha_0)_X}\downarrow & & \downarrow{\scriptstyle \theta} \\ SX & \xrightarrow[S!]{} & S1 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A_1\mathbf{X} & \xrightarrow{A_1!} & \mathbf{I}_a \\ & & \downarrow{\scriptstyle \mathbf{I}_\theta} \\ {\scriptstyle (\alpha_1)_\mathbf{X}}\downarrow & & \mathbf{I}_{S1} \\ & & \downarrow{\scriptstyle \mathfrak{e}_1} \\ S\mathbf{X} & \xrightarrow[S!]{} & S\mathbf{I}_1. \end{array}$$

  Since s and t strictly preserve pullbacks, its easy to see that A_1 and A_0, and similarly α_1 and α_0, are compatible with source and target. We aim to equip $A = (A_0, A_1)$ with the structure of a double homomorphism, and to show that $\alpha = (\alpha_0, \alpha_1)$ becomes a cartesian vertical transformation with respect to this structure. To do this, we must produce special natural isomorphisms

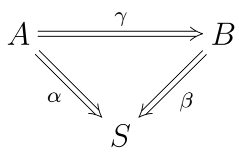
$$\mathfrak{m}_{\mathbf{X},\mathbf{Y}}\colon A\mathbf{X} \otimes A\mathbf{Y} \to A(\mathbf{X} \otimes \mathbf{Y}) \quad \text{and} \quad \mathfrak{e}_X\colon \mathbf{I}_{AX} \to A\mathbf{I}_X.$$

So consider the diagram:

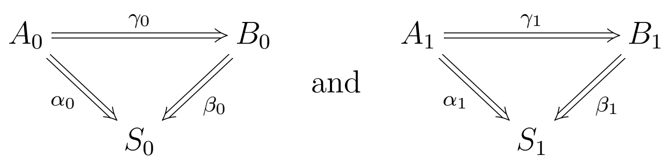


The front face is a pullback by definition; the back face by property (hps1). All the diagonal maps are isomorphisms, and the bottom and right faces commute by the coherence axioms for S and $\mathbb{L}$. Thus we induce a unique isomorphism $A\mathbf{X} \otimes A\mathbf{Y} \to A(\mathbf{X} \otimes \mathbf{Y})$ along the missing diagonal. Arguing identically for the unit, we induce a unique isomorphism $\mathbf{I}_{AX} \to A\mathbf{I}_X$. All required naturality and coherence now follows straightforwardly using the existing coherence and the universal property of pullback.

- **On maps**: suppose we have a map $\psi\colon (a, \theta) \to (b, \phi)$ in $L_0/S1$, with $G_0(a, \theta) = (A, \alpha)$ and $G_0(b, \phi) = (B, \beta)$. Then we must produce a map $\gamma\colon (A, \alpha) \to (B, \beta)$; that is, a vertical transformation $\gamma\colon A \Rightarrow B$ making the diagram

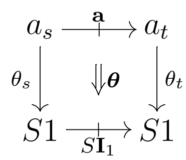


commute. Now, using the equivalences $L_0/S1 \simeq \mathit{Coll}(S_0)$ and $L_1/S\mathbf{I}_1 \simeq \mathit{Coll}(S_1)$ as before, we produce natural transformations γ_0 and γ_1 making

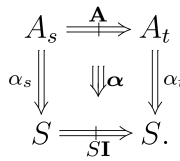


commute. We aim to show that $\gamma = (\gamma_0, \gamma_1)$ becomes a vertical transformation. Compatibility with source and target follows as before, whilst the other two axioms follow from the naturality of $\mathfrak{r}^{-1}$ and the universal property of pullback.

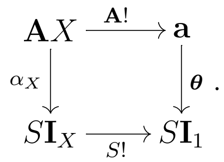
We now move on to G_1. Suppose we have an object



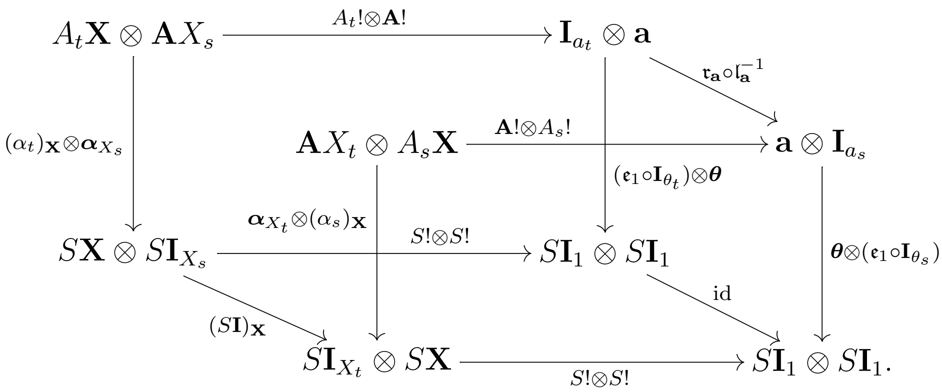
of $L_1/S\mathbf{I}_1$, with $G_0(a_s,\theta_s) = (A_s,\alpha_s)$ and $G_0(a_t,\theta_t) = (A_t,\alpha_t)$, say. Then we must produce an object $(\mathbf{A},\boldsymbol{\alpha}) \in \mathbb{C}oll(S)_1$ as follows:



Under the equivalence $L_1/S\mathbf{I}_1 \simeq Coll(S_1)$, we take $(\mathbf{a},\boldsymbol{\theta})$ to a functor $A\colon K_1 \to L_1$ and a cartesian natural transformation $\alpha\colon A \Rightarrow S_1$. Thus we specify the horizontal transformation $\mathbf{A}$ to have source A_s, target A_t and components functor $A_c = A\mathbf{I}\colon K_0 \to L_1$. Similarly, we take the modification $\boldsymbol{\alpha}$ to have source α_s, target α_t and central natural transformation $\alpha_c = \alpha\mathbf{I}\colon A\mathbf{I} \Rightarrow S\mathbf{I}\colon K_0 \to L_1$. Explicitly, $\mathbf{A}X$ and α_X will be the indicated arrows in the following pullback diagram:

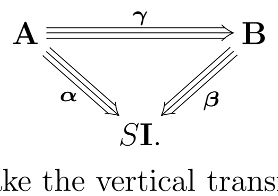


We must now specify the pseudonaturality maps for $\mathbf{A}$. So consider the diagram



The front and back faces are pullbacks by property (hps1) and the diagonal maps are all isomorphisms. It's easy to check that the bottom and right faces commute, and thus we induce a unique isomorphism along the missing diagonal, which will be the pseudonaturality map $A_{\mathbf{X}}$. Again, all required naturality and coherence follows easily using the existing naturality and coherence and the universal property of pullback.

We now give G_1 on maps. Given a map $\boldsymbol{\psi}\colon (\mathbf{a},\boldsymbol{\theta}) \to (\mathbf{b},\boldsymbol{\phi})$ in $K_1/S\mathbf{I}_1$, we must produce a map $\boldsymbol{\gamma}\colon (\mathbf{A},\boldsymbol{\alpha}) \to (\mathbf{B},\boldsymbol{\beta})$ of $\mathbb{C}oll(S)_1$, and thus a modification $\boldsymbol{\gamma}\colon \mathbf{A} \Rrightarrow \mathbf{B}$ fitting into the diagram



For its source and target, we take the vertical transformations

$$\gamma_s = G_0(\psi_s)\colon A_s \Rightarrow B_s \quad\text{and}\quad \gamma_t = G_0(\psi_t)\colon A_t \Rightarrow B_t.$$

For the central natural transformation, we apply once more the equivalence $L_1/S\mathbf{I}_1 \cong Coll(S_1)$ to get a commuting diagram

$$\begin{array}{ccc} A & \overset{\gamma}{\Longrightarrow} & B \\ & {\scriptstyle\alpha}\searrow\!\!\searrow \quad \swarrow\!\!\swarrow{\scriptstyle\beta} & \\ & S_1, & \end{array}$$

in the functor category $[L_1, L_1]$. We need a natural transformation $\gamma_c \colon A_c \Rightarrow B_c$, and from above we have $A_c = A\mathbf{I}$ and $B_c = B\mathbf{I}$; so we take $\gamma_c = \gamma\mathbf{I}$. This this provides coherent data for a modification follows by an argument similar to above. Finally, we note that we have

$$\alpha_c = \alpha\mathbf{I} = (\beta \circ \gamma)\mathbf{I} = \beta\mathbf{I} \circ \gamma\mathbf{I} = \beta_c \circ \gamma_c$$

as required. This completes the definition of G_1.

By construction, it is immediate that $tG_1 = G_0t$ and $sG_1 = G_0s$; so we need to show that (F_0, G_0) and (F_1, G_1) provide data for equivalences of categories. First note that if we choose pullbacks in L_0 and L_1 such that the pullback of identity arrows are identity arrows then we have

$$F_0G_0 = \mathrm{id}_{L_0/S1} \quad \text{and} \quad F_1G_1 = \mathrm{id}_{L_1/S\mathbf{I}_1}.$$

Conversely, it's an easy exercise using the universal property of pullback to construct natural isomorphisms $\eta_0 \colon \mathrm{id}_{\mathbb{C}oll(S)_0} \Rightarrow G_0F_0$ and $\eta_1 \colon \mathrm{id}_{\mathbb{C}oll(S)_1} \Rightarrow G_1F_1$, and to show that they are compatible with source and target maps as required. Thus we have all the requirements for Corollary 29, and so have an equivalence of double categories $\mathbb{C}oll(S) \simeq \mathbb{K}/S\mathbf{I}_1$. ☐

5 Monoidal double categories

To complete our exposition of the theory of double clubs, we need a suitable generalisation of *monoidal category* to the double category level. This is fairly straightforward: recall that the 2-category $\mathbf{DblCat}_\psi$ has finite products, given in the obvious way, and hence becomes a (cartesian) monoidal bicategory [6]. Thus we can define

Definition 31. A **monoidal double category** is a pseudomonoid [2, 16] in $\mathbf{DblCat}_\psi$.

However, this definition is too abstract to work with in practice; we use instead the following alternative characterisation, the proof of which is entirely routine:

Proposition 32. *Giving a monoidal double category $\mathbb{K}$ is equivalent to giving a double category $\mathbb{K}$ such that*

- *K_0 is a (not necessarily strict) monoidal category, with data $(\bullet_0, \ulcorner e \urcorner, \alpha_0, \lambda_0, \rho_0)$;*
- *K_1 is a (not necessarily strict) monoidal category, with data $(\bullet_1, \ulcorner \mathbf{e} \urcorner, \alpha_1, \lambda_1, \rho_1)$;*

- *The functors s and $t\colon K_1 \to K_0$ are strict monoidal;*
- *The functors $\mathbf{I}\colon K_0 \to K_1$ and $\otimes\colon K_{1\,s}{\times_t} K_1 \to K_1$ are strong monoidal (where $K_{1\,s}{\times_t} K_1$ acquires its monoidal structure via pullback along the strict monoidal functors s and t);*
- *The associativity and unitality natural transformations $\mathfrak{a}$, $\mathfrak{l}$ and $\mathfrak{r}$ for $\mathbb{K}$ are monoidal natural transformations.*

We note that the data making $\bullet$ and e strong monoidal amounts to giving invertible special maps in K_1 as follows:

$$k_{\mathbf{W},\mathbf{X},\mathbf{Y},\mathbf{Z}}\colon (\mathbf{W}\bullet\mathbf{X})\otimes(\mathbf{Y}\bullet\mathbf{Z}) \to (\mathbf{W}\otimes\mathbf{Y})\bullet(\mathbf{X}\otimes\mathbf{Z}),$$
$$u_{X,Y}\colon \mathbf{I}_{W\bullet X} \to \mathbf{I}_W \bullet \mathbf{I}_X,$$
$$k_{\mathbf{e}}\colon \mathbf{e}\otimes\mathbf{e} \to \mathbf{e} \quad \text{and} \quad u_e\colon \mathbf{I}_e \to \mathbf{e},$$

natural in all variables and obeying a number of coherence diagrams.

Example 33. The pseudo double category $\mathbb{C}at$ of Example 2 becomes a monoidal double category where $\bullet$ is given on objects by cartesian product of categories, extended in the evident way to vertical maps, horizontal maps and cells. More generally, the pseudo double category $\mathcal{V}$-$\mathbb{C}at$ becomes a monoidal double category where $\bullet$ is now given by tensor product of $\mathcal{V}$-categories.

We turn now to the apposite notion of *map* between two monoidal double categories. The obvious candidate is that of a *lax map of pseudomonoids* [2] in $\mathbf{DblCat}_\psi$. However, the underlying double morphism of such a map is necessarily a *homo*morphism, and this is not sufficiently general.

To overcome this, we observe that the 2-category **DblCat** also has finite products, and that the inclusion $\mathbf{DblCat}_\psi \to \mathbf{DblCat}$ preserves them. So we view a monoidal double category *a fortiori* as a pseudomonoid in **DblCat**, and define:

Definition 34. A **monoidal double morphism** between monoidal double categories $\mathbb{K}$ and $\mathbb{L}$ is a (lax) map of pseudomonoids $\mathbb{K} \to \mathbb{L}$ in **DblCat**.

Again, the following is entirely routine:

Proposition 35. *Giving a monoidal double morphism $F\colon \mathbb{K} \to \mathbb{L}$ is equivalent to giving a double morphism $F\colon \mathbb{K} \to \mathbb{L}$ such that*

- *F_0 and F_1 are lax monoidal functors;*
- *The equalities $sF_1 = F_0 s$ and $tF_1 = F_0 t$ hold as equalities of lax monoidal functors;*
- *The natural transformations*

$$\mathfrak{m}\colon F_1(-)\otimes F_1(?) \to F_1(-\otimes ?) \quad \textit{and} \quad \mathfrak{e}\colon \mathbf{I}_{F_0(-)} \to F_1(\mathbf{I}_{(-)})$$

  are lax monoidal natural transformations (where we observe that all the functors in question are indeed lax monoidal functors; for instance, $F_1(-)\otimes F_1(?)$ is the composite

$$K_1\ {}_s{\times_t}\ K_1 \xrightarrow{F_{1\,s}{\times_t}F_1} L_1\ {}_s{\times_t}\ L_1 \xrightarrow{\otimes} L_1$$

  which is the composite of a lax monoidal and a strong monoidal functor as required).

We can now define notions of *monoidal double homomorphism*, *opmonoidal double morphism*, *opmonoidal double opmorphism*, and so on. Let us also note the correct notion of vertical transformation between monoidal double morphisms:

Definition 36. A **monoidal vertical transformation** between monoidal double morphisms $F, G \colon \mathbb{K} \to \mathbb{L}$ is a pseudomonoid transformation $F \Rightarrow G$ in **DblCat**.

Proposition 37. *Giving a monoidal vertical transformation $\alpha \colon F \Rightarrow G$ is equivalent to giving a vertical transformation $\alpha \colon F \Rightarrow G$ such that α_0 and α_1 are monoidal transformations.*

Straightforwardly, monoidal double categories, monoidal double morphisms and monoidal vertical transformations form a 2-category **MonDblCat**, along with all the expected variants: $\mathbf{MonDblCat}_\psi$, **OpMonDblCat**, $\mathbf{OpMonDblCat}_o$, and so on.

5.1 The monoidal double category $[\mathbb{K}, \mathbb{K}]_\psi$

Given a small category $\mathbf{C}$, the endofunctor category $[\mathbf{C}, \mathbf{C}]$ acquires the structure of a monoidal category. We shall see in this section that a similar result holds for pseudo double categories, namely, that the endohom double category $[\mathbb{K}, \mathbb{K}]_\psi$ is naturally a monoidal double category.

Just as with transformations between morphisms of bicategories, there are two canonical choices for the composite of two horizontal transformations

$$\mathbf{A} \colon A_s \nRightarrow A_t \colon \mathbb{K} \to \mathbb{K} \quad \text{and} \quad \mathbf{B} \colon B_s \nRightarrow B_t \colon \mathbb{K} \to \mathbb{K},$$

namely

$$A_t\mathbf{B} \otimes \mathbf{A}B_s \quad \text{and} \quad \mathbf{A}B_s \otimes A_t\mathbf{B}.$$

As with the bicategorical case, it makes no material difference which we choose:

Proposition 38. *There are canonical invertible special modifications*

$$i_{\mathbf{A},\mathbf{B}} \colon A_t\mathbf{B} \otimes \mathbf{A}B_s \Rrightarrow \mathbf{A}B_s \otimes A_t\mathbf{B},$$

natural in $\mathbf{A}$ *and* $\mathbf{B}$.

Proof. We take $i_{\mathbf{A},\mathbf{B}}$ to have central natural transformation $A_{B_c(-)}$; so the component of $i_{\mathbf{A},\mathbf{B}}$ at X is given by

$$A_{\mathbf{B}X} \colon A_t\mathbf{B}X \otimes \mathbf{A}B_sX \to \mathbf{A}B_sX \otimes A_t\mathbf{B}X.$$

Visibly this is compatible with source and target, whilst the other modification axiom is a long diagram chase using the axioms for $\mathbf{A}$ and $\mathbf{B}$. For the naturality of these maps in $\mathbf{A}$ and $\mathbf{B}$, suppose we are given modifications $\boldsymbol{\alpha} \colon \mathbf{A} \Rrightarrow \mathbf{C}$ and $\boldsymbol{\beta} \colon \mathbf{B} \Rrightarrow \mathbf{D}$. Then we require the following diagrams to commute for all $X \in K_0$:

$$\begin{array}{ccccc} A_t\mathbf{B}X \otimes \mathbf{A}B_sX & \xrightarrow{A_t\boldsymbol{\beta}_X \otimes \mathbf{A}(\beta_s)_X} & A_t\mathbf{D}X \otimes \mathbf{A}D_sX & \xrightarrow{(\alpha_t)_{\mathbf{D}X} \otimes \boldsymbol{\alpha}_{D_sX}} & C_t\mathbf{D}X \otimes \mathbf{C}D_sX \\ \downarrow{\scriptstyle A_{\mathbf{B}X}} & & \downarrow{\scriptstyle A_{\mathbf{D}X}} & & \downarrow{\scriptstyle C_{\mathbf{D}X}} \\ \mathbf{A}B_sX \otimes A_t\mathbf{B}X & \xrightarrow[\mathbf{A}(\beta_s)_X \otimes A_t\boldsymbol{\beta}_X]{} & \mathbf{A}D_sX \otimes A_t\mathbf{D}X & \xrightarrow[\boldsymbol{\alpha}_{D_sX} \otimes (\alpha_t)_{\mathbf{D}X}]{} & \mathbf{C}D_sX \otimes C_t\mathbf{D}X. \end{array}$$

But the left-hand square is a naturality square for $A_{(-)}$ whilst the right-hand square is one of the axioms for $\boldsymbol{\alpha}$; and hence we are done. $\square$

Proposition 39. *The double category $[\mathbb{K},\mathbb{K}]_\psi$ is a monoidal double category.*

Proof.

- **Monoidal structure on** $[\mathbb{K},\mathbb{K}]_{v\psi}$: Observe that this is the hom-category $\mathbf{DblCat}_\psi(\mathbb{K},\mathbb{K})$ in the 2-category $\mathbf{DblCat}_\psi$, and hence is equipped with a strict monoidal structure;
- **Monoidal structure on** $[\mathbb{K},\mathbb{K}]_{h\psi}$: We take for the tensor unit $\mathbf{e}$, the object

$$\mathbf{e} = \mathbf{I}_{\mathrm{id}} \colon \mathrm{id} \Rrightarrow \mathrm{id}.$$

The tensor product is given as follows:

  - **On objects**: given $\mathbf{A}\colon A_s \Rrightarrow A_t$ and $\mathbf{B}\colon B_s \Rrightarrow B_t$, we take

$$\mathbf{A} \bullet \mathbf{B} = \mathbf{A}B_t \otimes A_s\mathbf{B} \colon A_sB_s \Rrightarrow A_tB_t.$$

Explicitly, this has components

$$(\mathbf{A} \bullet \mathbf{B})(X) = A_sB_sX \xrightarrow{A_s\mathbf{B}X} A_sB_tX \xrightarrow{\mathbf{A}B_tX} A_tB_tX.$$

  - **On maps**: Given $\boldsymbol{\alpha}\colon \mathbf{A} \Rrightarrow \mathbf{C}$ and $\boldsymbol{\beta}\colon \mathbf{B} \Rrightarrow \mathbf{D}$, we take

$$\boldsymbol{\alpha} \bullet \boldsymbol{\beta} = \boldsymbol{\alpha}\beta_t \otimes \alpha_s\boldsymbol{\beta} \colon \mathbf{A}B_t \otimes A_s\mathbf{B} \Rrightarrow \mathbf{C}D_t \otimes C_s\mathbf{D}.$$

The functoriality of $\bullet$ is immediate from the functoriality of $\otimes$ and of the whiskering operations. We must now exhibit the unitality and associativity coherence constraints in $[\mathbb{K},\mathbb{K}]_{v\psi}$. For unitality, we have that $\mathbf{e}\bullet\mathbf{A} = \mathbf{I}_{\mathrm{id}}A_t \otimes \mathbf{A}$ and $\mathbf{A} \bullet \mathbf{e} = \mathbf{A} \otimes A_s\mathbf{I}_{\mathrm{id}}$, and so we give $\rho_{\mathbf{A}}$ and $\lambda_{\mathbf{A}}$ by the special invertible modifications

$$\mathbf{A} \xRightarrow{\mathfrak{l}_{\mathbf{A}}} \mathbf{I}_{A_t} \otimes \mathbf{A} \xRightarrow{\mathrm{id}} \mathbf{I}_{\mathrm{id}}A_t \otimes \mathbf{A}$$
$$\text{and} \quad \mathbf{A} \xRightarrow{\mathfrak{r}_{\mathbf{A}}} \mathbf{A} \otimes \mathbf{I}_{A_s} \xRightarrow{\mathbf{A}\otimes\mathfrak{e}_{\mathbf{A}}} \mathbf{A} \otimes A_s\mathbf{I}_{\mathrm{id}}$$

respectively. The naturality of these in $\mathbf{A}$ follows from the naturality of $\mathfrak{l}$, $\mathfrak{r}$ and $\mathfrak{e}$. For the associativity modifications, suppose we are given $\mathbf{A}\colon A_s \Rrightarrow A_t$, $\mathbf{B}\colon B_s \Rrightarrow B_t$ and $\mathbf{C}\colon C_s \Rrightarrow C_t$. Now we have

$$\mathbf{A} \bullet (\mathbf{B} \bullet \mathbf{C}) = \mathbf{A}(B_tC_t) \otimes A_s(\mathbf{B}C_t \otimes B_s\mathbf{C})$$
$$\text{and } (\mathbf{A} \bullet \mathbf{B}) \bullet \mathbf{C} = (\mathbf{A}B_t \otimes A_s\mathbf{B})C_t \otimes (A_sB_s)\mathbf{C}.$$

Hence we take $\alpha_{\mathbf{A},\mathbf{B},\mathbf{C}}$ to be the special modification

$$\begin{array}{c}
\mathbf{A}(B_tC_t) \otimes A_s(\mathbf{B}C_t \otimes B_s\mathbf{C}) \\
\Big\Downarrow \scriptstyle \mathbf{A}(B_tC_t)\otimes \mathfrak{m}^{-1}_{\mathbf{B}C_t, B_s\mathbf{C}} \\
\mathbf{A}(B_tC_t) \otimes \big(A_s(\mathbf{B}C_t) \otimes A_s(B_s\mathbf{C})\big) \\
\Big\Downarrow \scriptstyle \mathrm{id} \\
(\mathbf{A}B_t)C_t \otimes \big((A_s\mathbf{B})C_t \otimes (A_sB_s)\mathbf{C}\big) \\
\Big\Downarrow \scriptstyle \mathfrak{a}_{(\mathbf{A}B_t)C_t, (A_s\mathbf{B})C_t, (A_sB_s)\mathbf{C}} \\
\big((\mathbf{A}B_t)C_t \otimes (A_s\mathbf{B})C_t\big) \otimes (A_sB_s)\mathbf{C} \\
\Big\Downarrow \scriptstyle \mathrm{id} \\
(\mathbf{A}B_t \otimes A_s\mathbf{B})C_t \otimes (A_sB_s)\mathbf{C}.
\end{array}$$

The naturality of these components in $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ follows from the naturality of $\mathfrak{m}$ and $\mathfrak{a}$; and a routine diagram chase using the coherence axioms for $\mathfrak{l}$, $\mathfrak{r}$, $\mathfrak{a}$, $\mathfrak{m}$ and $\mathfrak{e}$ shows that α, ρ and λ satisfy the associativity pentagon and the unit triangles.

- **s and t: $[\mathbb{K}, \mathbb{K}]_{h\psi} \to [\mathbb{K}, \mathbb{K}]_{v\psi}$ are strict monoidal**: this is immediate from above.

- **$\mathbf{I}$: $[\mathbb{K}, \mathbb{K}]_{v\psi} \to [\mathbb{K}, \mathbb{K}]_{h\psi}$ is strong monoidal**: We observe that $\mathbf{I}_e = \mathbf{e}$, so that $\mathbf{I}$ is strict monoidal with respect to the unit. For the binary tensor $\bullet$, we have $\mathbf{I}_F \bullet \mathbf{I}_G = \mathbf{I}_F G \otimes F\mathbf{I}_G$, and so we take $u_{F,G}\colon \mathbf{I}_{FG} \Rrightarrow \mathbf{I}_F \bullet \mathbf{I}_G$ to be the special invertible modification

$$\mathbf{I}_{FG} \overset{\mathfrak{l}_{\mathbf{I}_{FG}}}{\Longrightarrow} \mathbf{I}_{FG} \otimes \mathbf{I}_{FG} \overset{\mathrm{id}\otimes\mathfrak{e}_G}{\Longrightarrow} \mathbf{I}_F G \otimes F\mathbf{I}_G.$$

Again, naturality in F and G follows from naturality of $\mathfrak{e}$, and it's easy to check that the three diagrams making $\mathbf{I}$ strong monoidal commute.

- **$\otimes$: $[\mathbb{K}, \mathbb{K}]_{h\psi}\ {}_s\times_t\ [\mathbb{K}, \mathbb{K}]_{h\psi} \to [\mathbb{K}, \mathbb{K}]_{h\psi}$ is strong monoidal**: Since $\mathbf{I}_e = \mathbf{e}$, we can take

$$k_{\mathbf{e}}\colon \mathbf{e} \otimes \mathbf{e} \to \mathbf{e}$$

to be the canonical map $\mathfrak{r}^{-1}_{\mathbf{I_e}} = \mathfrak{l}^{-1}_{\mathbf{I_e}}$. Now, suppose we are given horizontal transformations

$$\begin{aligned} &\mathbf{A}\colon A_1 \nRightarrow A_2, && \mathbf{A}'\colon A_2 \nRightarrow A_3, \\ &\mathbf{B}\colon B_1 \nRightarrow B_2 && \text{and } \mathbf{B}'\colon B_2 \nRightarrow B_3. \end{aligned}$$

Then

$$(\mathbf{A}' \bullet \mathbf{B}') \otimes (\mathbf{A} \bullet \mathbf{B}) = (\mathbf{A}'B_3 \otimes A_2\mathbf{B}') \otimes (\mathbf{A}B_2 \otimes A_1\mathbf{B})$$

whilst

$$(\mathbf{A}' \otimes \mathbf{A}) \bullet (\mathbf{B}' \otimes \mathbf{B}) = (\mathbf{A}' \otimes \mathbf{A})B_3 \otimes A_1(\mathbf{B}' \otimes \mathbf{B}).$$

Therefore we take for $k_{\mathbf{A}',\mathbf{B}',\mathbf{A},\mathbf{B}} \colon (\mathbf{A}' \bullet \mathbf{B}') \otimes (\mathbf{A} \bullet \mathbf{B}) \to (\mathbf{A}' \otimes \mathbf{A}) \bullet (\mathbf{B}' \otimes \mathbf{B})$ the special invertible modification

$$\begin{array}{c}
(\mathbf{A}'B_3 \otimes A_2\mathbf{B}') \otimes (\mathbf{A}B_2 \otimes A_1\mathbf{B}) \\
\Big\Downarrow \mathfrak{a} \\
\big(\mathbf{A}'B_3 \otimes (A_2\mathbf{B}' \otimes \mathbf{A}B_2)\big) \otimes A_1\mathbf{B} \\
\Big\Downarrow (\mathbf{A}'B_3 \otimes i_{\mathbf{A},\mathbf{B}'}) \otimes A_1\mathbf{B} \\
\big(\mathbf{A}'B_3 \otimes (\mathbf{A}B_3 \otimes A_1\mathbf{B}')\big) \otimes A_1\mathbf{B}s \\
\Big\Downarrow \mathfrak{a} \\
(\mathbf{A}'B_3 \otimes \mathbf{A}B_3) \otimes (A_1\mathbf{B}' \otimes A_1\mathbf{B}) \\
\Big\Downarrow \mathrm{id} \otimes \mathfrak{m}_{\mathbf{B}',\mathbf{B}} \\
(\mathbf{A}' \otimes \mathbf{A})B_3 \otimes A_1(\mathbf{B}' \otimes \mathbf{B})
\end{array}$$

where the maps labelled $\mathfrak{a}$ are appropriate composites of associativity maps. The naturality of the displayed map in all variables follows from the naturality of $\mathfrak{a}$, i and $\mathfrak{m}$. It's now a diagram chase to check that the required coherence laws hold to make $\otimes$ strong monoidal.

- **The natural transformations $\mathfrak{a}$, $\mathfrak{l}$ and $\mathfrak{r}$ are strong monoidal transformations**: This is another routine diagram chase. ☐

5.2 Monoidal comma double categories

We now wish to mimic the result that, in the theory of clubs, tells us that $[\mathbf{C}, \mathbf{C}]/S$ acquires a natural structure of monoidal category. As there, we consider the lax limit of an arrow $\ulcorner \mathbf{X} \urcorner : 1 \to \mathbb{K}$, but this time in the 2-category **MonDblCat**. Such an arrow amounts to a *monoidal monad* in $\mathbb{K}$:

Definition 40. A **monoidal monad** in the monoidal double category $\mathbb{K}$ consists of:

- A monad $(\mathbf{X} \colon X \to\!\!\!\!\!\!\!\!\;\shortmid\;\; X, \mathfrak{m}, \mathfrak{e})$ in $\mathbb{K}$;
- Maps

$$\boldsymbol{\mu} \colon \mathbf{X} \bullet \mathbf{X} \to \mathbf{X}, \quad \boldsymbol{\eta} \colon \mathbf{e} \to \mathbf{X}$$
$$\mu \colon X \bullet X \to X, \quad \text{and} \quad \eta \colon e \to X$$

such that:

- $s(\boldsymbol{\mu}) = t(\boldsymbol{\mu}) = \mu$ and $s(\boldsymbol{\eta}) = t(\boldsymbol{\eta}) = \eta$;
- $(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\eta})$ is a monoid in the monoidal category K_1;
- (X, μ, η) is a monoid in the monoidal category K_0;

- The following diagrams commute:

$$\begin{array}{ccc}
(\mathbf{X} \bullet \mathbf{X}) \otimes (\mathbf{X} \bullet \mathbf{X}) & \xrightarrow{k_{\mathbf{X},\mathbf{X},\mathbf{X},\mathbf{X}}} & (\mathbf{X} \otimes \mathbf{X}) \bullet (\mathbf{X} \otimes \mathbf{X}) \\
\downarrow{\scriptstyle \boldsymbol{\mu}\otimes\boldsymbol{\mu}} & & \downarrow{\scriptstyle \mathfrak{m}\bullet\mathfrak{m}} \\
\mathbf{X} \otimes \mathbf{X} & & \mathbf{X} \bullet \mathbf{X} \\
\downarrow{\scriptstyle \mathfrak{m}} & & \downarrow{\scriptstyle \boldsymbol{\mu}} \\
\mathbf{X} & \xrightarrow[\mathrm{id}]{} & \mathbf{X}
\end{array}$$

$$\begin{array}{ccc}
\mathbf{I}_{X\bullet X} & \xrightarrow{u_{X,X}} & \mathbf{I}_X \bullet \mathbf{I}_X \\
\downarrow{\scriptstyle \mathbf{I}_\mu} & & \downarrow{\scriptstyle \mathfrak{e}\bullet\mathfrak{e}} \\
\mathbf{I}_X & & \mathbf{X} \bullet \mathbf{X} \\
\downarrow{\scriptstyle \mathfrak{e}} & & \downarrow{\scriptstyle \boldsymbol{\mu}} \\
\mathbf{X} & \xrightarrow[\mathrm{id}]{} & \mathbf{X}
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{e} \otimes \mathbf{e} & \xrightarrow{k_{\mathbf{e}}} & \mathbf{e} \\
\downarrow{\scriptstyle \boldsymbol{\eta}\otimes\boldsymbol{\eta}} & & \\
\mathbf{X} \otimes \mathbf{X} & & \downarrow{\scriptstyle \boldsymbol{\eta}} \\
\downarrow{\scriptstyle \mathfrak{m}} & & \\
\mathbf{X} & \xrightarrow[\mathrm{id}]{} & \mathbf{X}
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{I}_e & \xrightarrow{u_e} & \mathbf{e} \\
\downarrow{\scriptstyle \mathbf{I}_\eta} & & \\
\mathbf{I}_X & & \downarrow{\scriptstyle \boldsymbol{\eta}} \\
\downarrow{\scriptstyle \mathfrak{e}} & & \\
\mathbf{X} & \xrightarrow[\mathrm{id}]{} & \mathbf{X}.
\end{array}$$

Proposition 41. *Let* $\mathbb{K}$ *be a monoidal double category, and let* $(\mathbf{X}, \mathfrak{m}, \mathfrak{e}, \boldsymbol{\mu}, \boldsymbol{\eta})$ *be a monoidal monad in* $\mathbb{K}$*. Then the slice double category* $\mathbb{K}/\mathbf{X}$ *can be equipped with the structure of a monoidal double category in such a way as to become the lax limit of the arrow* $\ulcorner \mathbf{X} \urcorner : 1 \to \mathbb{K}$ *in* **MonDblCat***.*

Proof. We see that $\mathbf{X}$ and X are monoids in the respective monoidal categories K_1 and K_0, and therefore $K_1/\mathbf{X}$ and K_0/X become monoidal categories. It is straightforward to check that s and t are strict monoidal with respect to this structure; for example, given $(\mathbf{U}, \mathbf{f})$ and $(\mathbf{U}', \mathbf{f}')$ in $K_1/\mathbf{X}$, we have $(\mathbf{U}, \mathbf{f}) \bullet (\mathbf{U}', \mathbf{f}')$ given by

$$\mathbf{U} \bullet \mathbf{U}' \xrightarrow{\mathbf{f}\bullet\mathbf{f}'} \mathbf{X} \bullet \mathbf{X} \xrightarrow{\boldsymbol{\mu}} \mathbf{X},$$

whose image under s is the object

$$U_s \bullet U'_s \xrightarrow{f_s\bullet f'_s} X \bullet X \xrightarrow{\mu} X$$

which is $(U_s, f_s) \bullet (U'_s, f'_s)$ as required. It remains to specify the invertible transformations k and u and the invertible maps $k_{\mathbf{e}}$ and u_e; the latter lift straightforwardly from $\mathbb{K}$, and the former we give as follows:

$$k_{(\mathbf{U},\mathbf{f}),(\mathbf{U}',\mathbf{f}'),(\mathbf{V},\mathbf{g}),(\mathbf{V}',\mathbf{g}')} = k_{\mathbf{U},\mathbf{U}',\mathbf{V},\mathbf{V}'},$$
$$\text{and } u_{(U,f),(V,g)} = u_{U,V}.$$

That the required triangles commute for these to be maps in $K_1/\mathbf{X}$ follows from the coherence diagrams for $\mathbf{X}$; their naturality follows from the naturality of k and u for $\mathbb{K}$; and finally the coherence diagrams that they are required to satisfy follow using the coherence diagrams for $\mathbf{X}$ and $\mathbb{K}$. ☐

In order to use this result in our theory of double clubs, we shall need the following:

Proposition 42. *Let (S, μ, η) be a double monad on a double category $\mathbb{K}$. Then the monad $S\mathbf{I}$ in the monoidal double category $[\mathbb{K}, \mathbb{K}]_\psi$ is canonically a monoidal monad.*

Proof. S is a monad in $\mathbf{DblCat}_\psi$, and thus a monoid in $\mathbf{DblCat}_\psi(\mathbb{K}, \mathbb{K}) = [\mathbb{K}, \mathbb{K}]_{v\psi}$. We equip the object $S\mathbf{I} \in [\mathbb{K}, \mathbb{K}]_{h\psi}$ with monoid structure as follows. Recall that $S\mathbf{I}$ is in fact the monad $S\mathbf{I}_{\mathrm{id}_\mathbb{K}}$; so we give the unit $\boldsymbol{\eta} \colon \mathbf{I}_{\mathrm{id}_\mathbb{K}} \Rrightarrow S\mathbf{I}$ by the modification

$$\mathbf{I}_{\mathrm{id}_\mathbb{K}} \xRightarrow{\eta \mathbf{I}_{\mathrm{id}_\mathbb{K}}} S\mathbf{I}_{\mathrm{id}_\mathbb{K}}.$$

For the multiplication, observe first that we have $S\mathbf{I} \bullet S\mathbf{I} = (S\mathbf{I}_{\mathrm{id}_\mathbb{K}})S \otimes S(S\mathbf{I}_{\mathrm{id}_\mathbb{K}}) = S\mathbf{I}_S \otimes S(S\mathbf{I}_{\mathrm{id}_\mathbb{K}})$. Therefore we take for $\boldsymbol{\mu} \colon S\mathbf{I} \bullet S\mathbf{I} \Rrightarrow S\mathbf{I}$ the modification

$$S\mathbf{I}_S \otimes S(S\mathbf{I}_{\mathrm{id}_\mathbb{K}}) \xRightarrow{\mathfrak{m}_{\mathbf{I}_S, S\mathbf{I}_{\mathrm{id}_\mathbb{K}}}} S(\mathbf{I}_S \otimes S\mathbf{I}_{\mathrm{id}_\mathbb{K}}) \xRightarrow{S\mathfrak{l}^{-1}_{S\mathbf{I}_{\mathrm{id}_\mathbb{K}}}} SS\mathbf{I}_{\mathrm{id}_\mathbb{K}} \xRightarrow{\mu \mathbf{I}_{\mathrm{id}_\mathbb{K}}} S\mathbf{I}_{\mathrm{id}_\mathbb{K}}.$$

It's straightforward to check that this makes $S\mathbf{I}$ into a monoid in $[\mathbb{K}, \mathbb{K}]_{h\psi}$. Further, s and t send it to the monoid S in $[\mathbb{K}, \mathbb{K}]_{v\psi}$ as required. Finally, the diagrams expressing the compatibility of the monoid and monad structure on S are easily verified. ☐

Assembling the previous two results, we have:

Proposition 43. *Given a double monad (S, η, μ) on a double category $\mathbb{K}$, the slice double category $[\mathbb{K}, \mathbb{K}]_\psi / S\mathbf{I}$ has a natural structure of monoidal double category.*

6 Double clubs II

We now have enough pseudo double category theory under our belt to define the notion of a double club. First a few preliminaries:

Definition 44. Let $\mathbb{K}$ and $\mathbb{L}$ be double categories.

- We say that $\mathbb{K}$ is a **vertically full sub-double category** of $\mathbb{L}$ if there is a strict homomorphism $F \colon \mathbb{K} \to \mathbb{L}$ such that F_0 and F_1 exhibit K_0 and K_1 as full subcategories of L_0 and L_1.
- If $\mathbb{K}$ and $\mathbb{L}$ are monoidal double categories, we say that $\mathbb{K}$ is a **sub-monoidal double category** of $\mathbb{L}$ if there is a strict monoidal strict homomorphism $F \colon \mathbb{K} \to \mathbb{L}$ exhibiting K_0 and K_1 as subcategories of L_0 and L_1.

In particular, if $\mathbb{K}$ is a vertically full sub-double category of a monoidal double category $\mathbb{L}$, then $\mathbb{K}$ can be made into a sub-monoidal double category of $\mathbb{L}$ if and only the object sets of K_0 and K_1 are closed under the binary and nullary tensors on L_0 and L_1 respectively.

Definition 45. Let (S, η, μ) be a double monad on a double category $\mathbb{K}$. We say that S is a **double club** if:

- S has property (hps);

- $\mathbb{C}oll(S)$ is a sub-monoidal double category of $[\mathbb{K}, \mathbb{K}]_\psi/S\mathbf{I}$.

Note that this is simply the natural generalisation of Definition 19: the extra requirement that condition (hps) be satisfied is necessary to ensure that $\mathbb{C}oll(S)$ exists in the first place; in the plain category case, the existence of the 'category of collections' is automatic.

The above definition of a double club, though compact, is not very easy to work with: but as with plain clubs, there is a more hands-on description which greatly simplifies the task of applying the theory.

We begin by observing that if (S, η, μ) is a double monad on $\mathbb{K}$, then (S_0, η_0, μ_0) is a monad on K_0 and (S_1, η_1, μ_1) a monad on K_1. Therefore it makes sense to ask whether or not S_0 and S_1 are clubs in the sense of Section 3 on their respective categories, and once we have asked this, we may naturally ask whether this is sufficient to make S into a double club. In fact, as long as S has property (hps), the answer is yes:

Proposition 46. *If (S, η, μ) is a double monad on $\mathbb{K}$ such that:*

- *S has property (hps);*
- *S_0 and S_1 are clubs on the categories K_0 and K_1 respectively,*

then S is a double club.

Proof. We must check that $\mathbb{C}oll(S)$ is a sub-monoidal double category of $[\mathbb{K}, \mathbb{K}]_\psi/S\mathbf{I}$. Since $\mathbb{C}oll(S)$ is a vertically full sub-double category of $[\mathbb{K}, \mathbb{K}]_\psi/S\mathbf{I}$, it suffices to check that:

- $\mathbb{C}oll(S)_0$ is closed under the monoidal structure on $[\mathbb{K}, \mathbb{K}]_{v\psi}/S$;
- $\mathbb{C}oll(S)_1$ is closed under the monoidal structure on $[\mathbb{K}, \mathbb{K}]_{h\psi}/S\mathbf{I}$.

We begin with $\mathbb{C}oll(S)_0$. We have evident forgetful functors

$$\pi_i \colon [\mathbb{K}, \mathbb{K}]_{v\psi}/S \to [K_i, K_i]/S_i \quad (\text{for } i = 0, 1)$$

which are strict monoidal. Since S_0 and S_1 are clubs, $Coll(S_i)$ is closed under the monoidal structure on $[K_i, K_i]/S_i$. But an object A of $[\mathbb{K}, \mathbb{K}]_{v\psi}$ lies in $\mathbb{C}oll(S)_0$ just when its projections $\pi_i(A)$ lie in $Coll(S_i)$; and hence we see that $\mathbb{C}oll(S)_0$ is closed under the monoidal structure on $[\mathbb{K}, \mathbb{K}]_{v\psi}$ as required.

Moving on to $\mathbb{C}oll(S)_1$, we first show that the unit object $\boldsymbol{\eta} \colon \mathbf{I}_{\mathrm{id}_\mathbb{K}} \Rightarrow S\mathbf{I}$ of $[\mathbb{K}, \mathbb{K}]_{h\psi}$ lies in $\mathbb{C}oll(S)_1$. By Proposition 20 and the fact that S_0 and S_1 are clubs, we have that η_0 and η_1 are cartesian natural transformations; hence $\eta \colon \mathrm{id}_\mathbb{K} \Rightarrow S$ is a cartesian vertical transformation. It remains to show that the central natural transformation of $\boldsymbol{\eta}$ is cartesian, i.e., that diagrams of the following form are pullbacks:

$$\begin{array}{ccc} \mathbf{I}_X & \xrightarrow{\mathbf{I}_!} & \mathbf{I}_1 \\ {\scriptstyle \eta_{\mathbf{I}_X}} \big\downarrow & & \big\downarrow {\scriptstyle \eta_{\mathbf{I}_1}} \\ S\mathbf{I}_X & \xrightarrow[S\mathbf{I}_!]{} & S\mathbf{I}_1, \end{array}$$

which is just the cartesianness of η_0. We now show that $\mathbb{C}oll(S)_1$ is closed under the binary tensor product on $[\mathbb{K},\mathbb{K}]_{h\psi}$. So suppose we are given objects $(\mathbf{A},\boldsymbol{\alpha})$ and $(\mathbf{B},\boldsymbol{\beta})$ of $\mathbb{C}oll(S)_1$; then their tensor product is given by

$$\begin{array}{ccc} A_sB_s & \overset{\mathbf{A}\bullet\mathbf{B}}{\Longrightarrow} & A_tB_t \\ {\scriptstyle \alpha_s\beta_s}\Downarrow & \Downarrow\boldsymbol{\alpha}\bullet\boldsymbol{\beta} & \Downarrow{\scriptstyle \alpha_t\beta_t} \\ SS & \overset{S\mathbf{I}\bullet S\mathbf{I}}{\Longrightarrow} & SS \\ {\scriptstyle\mu}\Downarrow & \Downarrow\boldsymbol{\mu} & \Downarrow{\scriptstyle\mu} \\ S & \underset{S\mathbf{I}}{\Longrightarrow} & S, \end{array}$$

so it suffices to show that $\boldsymbol{\alpha}\bullet\boldsymbol{\beta}$ and $\boldsymbol{\mu}$ are cartesian modifications. We begin with $\boldsymbol{\alpha}\bullet\boldsymbol{\beta}$; the cartesianness of $\alpha_s\beta_s$ and $\alpha_t\beta_t$ follows from the fact that S_1 and S_0 are clubs on K_1 and K_0, and so it suffices to check that the central natural transformation of $\boldsymbol{\alpha}\bullet\boldsymbol{\beta}$ is cartesian. This central natural transformation has components

$$\mathbf{A}B_tX\otimes A_s\mathbf{B}X \xrightarrow{\boldsymbol{\alpha}_{B_tX}\otimes(\alpha_s)_{\mathbf{B}X}} S\mathbf{I}_{B_tX}\otimes S\mathbf{B}X \xrightarrow{S\mathbf{I}_{(\beta_t)_X}\otimes S\boldsymbol{\beta}_X} S\mathbf{I}_{SX}\otimes SS\mathbf{I}_X\,.$$

So, consider the following diagram:

$$\begin{array}{ccccc} & \mathbf{A}B_tX & \xrightarrow{\mathbf{A}B_t!} & \mathbf{A}B_t1 & \\ & {\scriptstyle\boldsymbol{\alpha}_{B_tX}}\downarrow & & \downarrow{\scriptstyle\boldsymbol{\alpha}_{B_t1}} & \\ & S\mathbf{I}_{B_tX} & \xrightarrow[S\mathbf{I}_{B_t!}]{} & S\mathbf{I}_{B_t1} & \\ & {\scriptstyle S\mathfrak{e}_X}\downarrow & & \downarrow{\scriptstyle S\mathfrak{e}_1} & \\ {\scriptstyle S\mathbf{I}_{(\beta_t)_X}} & SB_t\mathbf{I}_X & \xrightarrow[SB_t\mathbf{I}_!]{} & SB_t\mathbf{I}_1 & {\scriptstyle S\mathbf{I}_{(\beta_t)_1}} \\ & {\scriptstyle S(\beta_t)_{\mathbf{I}_X}}\downarrow & & \downarrow{\scriptstyle S(\beta_t)_{\mathbf{I}_1}} & \\ & SS\mathbf{I}_X & \xrightarrow[SS\mathbf{I}_!]{} & SS\mathbf{I}_1 & \\ & {\scriptstyle S\mathfrak{e}_X^{-1}}\downarrow & & \downarrow{\scriptstyle S\mathfrak{e}_1^{-1}} & \\ & S\mathbf{I}_{SX} & \xrightarrow[S\mathbf{I}_{S!}]{} & S\mathbf{I}_{S1}. & \end{array}$$

The top square is a pullback by cartesianness of α, the second and fourth are pullbacks since their vertical sides are isomorphisms, and the third square is a pullback by cartesianness of β_t and because S_1 preserves cartesian natural transformations into S_1. Therefore the outside edge of this diagram is a pullback. Similarly, considering the diagram

$$\begin{array}{ccc} A_s\mathbf{B}X & \xrightarrow{A_s\mathbf{B}!} & A_s\mathbf{B}1 \\ {\scriptstyle(\alpha_s)_{\mathbf{B}X}}\downarrow & & \downarrow{\scriptstyle(\alpha_s)_{\mathbf{B}1}} \\ S\mathbf{B}X & \xrightarrow[S\mathbf{B}!]{} & S\mathbf{B}1 \\ {\scriptstyle S\boldsymbol{\beta}_X}\downarrow & & \downarrow{\scriptstyle S\boldsymbol{\beta}_1} \\ SS\mathbf{I}_X & \xrightarrow[SS\mathbf{I}_!]{} & SS\mathbf{I}_1, \end{array}$$

the top square is a pullback by cartesianness of α_s, whilst the bottom square is a pullback by cartesianness of β and the fact that S_1 preserves cartesian transformations into S_1. Thus, forming the tensor product of these two diagrams and applying condition (hps1), we see therefore that the naturality squares for $(\boldsymbol{\alpha} \bullet \boldsymbol{\beta})_c$ are pullbacks as required.

Finally, we check that $\boldsymbol{\mu}$ is a cartesian modification. By Proposition 20 and the fact that S_0 and S_1 are clubs, we have that μ_0 and μ_1 are cartesian natural transformations; hence $\mu\colon SS \Rightarrow S$ is a cartesian vertical transformation. So we need only check that the central natural transformation of $\boldsymbol{\mu}$ is cartesian, for which we must check that the outer edge of the following diagram is a pullback:

$$\begin{array}{ccc}
S\mathbf{I}_{SX} \otimes SS\mathbf{I}_X & \xrightarrow{S\mathbf{I}_{S!} \otimes SS!} & S\mathbf{I}_{S1} \otimes SS\mathbf{I}_1 \\
{\scriptstyle \mathfrak{m}_{\mathbf{I}_{SX}, S\mathbf{I}_X}} \downarrow & & \downarrow {\scriptstyle \mathfrak{m}_{\mathbf{I}_{S1}, S\mathbf{I}_1}} \\
S(\mathbf{I}_{SX} \otimes S\mathbf{I}_X) & \xrightarrow{S(\mathbf{I}_{S!} \otimes S!)} & S(\mathbf{I}_{S1} \otimes S\mathbf{I}_1) \\
{\scriptstyle S\mathfrak{l}^{-1}_{S\mathbf{I}_X}} \downarrow & & \downarrow {\scriptstyle S\mathfrak{l}^{-1}_{S\mathbf{I}_1}} \\
SS\mathbf{I}_X & \xrightarrow{SS!} & SS\mathbf{I}_1 \\
{\scriptstyle \mu_{\mathbf{I}_X}} \downarrow & & \downarrow {\scriptstyle \mu_{\mathbf{I}_1}} \\
S\mathbf{I}_X & \xrightarrow[S!]{} & S\mathbf{I}_1 .
\end{array}$$

Now, the bottom square is a pullback by cartesianness of μ, whilst all other squares are pullbacks since they have isomorphisms along their vertical edges; hence the outer edge is a pullback as required. □

7 The double club for symmetric strict monoidal categories

In Example 6, we saw that the monad on **Cat** for symmetric strict monoidal categories extends to a double monad S on $\mathbb{C}at$. In Example 21, we saw that this monad on **Cat** is in fact a *club* on **Cat**. What we are now in a position to show is that the double monad S on $\mathbb{C}at$ is likewise a double club on $\mathbb{C}at$.

Using Proposition 46, this task is reduced to the following: firstly, checking that S_0 and S_1 are clubs on their respective categories, and secondly, showing that S has property (hps). We have already seen in Example 21 that S_0 is a club on $\mathbb{C}at_0 = \mathbf{Cat}$, and the following is a straightforward but tedious calculation from the definitions:

Proposition 47. *The monad (S_1, η_1, μ_1) is a club on $\mathbb{C}at_1$.*

Therefore it remains only to show that S satisfies property (hps), for which we shall use the following two propositions:

Proposition 48. *Suppose that*

$$\begin{array}{ccc} D & \xrightarrow{j} & C \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[f]{} & A \end{array}$$

is a pullback in $\mathbb{C}at_0$*; then so is*

$$\begin{array}{ccc} \mathbf{I}_D & \xrightarrow{\mathbf{I}_j} & \mathbf{I}_C \\ {\scriptstyle \mathbf{I}_k}\downarrow & & \downarrow{\scriptstyle \mathbf{I}_g} \\ \mathbf{I}_B & \xrightarrow[\mathbf{I}_f]{} & \mathbf{I}_A \end{array}$$

in $\mathbb{C}at_1$*.*

Proof. Viewing $\mathbb{C}at_1$ as $\mathbf{Cat}/\mathbf{2}$, we see that the functor $\mathbf{I}_{(\ )}\colon \mathbb{C}at_0 \to \mathbb{C}at_1$ sends D to $(D \times \mathbf{2}) \xrightarrow{\pi_2} \mathbf{2}$, and is thus right adjoint to the domain functor $\mathbf{Cat}/\mathbf{2} \to \mathbf{Cat}$. Thus $\mathbf{I}_{(\ )}$ preserves small limits and so *a fortiori* the result. ☐

Proposition 49. *Let* A *be a small groupoidal category and suppose we are given pullback diagrams*

$$(23) := \begin{array}{ccc} \mathbf{D}_{23} & \xrightarrow{\mathbf{j}_{23}} & \mathbf{C}_{23} \\ {\scriptstyle \mathbf{k}_{23}}\downarrow & & \downarrow{\scriptstyle \mathbf{g}_{23}} \\ \mathbf{B}_{23} & \xrightarrow[\mathbf{f}_{23}]{} & \mathbf{I}_A \end{array} \quad \textit{and} \quad (12) := \begin{array}{ccc} \mathbf{D}_{12} & \xrightarrow{\mathbf{j}_{12}} & \mathbf{C}_{12} \\ {\scriptstyle \mathbf{k}_{12}}\downarrow & & \downarrow{\scriptstyle \mathbf{g}_{12}} \\ \mathbf{B}_{12} & \xrightarrow[\mathbf{f}_{12}]{} & \mathbf{I}_A \end{array}$$

in $\mathbb{C}at_1$ *with*

$$s(12) = \begin{array}{ccc} D_1 & \xrightarrow{j_1} & C_1 \\ {\scriptstyle k_1}\downarrow & & \downarrow{\scriptstyle g_1} \\ B_1 & \xrightarrow[f_1]{} & A, \end{array} \qquad t(12) = \begin{array}{ccc} D_2 & \xrightarrow{j_2} & C_2 \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle g_2} \\ B_2 & \xrightarrow[f_2]{} & A, \end{array}$$

$$s(23) = \begin{array}{ccc} D_2 & \xrightarrow{j_2} & C_2 \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle g_2} \\ B_2 & \xrightarrow[f_2]{} & A, \end{array} \quad \textit{and} \quad t(23) = \begin{array}{ccc} D_3 & \xrightarrow{j_3} & C_3 \\ {\scriptstyle k_3}\downarrow & & \downarrow{\scriptstyle g_3} \\ B_3 & \xrightarrow[f_3]{} & A. \end{array}$$

Suppose further that the arrow $f_2\colon B_2 \to A$ *is a fibration; then the diagram*

$$(13) := \begin{array}{ccc} \mathbf{D}_{23} \otimes \mathbf{D}_{12} & \xrightarrow{\mathbf{j}_{23} \otimes \mathbf{j}_{12}} & \mathbf{C}_{23} \otimes \mathbf{C}_{12} \\ {\scriptstyle \mathbf{k}_{23} \otimes \mathbf{k}_{12}}\downarrow & & \downarrow{\scriptstyle \mathbf{g}_{23} \otimes \mathbf{g}_{12}} \\ \mathbf{B}_{23} \otimes \mathbf{B}_{12} & \xrightarrow[\mathbf{f}_{23} \otimes \mathbf{f}_{12}]{} & \mathbf{I}_A \otimes \mathbf{I}_A \end{array}$$

is also a pullback.

Proof. First some notation; we shall use b_i, c_i and d_i to denote typical elements of B_i, C_i and D_i (for $i = 1, \ldots, 3$), and similarly use a_i to denote elements of A, with the convention that

$$k_i(d_i) = b_i, \quad j_i(d_i) = c_i, \quad \text{and} \quad f_i(b_i) = a_i = g_i(c_i).$$

So now, let $\mathbf{E} = (E_1, E_2, E)$ be the pullback

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{\mathbf{j}'} & \mathbf{C}_{23} \otimes \mathbf{C}_{12} \\ {\scriptstyle \mathbf{k}'}\downarrow & & \downarrow{\scriptstyle \mathbf{g}_{23} \otimes \mathbf{g}_{12}} \\ \mathbf{B}_{23} \otimes \mathbf{B}_{12} & \xrightarrow[\mathbf{f}_{23} \otimes \mathbf{f}_{12}]{} & \mathbf{I}_A \otimes \mathbf{I}_A. \end{array}$$

The universal property of pullback induces a canonical arrow

$$\mathbf{u} = (u_1, u_2, u) \colon \mathbf{D}_{23} \otimes \mathbf{D}_{12} \to \mathbf{E}$$

in $\mathbb{C}at_1$. It suffices to show that this map is an isomorphism. Observe first that $s(13) = s(12)$ and $t(13) = t(23)$, and thus that these projections are pullback diagrams in **Cat**. Thus we may take it that $E_1 = D_1$ and $E_2 = D_3$, and that $u_1 = \mathrm{id}_{D_1}$ and $u_2 = \mathrm{id}_{D_3}$. Thus we need only concern ourselves with the 2-cell u; we shall exhibit an inverse v for this 2-cell. First, let us describe explicitly what u does. A typical element of $\mathbf{D}_{23} \otimes \mathbf{D}_{12}(d_3; d_1)$ looks like

$$\big((\alpha, \gamma) \otimes (\beta, \delta)\big) = \big((b_3, c_3) \overset{(\alpha,\gamma)}{\nrightarrow} (b_2, c_2)\big) \otimes \big((b_2, c_2) \overset{(\beta,\delta)}{\nrightarrow} (b_1, c_1)\big)$$

where $\alpha \colon b_3 \nrightarrow b_2$, $\beta \colon b_2 \nrightarrow b_1$, $\gamma \colon c_3 \nrightarrow c_2$, and $\delta \colon c_2 \nrightarrow c_1$ satisfy

$$a_3 \xrightarrow{f_{23}(\alpha)} a_2 = a_3 \xrightarrow{g_{23}(\gamma)} a_2 \quad \text{and} \quad a_2 \xrightarrow{f_{12}(\beta)} a_1 = a_2 \xrightarrow{g_{23}(\delta)} a_1,$$

whilst a typical element $\big((\alpha \otimes \beta), (\gamma \otimes \delta)\big)$ of $\mathbf{E}(d_3; d_1)$ looks like

$$\big((b_3 \overset{\alpha}{\nrightarrow} b) \otimes (b \overset{\beta}{\nrightarrow} b_1), (c_3 \overset{\gamma}{\nrightarrow} c) \otimes (c \overset{\delta}{\nrightarrow} c_1)\big)$$

where

$$a_3 \xrightarrow{f_{23}(\alpha)} f_2(b) \xrightarrow{f_{12}(\beta)} a_1 = a_3 \xrightarrow{g_{23}(\gamma)} g_2(c) \xrightarrow{g_{12}(\delta)} a_1$$

in A. Then the 2-cell u has components given by

$$\begin{aligned} u_{d_3,d_1} \colon \mathbf{D}_{23} \otimes \mathbf{D}_{12}(d_3; d_1) &\to \mathbf{E}(d_3; d_1) \\ \big((\alpha, \gamma) \otimes (\beta, \delta)\big) &\mapsto (\alpha \otimes \beta, \gamma \otimes \delta). \end{aligned}$$

Now let us construct the promised inverse v for this 2-cell. Suppose we are given an element $(\alpha \otimes \beta, \gamma \otimes \delta) \in \mathbf{E}(d_3; d_1)$; we must send this to an element of $\mathbf{D}_{23} \otimes \mathbf{D}_{12}(d_3; d_1)$. So consider the map

$$\psi := f_2(b) \xrightarrow{f_{23}(\alpha)^{-1}} a_3 \xrightarrow{g_{23}(\gamma)} g_2(c)$$

in A. The functor $f_2 \colon B_2 \to A$ is a fibration and A is a groupoid; thus f_2 is also a *co*fibration, and so we can lift the displayed map to a cocartesian arrow $\hat{\psi} \colon b \to \psi^* b$ in B_2; and since ψ is invertible, so is $\hat{\psi}$. So now we set $v\big((\alpha \otimes \beta, \gamma \otimes \delta)\big)$ to be

$$\big((b_3, c_3) \xrightarrow{(\hat{\psi}\alpha, \gamma)} (\psi^* b, c)\big) \otimes \big((\psi^* b, c) \xrightarrow{(\beta\hat{\psi}^{-1}, \delta)} (b_1, c_1)\big).$$

For this to be well-defined we need to check firstly that it does indeed map into $\mathbf{D}_{23} \otimes \mathbf{D}_{12}(d_3; d_1)$; and secondly that it is independent of the choice of representative for $(\alpha \otimes \beta, \gamma \otimes \delta)$, both of which are fairly tedious calculations which we therefore omit. We must also check that v is indeed inverse to u. We have $u\big((\alpha, \gamma) \otimes (\beta, \delta)\big) = (\alpha \otimes \beta, \gamma \otimes \delta)$, and thus $v\big(u\big((\alpha, \gamma) \otimes (\beta, \delta)\big)\big)$ is given by

$$\big((b_3, c_3) \xrightarrow{(\hat{\psi}\alpha, \gamma)} (\psi^* b_2, c_2)\big) \otimes \big((\psi^* b_2, c_2) \xrightarrow{(\beta\hat{\psi}^{-1}, \delta)} (b_1, c_1)\big),$$

where $\psi := g_{23}(\gamma) \circ f_{23}(\alpha)^{-1}$. But by definition of $\mathbf{D}_{23} \otimes \mathbf{D}_{12}$, we have $f_{23}(\alpha) = g_{23}(\gamma) \colon a_3 \to a_2$, and thus

$$vu\big((\alpha, \gamma) \otimes (\beta, \delta)\big) = \big((\alpha, \gamma) \otimes (\beta, \delta)\big)$$

as required. Conversely, given $(\alpha \otimes \beta, \gamma \otimes \delta)$ in $\mathbf{E}(d_3; d_1)$, we have that

$$\begin{aligned} uv(\alpha \otimes \beta, \gamma \otimes \delta) &= \big((\hat{\psi}^{-1}\alpha) \otimes (\beta\hat{\psi}), \gamma \otimes \delta\big) \\ &= \big(\alpha \otimes \beta, \gamma \otimes \delta\big) \end{aligned}$$

as required. ☐

Corollary 50. *The homomorphism S satisfies property (hps).*

Proof. Condition (hps2) follows trivially from Proposition 48. For (hps1), suppose we are given horizontally composable pullbacks

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\mathbf{p}_1} & \mathbf{B} \\ {\scriptstyle \mathbf{p}_2}\downarrow & & \downarrow{\scriptstyle \mathbf{f}} \\ S\mathbf{C} & \xrightarrow[S!]{} & S\mathbf{I}_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{A}' & \xrightarrow{\mathbf{p}'_1} & \mathbf{B}' \\ {\scriptstyle \mathbf{p}'_2}\downarrow & & \downarrow{\scriptstyle \mathbf{f}'} \\ S\mathbf{C}' & \xrightarrow[S!]{} & S\mathbf{I}_1, \end{array}$$

in $\mathbb{C}at_1$. Then consider the diagram

$$\begin{array}{ccc} \mathbf{A}' \otimes \mathbf{A} & \xrightarrow{\mathbf{p}'_1 \otimes \mathbf{p}_1} & \mathbf{B}' \otimes \mathbf{B} \\ {\scriptstyle \mathbf{p}'_2 \otimes \mathbf{p}_2}\downarrow & & \downarrow{\scriptstyle \mathbf{f}' \otimes \mathbf{f}} \\ S\mathbf{C}' \otimes S\mathbf{C} & \xrightarrow[S! \otimes S!]{} & S\mathbf{I}_1 \otimes S\mathbf{I}_1 \end{array}$$

We observe that $S1$ is a groupoid in **Cat**, and that the arrow $S! \colon SC_t \to S1$ in **Cat** is a fibration. We have an isomorphism $S\mathbf{I}_1 \cong \mathbf{I}_{S1}$, and so can replace the bottom-right vertex with $\mathbf{I}_{S1} \otimes \mathbf{I}_{S1}$; we now apply Proposition 49 to see that this square a pullback as required. ☐

Corollary 51. *The double monad* (S, η, μ) *is a double club on* $\mathbb{C}at$.

Proof. By Proposition 50, S has property (hps); and by Proposition 47, S_0 and S_1 are clubs on their respective categories. Therefore, by Proposition 46, (S, η, μ) is a double club on $\mathbb{C}at$. ☐

In Example 21, we also considered the clubs on **Cat** for non-symmetric monoidal categories, and for categories with finite products; in Example 6, we remarked that they extended to double monads on $\mathbb{C}at$. We leave it as an exercise to the reader to show that these double monads are in fact double clubs.

Appendix A: Double equivalences

We aim in this section to give an elementary characterisation of equivalences in $\mathbf{DblCat}_\psi$. In fact, for very little extra effort, we can garner significant extra generality by giving a characterisation of adjunctions in **DblCat**. A well-known result in the theory of monoidal categories [11] says that to give an adjunction in **MonCat**, the 2-category of monoidal categories, lax monoidal functors and monoidal transformations, is to give an adjunction between the underlying ordinary categories in **Cat** for which the left adjoint is strong monoidal.

We shall produce a direct generalisation of this to pseudo double categories, for which we need an analogue of 'underlying ordinary category'; more precisely, we need an appropriate analogue of the 2-category **Cat**:

Definition 52. We write **DblGph** for the 2-category $[\bullet \rightrightarrows \bullet, \mathbf{Cat}]$.

There is an evident 2-functor $U\colon \mathbf{DblCat} \to \mathbf{DblGph}$ which forgets horizontal structure, and so we may speak of the 'underlying double graph' of a double category.

Proposition 53. *To give an adjunction* $F \dashv G\colon \mathbb{L} \to \mathbb{K}$ *in* **DblCat** *is equivalent to giving an adjunction* $F \dashv G\colon U\mathbb{L} \to U\mathbb{K}$ *in* **DblGph** *together with the structure of a double homomorphism on* F.

Let us spell out explicitly what the right hand side of the above amounts to:

- A double homomorphism $F\colon \mathbb{K} \to \mathbb{L}$;
- A map of double graphs $G\colon \mathbb{L} \to \mathbb{K}$;
- Adjunctions $F_0 \dashv G_0$ and $F_1 \dashv G_1$ with unit and counit (η_0, ϵ_0) and (η_1, ϵ_1) respectively,

such that $s\epsilon_1 = \epsilon_0 s$, $t\epsilon_1 = \epsilon_0 t$, $s\eta_1 = \eta_0 s$ and $t\eta_1 = \eta_0 t$.

Proof. On an abstract level, this proof runs as follows: the 2-functor $U\colon \mathbf{DblCat} \to \mathbf{DblGph}$ has a left 2-adjoint F, which gives the 'free double category' on a given double graph. Now, the 2-category of strict algebras and strict algebra maps for the induced monad UF on **DblGph** is precisely the 2-category of *strict* double categories, whilst the 2-category of *pseudo*-algebras and lax algebra maps is *almost* the 2-category **DblCat**; more precisely, it is the 2-category of 'unbiased' (in the sense of [14]) pseudo double categories, which come equipped with n-ary horizontal composition functors for all n. As in the bicategorical case, it is not too hard to show that this notion is essentially equivalent to the 'biased' notion of pseudo double category that we have adopted.

Now, the 2-category **DblGph** is complete and cocomplete as a 2-category, and hence by Section 6.4 of [1], there is a 2-monad T' on **DblGph** whose *strict* algebras are precisely the *pseudo* algebras for the composite monad $T = UF$. Thus, we have a 2-monad T' on **DblGph** whose category of strict algebras and lax algebra maps can be identified with **DblCat**.

But now we are in a position to apply Kelly's 'doctrinal adjunction'; by Theorem 1.5 of [11], to give an adjunction in **DblCat** is precisely to give an adjunction between the underlying objects of **DblGph** for which the left adjoint is a pseudo map of T'-algebras; and to give such a map is essentially the same thing as giving a homomorphism of pseudo double categories.

Now, there are many details missing from the above, and rather than attempt to fill them in, it will be easier to give a direct proof following [11]. So, suppose first we are given an adjunction $UF \dashv UG$ in **DblGph** for which the left adjoint is a double homomorphism; then it suffices to equip G with comparison transformations $\mathfrak{m}$ and $\mathfrak{e}$, and to show that $\eta = (\eta_0, \eta_1)$ and $\epsilon = (\epsilon_0, \epsilon_1)$ become vertical transformations with respect to this data. So, suppose that F has comparison transformations

$$\begin{array}{ccc} K_1 \,{}_s\!\times_t K_1 & \xrightarrow{F_1 {}_s\times_t F_1} & L_1 \,{}_s\!\times_t L_1 \\ {\scriptstyle\otimes}\downarrow & \Downarrow \mathfrak{m} & \downarrow{\scriptstyle\otimes} \\ K_1 & \xrightarrow[F_1]{} & L_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} K_0 & \xrightarrow{F_0} & L_0 \\ {\scriptstyle\mathbf{I}}\downarrow & \Downarrow \mathfrak{e} & \downarrow{\scriptstyle\mathbf{I}} \\ K_1 & \xrightarrow[F_1]{} & L_1 \end{array}$$

Then we give the comparison transformations for G as the mates

$$\begin{array}{ccc} L_1 \,{}_s\!\times_t L_1 & \xrightarrow{G_1 {}_s\times_t G_1} & K_1 \,{}_s\!\times_t K_1 \\ {\scriptstyle\otimes}\downarrow & \Downarrow \overline{\mathfrak{m}^{-1}} & \downarrow{\scriptstyle\otimes} \\ L_1 & \xrightarrow[G_1]{} & K_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} L_0 & \xrightarrow{G_0} & K_0 \\ {\scriptstyle\mathbf{I}}\downarrow & \Downarrow \overline{\mathfrak{e}^{-1}} & \downarrow{\scriptstyle\mathbf{I}} \\ L_1 & \xrightarrow[G_1]{} & K_1 \end{array}$$

of $\mathfrak{m}^{-1}$ and $\mathfrak{e}^{-1}$ under the adjunctions $F_0 \dashv G_0$, $F_1 \dashv G_1$ and $F_1 \,{}_s\!\times_t F_1 \dashv G_1 \,{}_s\!\times_t G_1$. Explicitly, the components of these transformations at $(\mathbf{X}, \mathbf{Y})$ and X respectively are given as follows:-

$$\begin{array}{c} G\mathbf{X} \otimes G\mathbf{Y} \\ \downarrow {\scriptstyle \eta_{G\mathbf{X}\otimes G\mathbf{Y}}} \\ GF(G\mathbf{X} \otimes G\mathbf{Y}) \\ \downarrow {\scriptstyle G\mathfrak{m}^{-1}_{G\mathbf{X},G\mathbf{Y}}} \\ G(FG\mathbf{X} \otimes FG\mathbf{Y}) \\ \downarrow {\scriptstyle G(\epsilon_{\mathbf{X}} \otimes \epsilon_{\mathbf{Y}})} \\ G(\mathbf{X} \otimes \mathbf{Y}) \end{array} \quad \text{and} \quad \begin{array}{c} \mathbf{I}_{GX} \\ \downarrow {\scriptstyle \eta_{\mathbf{I}_{GX}}} \\ GF(\mathbf{I}_{GX}) \\ \downarrow {\scriptstyle G\mathfrak{e}_X^{-1}} \\ GFG\mathbf{I}_X \\ \downarrow {\scriptstyle G\epsilon_{\mathbf{I}_X}} \\ G\mathbf{I}_X. \end{array}$$

That this data is coherent follows automatically from the coherence axioms for F and the functoriality of mates, and it's now a straightforward exercise in the calculus of mates, following [11], to show that $\eta = (\eta_0, \eta_1)$ and $\epsilon = (\epsilon_0, \epsilon_1)$ become vertical transformations with respect to this data. Thus we have an adjunction in **DblCat** as required.

Conversely, any adjunction (F, G, η, ϵ) in **DblCat** gives rise to the data specified above; we need only check that F is a homomorphism, i.e., that its special comparison maps are invertible. Suppose that the comparison maps for G are $\mathfrak{m}'$ and $\mathfrak{e}'$; then it's easy to check that their mates $\overline{\mathfrak{m}'}$ and $\overline{\mathfrak{e}'}$ furnish us with inverses for $\mathfrak{m}'$ and $\mathfrak{e}'$ (explicitly, these inverses are given by

$$\begin{array}{c} F(\mathbf{X} \otimes \mathbf{Y}) \\ \downarrow {\scriptstyle F(\eta_{\mathbf{X}} \otimes \eta_{\mathbf{Y}})} \\ F(GF\mathbf{X} \otimes GF\mathbf{Y}) \\ \downarrow {\scriptstyle F\mathfrak{m}'_{F\mathbf{X},F\mathbf{Y}}} \\ FG(F\mathbf{X} \otimes F\mathbf{Y}) \\ \downarrow {\scriptstyle \epsilon_{F\mathbf{X} \otimes F\mathbf{Y}}} \\ F\mathbf{X} \otimes F\mathbf{Y} \end{array} \quad \text{and} \quad \begin{array}{c} F\mathbf{I}_X \\ \downarrow {\scriptstyle F\mathbf{I}_{\eta_X}} \\ F\mathbf{I}_{GFX} \\ \downarrow {\scriptstyle F\mathfrak{e}'_X} \\ FG\mathbf{I}_{FX} \\ \downarrow {\scriptstyle \epsilon_{\mathbf{I}_{FX}}} \\ \mathbf{I}_{FX}.\ ) \end{array}$$

The only thing remaining to check is that these two processes are mutually inverse. Suppose we are given an adjunction (F, G, η, ϵ) in **DblCat**; then we must show that we can reconstruct this adjunction from the underlying adjunction in **DblGph** together with the data for F.

This amounts to checking that the special comparison maps we produce for G are the ones we started with; but this is immediate, since we take them to be $\overline{\mathfrak{m}^{-1}}$ and $\overline{\mathfrak{e}^{-1}}$, which are $\overline{\overline{\mathfrak{m}'}} = \mathfrak{m}'$ and $\overline{\overline{\mathfrak{e}'}} = \mathfrak{e}'$ as required. ☐

Corollary 54. *Suppose we are given double categories $\mathbb{K}$ and $\mathbb{L}$, and:*

- *A double homomorphism $F\colon \mathbb{K} \to \mathbb{L}$;*
- *A map of double graphs $G\colon \mathbb{L} \to \mathbb{K}$*

together with natural isomorphisms $\eta_i\colon \mathrm{id}_{K_i} \cong G_i F_i$ and $\epsilon_i\colon F_i G_i \cong \mathrm{id}_{K_i}$ $(i = 0, 1)$, such that such that $s\epsilon_1 = \epsilon_0 s$, $t\epsilon_1 = \epsilon_0 t$, $s\eta_1 = \eta_0 s$ and $t\eta_1 = \eta_0 t$. Then $\mathbb{K}$ and $\mathbb{L}$ are equivalent in $\mathbf{DblCat}_\psi$.

Proof. To give this data is to give an equivalence in **DblGph**, so by replacing ϵ_1 and ϵ_0, we can make this into an *adjoint* equivalence in **DblGph**. Now, applying the previous result, we get an (adjoint) equivalence in **DblCat**; but now we note that the comparison special maps for G will be invertible, since they are constructed from a composite of invertible maps, and hence that our equivalence is an equivalence in $\mathbf{DblCat}_\psi$ as well. ☐

Appendix B: Whiskering and double clubs

We have defined the concept of double club in terms of closure under the structure of monoidal double category. However, we may also ask about closure under the 'whiskering' operations of Section 2. *Prima facie*, this may appear to be a strictly stronger requirement, but in fact it follows from the definition of double club given above.

We begin with a preliminary general result on endohom double categories. We saw how to construct the monoidal structure on $[\mathbb{K}, \mathbb{K}]_\psi$ using the whiskering operations $G(\text{–})$ and $(\text{–})G$. We can also to a certain extent go in the other direction, and derive something like the whiskering homomorphisms from the monoidal structure on $[\mathbb{K}, \mathbb{K}]_\psi$. Indeed, given a homomorphism $G\colon \mathbb{K} \to \mathbb{K}$, we obtain homomorphisms

$$(\text{–}) \bullet \mathbf{I}_G \colon [\mathbb{K}, \mathbb{K}]_\psi \cong [\mathbb{K}, \mathbb{K}]_\psi \times 1 \xrightarrow{\mathrm{id} \times \ulcorner \mathbf{I}_G \urcorner} [\mathbb{K}, \mathbb{K}]_\psi \times [\mathbb{K}, \mathbb{K}]_\psi \xrightarrow{\bullet} [\mathbb{K}, \mathbb{K}]_\psi$$
$$\mathbf{I}_G \bullet (\text{–}) \colon [\mathbb{K}, \mathbb{K}]_\psi \cong 1 \times [\mathbb{K}, \mathbb{K}]_\psi \xrightarrow{\ulcorner \mathbf{I}_G \urcorner \times \mathrm{id}} [\mathbb{K}, \mathbb{K}]_\psi \times [\mathbb{K}, \mathbb{K}]_\psi \xrightarrow{\bullet} [\mathbb{K}, \mathbb{K}]_\psi.$$

And these homomorphisms approximate the operation of whiskering by G in the following sense:

Proposition 55. *There are canonical invertible vertical transformations*

$$l_G \colon G(\text{–}) \Rightarrow \mathbf{I}_G \bullet (\text{–}) \quad \textit{and} \quad r_G \colon (\text{–})G \Rightarrow (\text{–}) \bullet \mathbf{I}_G$$

which are natural in G.

Proof. We have $\big(G(\text{–})\big)_0 = \big(\mathbf{I}_G \bullet (\text{–})\big)_0$ and $\big((\text{–})G)_0 = \big((\text{–}) \bullet \mathbf{I}_G\big)_0$, so we can take $(l_G)_0$ and $(r_G)_0$ to be identity natural transformations. For $(l_G)_1$ and $(r_G)_1$, observe that we have

$$\big(\mathbf{I}_G \bullet (\text{–})\big)_1 = \mathbf{I}_G(\text{–})_t \otimes G(\text{–}) = \mathbf{I}_{G(\text{–})_t} \otimes G(\text{–})$$
$$\text{and } \big((\text{–}) \bullet \mathbf{I}_G\big)_1 = (\text{–})G \otimes (\text{–})_s \mathbf{I}_G.$$

Therefore we take $(l_G)_1$ to be the natural transformation

$$(l_G)_1 = G(\text{–}) \xRightarrow{\mathfrak{l}_{G(\text{–})}} \mathbf{I}_{G(\text{–})_t} \otimes G(\text{–})$$

and $(r_G)_1$ to be the natural transformation

$$(r_G)_1 = (\text{–})G \xRightarrow{\mathfrak{r}_{(\text{–})G}} (\text{–})G \otimes \mathbf{I}_{(\text{–})_s G} \xRightarrow{\mathrm{id} \otimes \mathfrak{e}_G} (\text{–})G \otimes (\text{–})_s \mathbf{I}_G.$$

It's now routine diagram chasing to check that l and r satisfy all the required axioms for a vertical transformation, and that they are natural in G as required. ☐

Proposition 56. *Let S be a double club, and let (A, α) be an object of $\mathbb{C}oll(S)$. Then the whiskering homomorphisms*

$$(\text{–})A \colon [\mathbb{K}, \mathbb{K}]_\psi \to [\mathbb{K}, \mathbb{K}]_\psi \quad \textit{and} \quad A(\text{–}) \colon [\mathbb{K}, \mathbb{K}]_\psi \to [\mathbb{K}, \mathbb{K}]_\psi$$

lift to homomorphisms

$$(\text{–})(A, \alpha) \colon \mathbb{C}oll(S) \to \mathbb{C}oll(S) \quad \textit{and} \quad (A, \alpha)(\text{–}) \colon \mathbb{C}oll(S) \to \mathbb{C}oll(S).$$

Proof. We give the details for $(A, \alpha)(–)$, since $(–)(A, \alpha)$ follows similarly. Following Proposition 55, we have the homomorphism $\mathbf{I}_{(A,\alpha)} \bullet (–) \colon \mathbb{C}oll(S) \to \mathbb{C}oll(S)$; further we have the invertible special vertical transformation

$$l_A \colon A(–) \Rightarrow \mathbf{I}_A \bullet (–) \colon \mathbb{K} \to \mathbb{K}$$

So we give $(A, \alpha)(–)$ as follows. Its component $\big((A, \alpha)(–)\big)_0 \mathbb{C}oll(S)_0 \to \mathbb{C}oll(S)_0$ is simply $\big(\mathbf{I}_{(A,\alpha)} \bullet (–)\big)_0 = (A, \alpha) \bullet (–)$, whilst $\big((A, \alpha)(–)\big)_1 \colon \mathbb{C}oll(S)_1 \to \mathbb{C}oll(S)_1$ is given as follows:

- **On objects**: given $(\mathbf{B}, \boldsymbol{\beta})$ in $\mathbb{C}oll(S)_1$, we take $(A, \alpha)(\mathbf{B}, \boldsymbol{\beta})$ to be the modification

  $$A\mathbf{B} \xRightarrow{(l_A)_{\mathbf{B}}} \mathbf{I}_A \bullet \mathbf{B} \xRightarrow{\mathbf{I}_\alpha \bullet \boldsymbol{\beta}} \mathbf{I}_S \bullet S\mathbf{I} \xRightarrow{\mathfrak{e} \bullet S\mathbf{I}} S\mathbf{I} \bullet S\mathbf{I} \xRightarrow{\mathfrak{m}} S\mathbf{I}.$$

  The first modification above is cartesian since it is invertible, whilst the remaining composite is $\mathbf{I}_{(A,\alpha)} \bullet (\mathbf{B}, \boldsymbol{\beta})$, and hence cartesian since S is a double club; thus the entire composite is cartesian as required.

- **On maps**: given $\boldsymbol{\delta} \colon (\mathbf{B}, \boldsymbol{\beta}) \to (\mathbf{C}, \boldsymbol{\gamma})$, we take $(A, \alpha)(\boldsymbol{\delta})$ to be given by

  $$A\boldsymbol{\delta} \colon (A, \alpha)(\mathbf{B}, \boldsymbol{\beta}) \to (A, \alpha)(\mathbf{C}, \boldsymbol{\gamma}).$$

  That this map is compatible with the projections down to $S\mathbf{I}$ is an easy diagram chase.

It's immediate that these definitions are compatible with source and target; it remains to give the comparison maps $\mathfrak{m}$ and $\mathfrak{e}$, for which we simply take

$$\mathfrak{e}_{(B,\beta)} = \mathfrak{e}_B \colon \mathbf{I}_{AB} \Rrightarrow A\mathbf{I}_B$$
$$\text{and } \mathfrak{m}_{(\mathbf{B},\boldsymbol{\beta}),(\mathbf{B}',\boldsymbol{\beta}')} = \mathfrak{m}_{\mathbf{B},\mathbf{B}'} \colon A\mathbf{B} \otimes A\mathbf{B}' \to A(\mathbf{B} \otimes \mathbf{B}').$$

That these maps are compatible with the projections down to $S\mathbf{I}$ is another straightforward diagram chase, whilst the coherence axioms for $\mathfrak{m}$ and $\mathfrak{e}$ follows from those for $A(–)$ on $[\mathbb{K}, \mathbb{K}]_\psi$. □

For completeness, we also observe the following:

Proposition 57. *Let S be a double club, and let $\gamma \colon (A, \alpha) \to (B, \beta)$ be a vertical arrow of $\mathbb{C}oll(S)$. Then the whiskering vertical transformations*

$$(–)\gamma \colon (–)A \Rightarrow (–)B \quad \textit{and} \quad \gamma(–) \colon A(–) \Rightarrow B(–)$$

lift to vertical transformations

$$(–)\gamma \colon (–)(A, \alpha) \Rightarrow (–)(B, \beta) \quad \textit{and} \quad \gamma(–) \colon (A, \alpha)(–) \Rightarrow (B, \beta)(–).$$

The proof is straightforward: one must simply show that the components of $\gamma(–)$ and $(–)\gamma$ are compatible with the projections down to $S\mathbf{I}$.

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