COALGEBRAS GOVERNING BOTH WEIGHTED HURWITZ PRODUCTS AND THEIR POINTWISE TRANSFORMS

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Abstract. We give further insights into the weighted Hurwitz product and the weighted tensor product of Joyal species. Our first group of results relate the Hurwitz product to the pointwise product, including the interaction with Rota–Baxter operators. Our second group of results explain the first in terms of convolution with suitable bialgebras, and show that these bialgebras are in fact obtained in a particularly straightforward way by freely generating from pointed coalgebras. Our third group of results extend this from linear algebra to two-dimensional linear algebra, deriving the existence of weighted Hurwitz monoidal structures on the category of species using convolution with freely generated bimonoidales. Our final group of results relate Hurwitz monoidal structures with equivalences of Dold–Kan type.

1. Introduction

This paper continues the investigations of [40] into the λ-Hurwitz products of [21, 22]. Given a ring k, an element $\lambda \in$ k, and a k-algebra A, the *λ-Hurwitz product* is a certain multiplication $\cdot^\lambda$ on the set $A^{\mathbb{N}}$ which, together with the pointwise linear structure, endows it with the structure of a k-algebra $G_\lambda A$. This algebra has a universal role: it is the cofree λ-differential algebra on A. Here, a *λ-differential algebra* is a k-algebra equipped with a *λ-weighted derivation*—a k-linear endomorphism ∂ satisfying

$$\partial(1) = 0 \qquad \text{and} \qquad \partial(ab) = (\partial a)b + a(\partial b) + \lambda(\partial a)(\partial b)\ . \tag{1.1}$$

When $\lambda = 0$, of course, we re-find the classical notion of derivation and differential algebra; when $\lambda \neq 0$, we have variants on these notions apt for the study of *difference* rather than differential equations.

Our first set of results clarify the relation between the pointwise and λ-Hurwitz products on $A^{\mathbb{N}}$. We exhibit algebra morphisms $\gamma\colon (A^{\mathbb{N}}, \cdot^\lambda) \to (A^{\mathbb{N}}, \text{pointwise})$, which are algebra *isomorphisms* whenever $\lambda \in$ k is invertible. We relate these to Lagrange interpolation, and also to weighted *Rota–Baxter algebras* [3, 36]: these are k-algebras endowed with a *weighted Rota–Baxter operator*—a k-linear endomorphism P satisfying the equation:

$$P(a)P(b) = P(P(a)b + aP(b) + \lambda ab)\ . \tag{1.2}$$

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Just as derivations encode abstract differentation, so weighted Rota–Baxter operators encode abstract integration (when $\lambda = 0$) or summation (when $\lambda \neq 0$). Following [21, 22], we show that, when A is a weighted Rota–Baxter algebra, γ above lifts to a homomorphism of Rota–Baxter algebras; in particular, when λ is invertible, this establishes an isomorphism between two canonical Rota–Baxter algebra structures on $A^{\mathbb{N}}$ arising from a given one on A.

Our second set of results explain the first in terms of more basic data. The assignation $A \mapsto G_\lambda A$ underlies a comonad on the category of k-algebras; in fact, this may be seen as induced by convolution with a k-bialgebra $C(\lambda)_\infty$. There is another comonad H on this category given by $A \mapsto (A^{\mathbb{N}}, \text{pointwise})$, which is again induced by convolution with a bialgebra D_∞. Now the k-algebra morphism $\gamma \colon G_\lambda A \to HA$ can be seen as induced under convolution by a morphism of bialgebras $D^\infty \to C(\lambda)_\infty$. In fact, more is true. The comonads G_λ and H are both *cofree* on a copointed endofunctor; correspondingly, the bialgebras D_∞ and $C(\lambda)_\infty$ are free on pointed coalgebras D and $C(\lambda)$, and in these terms, the bialgebra morphism $D_\infty \to C(\lambda)_\infty$ can be seen as generated by the (much simpler) datum of a morphism of pointed coalgebras $D \to C(\lambda)$.

The remaining contributions of this paper are concerned with "categorifications" of the preceding ones. Rather than considering modules over a commutative ring k, we consider categories enriched over a suitable symmetric monoidal base $\mathscr{V}$ admitting a suitable class of colimits that play the role of "addition". Rather than (commutative) k-algebra structure, we consider (symmetric) monoidal structure on our $\mathscr{V}$-categories; and rather than *coalgebra* structure, we consider *comonoidale* structure in a suitable monoidal bicategory of $\mathscr{V}$-categories. As described in [40], there is in this setting a "categorification" of the λ-Hurwitz product found on $A^{\mathbb{N}}$ for any k-algebra A to a Λ-*Hurwitz monoidal structure* on $\mathscr{A}^{\mathfrak{S}}$ (where $\mathfrak{S}$ is the category of finite sets and bijections) for any "$\mathscr{V}$-algebra" $\mathscr{A}$. Inspired by the constructions of the preceding sections, our third set of results exhibit this Λ-Hurwitz monoidal structure as induced by convolution with a "$\mathscr{V}$-bialgebra", and show that this bialgebra may in fact be obtained as the free bialgebra generated by a pointed $\mathscr{V}$-coalgebra. Analogously to before, we obtain a comparison between the Hurwitz monoidal structure on $\mathscr{A}^{\mathfrak{S}}$ and the pointwise one, which, once again, may be seen as freely generated from a morphism of pointed $\mathscr{V}$-coalgebras.

The final set of results in this paper explain the link between categorified Hurwitz tensor products and *equivalences of Dold–Kan type*. The classical Dold–Kan equivalence is that between simplicial abelian groups and chain complexes of abelian groups, the simpler direction of which is the functor $N \colon [\Delta^{\mathrm{op}}, \mathbf{Ab}] \to \mathbf{Ch}$ sending each simplicial abelian group to its normalized Moore complex. However, when we equip $[\Delta^{\mathrm{op}}, \mathbf{Ab}]$ with the pointwise tensor product, and $\mathbf{Ch}$ with its classical tensor product, the functor N is *not* strong monoidal, though it is lax and oplax monoidal in a compatible manner: see [1, Chapter 5]. This means that transporting the pointwise monoidal structure on $[\Delta^{\mathrm{op}}, \mathbf{Ab}]$ across this equivalence yields a new tensor product on chain complexes, and as we will see, the formula for this is precisely the $\lambda = 1$ case of a Hurwitz-style tensor[1].

In fact, we will show something more general than this. Recent work such as [35, 37, 8] has established various generalisations of the classical Dold–Kan equivalence; in [33]

[1]The explicit calculation of this tensor product appears in unpublished work of Lack and Hess.

is described a general framework for obtaining such equivalences, which, starting from a category $\mathscr{P}$ equipped with suitable extra structure, derives a category with zero morphisms $\mathscr{D}$ and an equivalence of functor categories $[\mathscr{P}, \mathbf{Ab}] \simeq [\mathscr{D}, \mathbf{Ab}]_{\mathrm{pt}}$ (here the subscript "pt" indicates the restriction to zero-map preserving functors). Our fourth main result shows that, in this setting, the pointwise tensor product on $[\mathscr{P}, \mathbf{Ab}]$ always transports to a Hurwitz-style monoidal structure on $[\mathscr{D}, \mathbf{Ab}]_{\mathrm{pt}}$; while our fifth and final result shows that certain important examples of equivalences arising in this way, may, as before, be seen as induced by convolution with a map of "**Ab**-bialgebras" freely generated from a map of pointed **Ab**-coalgebras.

2. Preliminaries

Throughout this paper, k will be a commutative $\mathbb{Q}$-algebra. Given natural numbers $n, m_1, \dots, m_r$, we define the *multinomial coefficient*

$$\binom{n}{m_1, \dots, m_r} = \frac{n!}{m_1! \cdots m_r!} \ .$$

Usually this is for $\Sigma_i m_i = n$, so that this coefficient gives the number of ways of partitioning a set of cardinality n into disjoint subsets of cardinalities $m_1, \dots, m_r$. We extend this definition to all integers by declaring $k!$ to be zero for any $k < 0 \in \mathbb{Z}$. Of course, we write $\binom{n}{r}$ as usual for $\binom{n}{r, n-r}$; more generally, for a k-algebra A and $x \in A$, we define the *binomial coefficients* of x to be the elements of A given by:

$$\binom{x}{0} = 1 \qquad \text{and} \qquad \binom{x}{r} = \frac{x(x-1)\cdots(x-r+1)}{r!} \qquad \text{for } 0 < r \in \mathbb{N} \ .$$

The following elementary result is classical.

Lemma 2.1. *Let $p, q \in \mathbb{N}$.*

(i) If $n \in \mathbb{N}$, then

$$\binom{n}{p}\binom{n}{q} = \sum_{\substack{u+r+s+t=n \\ p=r+t,\, q=s+t}} \binom{n}{u, r, s, t} \ ;$$

(ii) If A is a k*-algebra and $x \in A$, then*

$$\binom{x}{p}\binom{x}{q} = \sum_t \binom{p+q-t}{p-t, q-t, t}\binom{x}{p+q-t} \ .$$

Proof. For a finite set X, we have bijections between any two of the sets

$$\{\, P, Q \subseteq X : |P| = p, |Q| = q \,\} \ ,$$
$$\{\, U, R, S, T \subseteq X : X = U + R + S + T, |R+T| = p, |S+T| = q \,\} \ ,$$
$$\text{and} \quad \{\, W, R, S, T \subseteq X : W = R + S + T, |R+T| = p, |S+T| = q \,\} \ .$$

This proves (i) and also (ii) in the case where $x \in \mathbb{N}$. The general case of (ii) proceeds by a straightforward induction on p. □

We will also require the following simple combinatorial identity.

Lemma 2.2. *For any* k*-algebra* A *and sequence* $a_0, a_1, \dots$ *in* A*, it holds that:*

$$a_n = \sum_{r+s+t=n} \binom{n}{r,s,t} (-1)^s a_t \ .$$

Proof. Let $\Sigma \colon A^{\mathbb{N}} \to A^{\mathbb{N}}$ be the suspension operator $\Sigma(a)_n = a_{n+1}$. As the operators $1, -1, \Sigma$ commute with each other, we have by the trinomial formula that

$$\Sigma^{\circ n} = (1 - 1 + \Sigma)^{\circ n} = \sum_{r+s+t=n} \binom{n}{r,s,t} 1^{\circ r} \circ (-1)^{\circ s} \circ \Sigma^{\circ t} \ .$$

Applying this operator identity to $a \in A^{\mathbb{N}}$ and evaluating at 0 yields the result. □

3. The λ-weighted Hurwitz product

Throughout this section, we fix $\lambda \in$ k and a k-algebra A. The *λ-weighted Hurwitz product* on $A^{\mathbb{N}}$ [21, §2.3] is defined by the equation

$$(a \cdot^{\lambda} b)_n = \sum_{n=r+s+t} \binom{n}{r,s,t} \lambda^t a_{r+t} b_{s+t} \ . \tag{3.3}$$

This product has $(1, 0, 0, \dots)$ as neutral element; taken together with the pointwise k-linear structure we obtain a k-algebra $(A^{\mathbb{N}}, \cdot^{\lambda})$, which is commutative whenever A is so. This formula restricts to A^{ℓ} regarded as the linear subspace of $A^{\mathbb{N}}$ comprising those a with $a_n = 0$ for $n \geqslant \ell$; moreover, we can recapture $A^{\mathbb{N}}$ and its algebra structure from the A^{ℓ}'s as the limit of the chain $\dots \to A^n \to \dots \to A^1 \to A^0$, where each map $A^{j+1} \to A^j$ sets a_j to zero. In light of this, we may consider that $A^{\mathbb{N}} = A^{\ell}$ for $\ell = \infty$.

Our first result relates the λ-weighted algebra structure on each A^{ℓ} to the pointwise one. Its first part allows us to focus attention on the case $\lambda = 1$ as occurring in [36]; the second reduces that to the pointwise case.

Proposition 3.1. *Let* $\ell \in \mathbb{N} \cup \{\infty\}$.

(i) There is a k*-algebra morphism* $\hat{\lambda} \colon (A^{\ell}, \cdot^{\lambda}) \to (A^{\ell}, \cdot^{1})$ *defined by* $\hat{\lambda}(a)_n = \lambda^n a_n$*;*

(ii) There is a k*-algebra isomorphism* $\theta_{\ell} \colon (A^{\ell}, \cdot^{1}) \to (A^{\ell}, \text{pointwise})$ *defined by*

$$\theta_{\ell}(a)_n = \sum_m \binom{n}{m} a_m \ ;$$

(iii) There is a k*-algebra morphism* $\gamma_{\ell} \colon (A^{\ell}, \cdot^{\lambda}) \to (A^{\ell}, \textit{pointwise})$ *defined by*

$$\gamma_{\ell}(a)_n = \sum_m \binom{n}{m} \lambda^m a_m \ ,$$

which is an isomorphism whenever λ *is invertible in* k.

Proof. (iii) is immediate from (i) and (ii). For (i), clearly $\hat{\lambda}$ is linear and preserves the multiplicative unit; we conclude since

$$\begin{aligned} \hat{\lambda}(a \cdot^{\lambda} b)_n &= \sum_{n=r+s+t} \binom{n}{r,s,t} \lambda^{n+t} a_{r+t} b_{s+t} \\ &= \sum_{n=r+s+t} \binom{n}{r,s,t} \lambda^{r+t} a_{r+t} \lambda^{s+t} b_{s+t} = (\hat{\lambda} a \cdot^{1} \hat{\lambda} b)_n \ . \end{aligned}$$

For (ii), clearly θ_ℓ is linear and preserves 1; for the multiplication, we calculate that:

$$\begin{aligned}
\theta_\ell(a \cdot^1 b)_n &= \sum_m \sum_{r+s+t=m} \binom{n}{m}\binom{m}{r,s,t} a_{r+t} b_{s+t} \\
&= \sum_{u+r+s+t=n} \binom{n}{u,r,s,t} a_{r+t} b_{s+t} \\
&= \sum_{p,q=0}^{n} \binom{n}{p}\binom{n}{q}\, a_p b_q = (\theta_\ell(a)\theta_\ell(b))_n
\end{aligned}$$

using Lemma 2.1 at the third step. Now θ_ℓ is invertible since it is represented on the standard basis of A^ℓ by a triangular matrix with 1's along the main diagonal; a direct calculation hinging on Lemma 2.2 shows that an explicit inverse $\bar{\theta}_\ell$ is given by

$$\bar{\theta}_\ell(a)_m = \sum_n \binom{m}{n} (-1)^{m-n} a_n \ . \qquad \square$$

We may also understand the ring isomorphism of Proposition 3.1(ii) in terms of Lagrange interpolation. Write $A[x]$ for the k-algebra of polynomials in indeterminate x with coefficients in A. For each natural number ℓ, we consider the quotient ring $A[x]/\binom{x}{\ell}$. In this ring we have $\binom{x}{n} = 0$ for all $n \geq \ell$. When $\ell = \infty$, we define $A[x]/\binom{x}{\ell}$ to be the limit of the sequence of quotient maps $\cdots \to A[x]/\binom{x}{n} \to \cdots \to A[x]/\binom{x}{0}$.

Proposition 3.2. *Let $\ell \in \mathbb{N} \cup \{\infty\}$.*

(i) There is a k*-algebra morphism $\psi_\ell \colon (A^\ell, \cdot^1) \longrightarrow A[x]/\binom{x}{\ell}$ defined by*

$$\psi_\ell(a) = \sum_n a_n \binom{x}{n} \ ;$$

(ii) There is a k*-algebra isomorphism $\varphi_\ell \colon A[x]/\binom{x}{\ell} \longrightarrow (A^\ell, \text{pointwise})$ defined by $\varphi_\ell(f)_n = f(n)$ for $0 \leqslant n \leqslant \ell$. Moreover, the following triangle commutes, implying ψ_ℓ invertible.*

$$\begin{array}{ccc}
(A^\ell, \cdot^1) & \xrightarrow{\quad\psi_\ell\quad} & A[x]/\binom{x}{\ell} \\
& {\scriptstyle\theta_\ell}\searrow \quad \swarrow{\scriptstyle\varphi_\ell} & \\
& (A^\ell, \text{pointwise}) &
\end{array}$$

Proof. For (i), ψ_ℓ is clearly linear and preserves 1; we conclude since

$$\begin{aligned}\psi_\ell(a \cdot^1 b) &= \sum_n \sum_{r+s+t=n} \binom{n}{r,s,t} a_{r+t}b_{s+t}\binom{x}{n} \\ &= \sum_{r,s,t} \binom{r+s+t}{r,s,t}\binom{x}{r+s+t} a_{r+t}b_{s+t} \\ &= \sum_{p,q,t} \binom{p+q-t}{p-t,q-t,t}\binom{x}{p+q-t} a_p b_q \\ &= \sum_{p,q} a_p b_q \binom{x}{p}\binom{x}{q} = \psi_\ell(a)\psi_\ell(b)\end{aligned}$$

using Lemma 2.1 at the fourth step. When $\ell < \infty$, (ii) follows since, by the Chinese Remainder Theorem for rings (see [19] for example), the homomorphism

$$A[x] \to \prod_{0 \le n < \ell} A[x]/(x-n) \cong (A^\ell, \text{pointwise})$$

obtained by pairing together the canonical quotient maps is itself a quotient, with as kernel the ideal generated by $x(x-1)\cdots(x-\ell+1)$. The case $\ell = \infty$ follows on passing to the limit. □

We now relate these results to Rota–Baxter operations. As in the introduction, a *Rota–Baxter operator of weight* λ on a k-algebra A is a k-linear map $P\colon A \to A$ satisfying (1.2). Note that the zero operator is always a Rota–Baxter operator.

Proposition 3.3. *Let $\ell \in \mathbb{N} \cup \{\infty\}$. Each Rota–Baxter operator P of weight λ on A lifts to one $\bar{P}$ on $(A^\ell, \cdot^\lambda)$, as defined left below, and to one $\tilde{P}$ on $(A^\ell, \text{pointwise})$, as defined right below.*

$$\bar{P}(a)_n = \begin{cases} P(a_0) & \text{for } n = 0; \\ a_{n-1} & \text{for } n > 0, \end{cases} \qquad\qquad \tilde{P}(a)_n = P(a_0) + \lambda \sum_{i<n} a_i \ .$$

Moreover, the map γ_ℓ of Proposition 3.1(iii) is a map of Rota–Baxter algebras in the sense that $\gamma_\ell \bar{P} = \tilde{P}\gamma_\ell$.

Proof. $\bar{P}$ is a Rota–Baxter operator by [22, Proposition 3.8]. For $\tilde{P}$, consider the difference operator $\partial\colon A^\ell \to A^\ell$ defined by $\partial(a)_n = a_{n+1} - a_n$ (taking $a_{\ell+1} = 0$ when $\ell < \infty$). We have $\partial \circ \tilde{P} = \lambda$ and $\partial(ab) = (\partial a)b + a(\partial b) + (\partial a)(\partial b)$ under the pointwise product. Clearly $a = b \in A^{\mathbb{N}}$ just when $a_0 = b_0$ and $\partial(a) = \partial(b)$; thus, since

$$\begin{aligned}\tilde{P}(a)\tilde{P}(b)_0 = P(a_0)P(b_0) = P(P(a_0)b_0 + a_0P(b_0) + \lambda a_0 b_0) = P(P(a)b + aP(b) + \lambda ab)_0 \\ \text{and} \quad \partial(\tilde{P}(a)\tilde{P}(b)) = \lambda a\tilde{P}(b) + \lambda\tilde{P}(b)a + \lambda^2 ab = \partial\tilde{P}(a\tilde{P}(b) + \tilde{P}(b)a + \lambda ab)\end{aligned}$$

we conclude that $\tilde{P}$ is a Rota–Baxter operator as required. To see that γ_ℓ is a map of Rota–Baxter algebras, we calculate similarly that $\gamma_\ell\bar{P}(a)_0 = P(a_0) = \tilde{P}\gamma_\ell(a)_0$, and that

$$\begin{aligned}\partial\gamma_\ell \bar{P}(a)_n &= \sum_m \binom{n+1}{m}\lambda^m(\bar{P}a)_m - \sum_m \binom{n}{m}\lambda^m(\bar{P}a)_m \\ &= \sum_m \binom{n}{m-1}\lambda^m(\bar{P}a)_m = \sum_m \binom{n}{m}\lambda^{m+1}(\bar{P}a)_{m+1} \\ &= \lambda\sum_m \binom{n}{m}\lambda^m a_m = \lambda\gamma_\ell(a)_n = \partial\tilde{P}\gamma_\ell(a)_n\ . \end{aligned}$$ □

4. Comonadic aspects

As described in the introduction, the algebra $G_\lambda A = (A^{\mathbb{N}}, \cdot^\lambda)$ associated to any k-algebra A is in fact the cofree λ-weighted differential algebra on A. To be precise about this, we consider the category $\mathbf{Dif}_\lambda$ of λ-weighted differential k-algebras; as in the introduction, the objects of this category are k-algebras equipped with a k-linear endomorphism ∂ satisfying (1.1), while the morphisms are maps of k-algebras preserving ∂.

Proposition 4.1. *For any* k*-algebra A, the operator $\partial\colon A^{\mathbb{N}} \to A^{\mathbb{N}}$ with $(\partial a)_n = a_{n+1}$ makes $G_\lambda A$ into a λ-weighted differential algebra. The algebra morphism $G_\lambda A \to A$ given by taking 0th components exhibits $(G_\lambda A, \partial)$ as the value at A of a right adjoint R to the forgetful functor $U\colon \mathbf{Dif}_\lambda \to$ k-$\mathbf{Alg}$. The adjunction $U \dashv R$ is comonadic.*

Proof. This is [21, Propositions 2.7 and 2.8] and [22, Theorem 3.5]. □

There is a corresponding (well-known) result for the algebra $HA = (A^{\mathbb{N}}, \text{pointwise})$. Writing k$[x]$-$\mathbf{Alg}$ for the category of k-algebras equipped with an algebra endomorphism, we have:

Proposition 4.2. *For any* k*-algebra A, the operator $\partial\colon A^{\mathbb{N}} \to A^{\mathbb{N}}$ with $(\partial a)_n = a_{n+1}$ is a ring endomorphism of HA. The algebra morphism $HA \to A$ given by taking 0th component exhibits (HA, ∂) as the value at A of a right adjoint S to the forgetful functor $V\colon$ k$[x]$-$\mathbf{Alg} \to$ k-$\mathbf{Alg}$. The adjunction $V \dashv S$ is comonadic.*

Proof. If $(B, \varphi\colon B \to B) \in$ k$[x]$-$\mathbf{Alg}$ and $f\colon V(B,\varphi) \to A$ is a map of k-algebras, then the corresponding map $\bar{f}\colon (B,\varphi) \to (HA, \partial)$ is defined by $\bar{f}(b)_n = f(\varphi^n(b))$. This gives adjointness; comonadicity follows as V preserves colimits and is conservative. □

We may now strengthen Proposition 3.1 so as to incorporate the induced comonad structures on $G_\lambda = UR$ and $H = VS$.

Proposition 4.3. *The algebra map $\gamma = \gamma_\infty\colon G_\lambda A \to HA$ of Proposition 3.1(iii) is the component at A of a comonad morphism $G_\lambda \to H$.*

Proof. If the k-algebra A bears a λ-weighted differential ∂, then we have a ring endomorphism of A given by the operator $\varphi = 1 + \lambda\partial$. Indeed, k-linearity and preservation of the unit are clear; as for multiplication, we have $\varphi(a)\varphi(b)$ equal to

$$(a + \lambda(\partial a))(b + \lambda(\partial b)) = ab + \lambda((\partial a)b + a(\partial b) + \lambda(\partial a)(\partial b)) = ab + \lambda\partial(ab) = \varphi(ab)$$

as required. The assignation $(A, \partial) \mapsto (A, 1 + \lambda\partial)$ is thus the action on objects of a functor $F\colon \mathbf{Dif}_\lambda \to$ k$[x]$-$\mathbf{Alg}$ commuting with the forgetful functors to k-$\mathbf{Alg}$.

Any such functor induces a comonad morphism $G_\lambda \to H$—see [5, Lemma 4.5.1], for example—which in this case is obtained as follows. Take the cofree λ-differential algebra $(G_\lambda A, \partial)$, the induced $\mathrm{k}[x]$-algebra $(G_\lambda A, 1 + \lambda\partial)$, and the 0th component homomorphism $\varepsilon \colon V(G_\lambda A, 1+\lambda\partial) \to A$. The comonad morphism in question now has its A-component given by the map of k-algebras underlying $\bar{\varepsilon} \colon (G_\lambda A, 1+\lambda\partial) \to (HA, \partial)$. From Proposition 4.2 above, we have that

$$\bar{\varepsilon}(a)_n = \varepsilon(1+\lambda\partial)^{\circ n}(a) = \sum_m \binom{n}{m} \lambda^m \varepsilon \partial^m(a) = \sum_m \binom{n}{m} \lambda^m a_m = \gamma(a)_n$$

since the operators 1 and $\lambda\partial$ commute; so $\gamma = \bar{\varepsilon}$ is the component at A of a comonad morphism, as claimed. □

Recall that a comonad P on a category $\mathscr{C}$ is said to be *cofree* on a copointed endofunctor $(T, \varepsilon \colon T \Rightarrow \mathrm{id})$ of $\mathscr{C}$ if P is the value at (T, ε) of a right adjoint to the forgetful functor $\mathbf{Cmd}(\mathscr{C}) \to [\mathscr{C}, \mathscr{C}]/\mathrm{id}$ from comonads to copointed endofunctors.

Proposition 4.4. *The comonads G_λ and H are cofree on copointed endofunctors, and the comonad map $\gamma \colon G_\lambda \to H$ is cofree on a map of copointed endofunctors.*

Proof. For G_λ, consider the copointed endofunctor (S, σ) with $SA = (A^2, \cdot^\lambda)$ and σ given by the first projection. To endow a k-algebra A with a homomorphism $a \colon A \to SA$ satisfying $\sigma a = 1_A$ is easily the same as endowing it with a λ-weighted differential; whence the category (S, σ)-$\mathbf{Coalg}$ of coalgebras for this copointed endofunctor is isomorphic over k-$\mathbf{Alg}$ to the category $\mathbf{Dif}_\lambda$. It follows by [30, Proposition 22.2] that the comonad G_λ induced by the adjunction $R \vdash U \colon \mathbf{Dif}_\lambda \to$ k-$\mathbf{Alg}$ is the cofree comonad on (S, σ). The same argument pertains for H on considering the copointed endofunctor (T, τ) with $TA = (A^2, \text{pointwise})$ and τ given again by the first projection. Finally, the maps $\gamma_2 \colon (A^2, \cdot^\lambda) \to (A^2, \text{pointwise})$ of Proposition 3.1(iii) are the components of a pointed endofunctor map $(S, \sigma) \to (T, \tau)$, composition with which induces the functor $F \colon \mathbf{Dif}_\lambda \to \mathrm{k}[x]$-$\mathbf{Alg}$ of the preceding proof; whence $\gamma \colon G_\lambda \to H$ is induced as the cofree comonad morphism on γ_2. □

5. Coalgebraic aspects

As anticipated in the introduction, we may understand the constructions of the preceding section more straightforwardly using convolution. Recall that, if (C, ε, δ) is a k-coalgebra and (A, η, μ) is a k-algebra, the k-linear hom $[C, A]$ becomes a k-algebra $[C, A]$ under *convolution*, with unit $e = \eta\varepsilon \colon C \to \mathrm{k} \to A$ and product $f * g = \mu(f \otimes g)\delta$.

As a first application, we consider the coalgebra $C(\lambda)$ whose underlying k-module is free on $\{e, d\}$ with counit and comultiplication defined by:

$$\varepsilon(e) = 1 \ , \ \varepsilon(d) = 0 \ , \ \delta(e) = e \otimes e \ , \ \delta(d) = d \otimes e + e \otimes d + \lambda \, d \otimes d \ .$$

On the other hand, we have the coalgebra D with the same underlying k-module but the "set-like" coalgebra structure given by

$$\varepsilon(e) = \varepsilon(d) = 1 \ , \ \delta(e) = e \otimes e \ , \ \delta(d) = d \otimes d \ .$$

Moreover, there is a coalgebra morphism $\xi \colon D \to C(\lambda)$ with $\xi(e) = e$ and $\xi(d) = \lambda d + e$. It is now direct from the definitions that:

Proposition 5.1. *For any* k*-algebra A, there are isomorphisms $[C(\lambda), A] \cong (A^2, \cdot^\lambda)$ and $[D, A] \cong (A^2, pointwise)$, and modulo these, $[\xi, A] = \gamma_2 \colon (A^2, \cdot^\lambda) \to (A^2, pointwise)$.*

In fact, the coalgebras $C(\lambda)$ and D are pointed by the maps $\eta \colon \mathrm{k} \to C(\lambda)$ and $\eta \colon \mathrm{k} \to D$ with $\eta(1) = e$, and $\xi \colon D \to C(\lambda)$ is a map of pointed coalgebras. This structure transports under convolution to make γ_2 into the map of *copointed* endofunctors $(S, \sigma) \to (T, \tau)$ of Proposition 4.4. We saw in that Proposition that the comonads G_λ and H and comonad morphism $\gamma \colon G_\lambda \to H$ may be derived from these data using cofreeness; our next result reconstructs this purely in the world of coalgebras.

We first recall the construction of the free k-bialgebra on a pointed k-coalgebra (E, η). For each $\ell > 1$, define the coalgebra E_ℓ as the joint coequaliser $\gamma \colon E^{\otimes \ell} \to E_\ell$ of the ℓ coalgebra morphisms

$$\eta \otimes 1 \otimes \cdots \otimes 1 \ , \ 1 \otimes \eta \otimes \cdots \otimes 1 \ , \ \ldots \ , \ 1 \otimes 1 \otimes \cdots \otimes \eta \colon E^{\otimes(\ell-1)} \to E^{\otimes \ell} \ .$$

Of course, we also can put $E_0 = \mathrm{k}$ and $E_1 = E$. Then for each $\ell \geqslant 0$ we have a unique coalgebra morphism $\zeta \colon E_\ell \to E_{\ell+1}$ such that the square

$$\begin{array}{ccc} E^{\otimes \ell} & \xrightarrow{\ \gamma\ } & E_\ell \\ {\scriptstyle 1 \otimes \cdots \otimes 1 \otimes \eta} \downarrow & & \downarrow {\scriptstyle \zeta} \\ E^{\otimes(\ell+1)} & \xrightarrow{\ \gamma\ } & E_{\ell+1} \end{array}$$

commutes. Now the free monoid construction in [15] shows that the free bialgebra E_∞ on the pointed coalgebra E is obtained as the colimit of the chain

$$E_0 \xrightarrow{\zeta = \eta} E_1 \xrightarrow{\zeta} E_2 \xrightarrow{\zeta} \ldots \xrightarrow{\zeta} E_\ell \xrightarrow{\zeta} E_{\ell+1} \xrightarrow{\zeta} \ldots \ .$$

We now apply this to the coalgebras D and $C(\lambda)$. As k-modules, $D^{\otimes \ell}$ and $C(\lambda)^{\otimes \ell}$ are both free on the basis $\{e, d\}^\ell$, which we think of as the set of words W of length ℓ in the letters e and d; while D_ℓ and $C(\lambda)_\ell$ are obtained by quotienting out by the relation $WeW' \sim eWW'$ on basis words. They are thus vector spaces of dimension $\ell + 1$ with basis elements $e^r d^s$ where $r + s = \ell$. Omitting to write the e^r term, we can thus use the isomorphic basis $\{d_s \mid 0 \leqslant s \leqslant \ell\}$.

As a coalgebra D_ℓ, like D, is "set-like" with respect to these basis vectors: that is $\varepsilon(d_s) = 1$ and $\delta(d_s) = d_s \otimes d_s$ for each $0 \leqslant s \leqslant \ell$. On the other hand:

Proposition 5.2. *The coalgebra $C(\lambda)_\ell$, with respect to its basis $\{d_s \mid 0 \leqslant s \leqslant \ell\}$, has $\varepsilon(d_0) = 1$, $\varepsilon(d_n) = 0$ for $0 < n \leq \ell$, and for $0 \leq n \leq \ell$*

$$\delta(d_n) = \sum_{n = r+s+t} \binom{n}{r, s, t} \lambda^t d_{r+t} \otimes d_{s+t} \ .$$

Proof. In the coalgebra $C(\lambda)^{\otimes \ell}$, we have for all $m + n = \ell$ that

$$\delta(e^m d^n) = (e \otimes e)^m (d \otimes e + e \otimes d + \lambda d \otimes d)^n \ .$$

We cannot expand binomially since $d \otimes e$ and $e \otimes d$ and $\lambda d \otimes d$ do not commute in $C(\lambda)^\ell \otimes C(\lambda)^\ell$; but they do after applying $\gamma \otimes \gamma \colon C(\lambda)^\ell \otimes C(\lambda)^\ell \to C(\lambda)_\ell \otimes C(\lambda)_\ell$, and so we find that

$$\begin{aligned}(\gamma\otimes\gamma)\delta(e^m d^n) &= \sum_{n=r+s+t}\binom{n}{r,s,t}(e\otimes e)^m(d\otimes e)^r(e\otimes d)^s\lambda^t(d\otimes d)^t \\ &= \sum_{n=r+s+t}\binom{n}{r,s,t}\lambda^t(e^{m+s}d^{r+t}\otimes e^{m+r}d^{s+t})\ .\end{aligned}$$

This implies the result for comultiplication; the counit case is left as an exercise. ☐

Moreover, the pointed coalgebra morphism $\xi\colon D\to C(\lambda)$ induces for each $\ell\in\mathbb{N}$ a coalgebra morphism $\xi_\ell\colon D_\ell\to C(\lambda)_\ell$, unique such that the square

$$\begin{array}{ccc} D^{\otimes\ell} & \xrightarrow{\gamma} & D_\ell \\ {\scriptstyle\xi^{\otimes\ell}}\downarrow & & \downarrow{\scriptstyle\xi_\ell} \\ C(\lambda)^{\otimes(\ell)} & \xrightarrow{\gamma} & C(\lambda)_{\ell+1}\end{array}$$

commutes; passing to the colimit, we obtain a map of k-bialgebras $\xi_\infty\colon D_\infty\to C(\lambda)_\infty$. Arguing as in the preceding proof, we find for finite ℓ that

$$\xi_\ell(d_n)=\gamma\xi^{\otimes\ell}(e^k d^n)=e^k(\lambda d+e)^n=e^k\sum_m\binom{n}{m}\lambda^m d^m e^{n-m}=\sum_m\lambda^m\binom{n}{m}d_m$$

which now yields the following result generalising Proposition 5.1.

Proposition 5.3. *For any* k*-algebra A and $\ell\in\mathbb{N}\cup\{\infty\}$, there are isomorphisms $[C(\lambda)_\ell,A]\cong(A^{\ell+1},\cdot^\lambda)$ and $[D_\ell,A]\cong(A^{\ell+1},\textit{pointwise})$; modulo these $[\xi_\ell,A]=\gamma_{\ell+1}$.*

Proof. For finite ℓ, we simply compare the preceding formulae with Proposition 3.1; for $\ell=\infty$, we observe that convolution $[–,A]\colon$ k-**Coalg** $\to$ k-**Alg**$^{\mathrm{op}}$ preserves colimits. ☐

The functor k-**Coalg** $\to$ [k-**Alg**, k-**Alg**]$^{\mathrm{op}}$ sending C to $[C,–]$ carries tensor product of coalgebras to composition of endofunctors, and so carries each k-bialgebra C to a comonad $[C,–]$ on k-**Alg**. Of course, for the bialgebras $C(\lambda)_\infty$ and D_∞, the associated comonads are G_λ and H; this follows from Proposition 4.4 and the fact that the convolution functor k-**Coalg** $\to$ [k-**Alg**, k-**Alg**]$^{\mathrm{op}}$ sends each free bialgebra sequence in k-**Coalg** to a cofree comonad sequence in [k-**Alg**, k-**Alg**].

In light of these investigations, we may wonder whether there are other kinds of k-linear "derivation" which a k-algebra can bear, satisfying some different kind of "Leibniz identity". Our next result denies this possibility.

Proposition 5.4. *All pointed* k*-coalgebras whose underlying module is free of rank* 2 *are isomorphic to $C(\lambda)$ for some λ.*

Proof. Suppose $(C',\eta\colon \mathrm{k}\to C')$ is a pointed coalgebra with basis $\{e',d'\}$ such that $\eta(1)=e'$. Since η is a coalgebra morphism, $\varepsilon(e')=1$ and $\delta(e')=e'\otimes e'$. Suppose that $\varepsilon(d')=\gamma$; by making the change of basis $e=e'$ and $d=d'-\gamma e$ we have that

$$\varepsilon(e)=1\ ,\ \varepsilon(d)=0\ ,\ \delta(e)=e\otimes e\ .$$

Suppose $\delta(d)=\rho e\otimes e+\sigma e\otimes d+\tau d\otimes e+\upsilon d\otimes d$. From the counit properties $(\varepsilon\otimes 1)\delta=1$ and $(1\otimes\varepsilon)\delta=1$, we deduce $d=\rho e+\sigma d=\rho e+\tau d$ and so $\sigma=\tau=1$ and $\rho=0$. It is easily checked that the coassociativity condition is now automatic. So we have our result with $\upsilon=\lambda$. ☐

6. The Λ-weighted tensor product of species

In [40] is described a "categorification" of the λ-Hurwitz product on k-algebras; in the rest of this paper, we give corresponding "categorifications" of the results of the previous sections. To give these generalisations, we replace the commutative $\mathbb{Q}$-algebra k with a complete and cocomplete symmetric monoidal closed category $\mathscr{V}$, replace k-modules with what we shall call *$\mathscr{V}$-vector spaces*—$\mathscr{V}$-categories admitting finite coproducts and *$\mathscr{V}$-tensors* [31, §3.7]—and replace k-algebras by *$\mathscr{V}$-algebras*, that is, $\mathscr{V}$-vector spaces with a monoidal structure which preserves finite coproducts and $\mathscr{V}$-tensors in each variable separately. We call a $\mathscr{V}$-algebra *symmetric* when its monoidal structure is so[2].

Throughout the rest of this section, we fix some $\Lambda \in \mathscr{V}$. For any (symmetric) $\mathscr{V}$-algebra $\mathscr{A}$, there is a now (symmetric) $\mathscr{V}$-algebra structure on $\mathscr{A}^{\mathbb{N}}$ with unit $J = (I, 0, \dots)$ and binary tensor $*_\Lambda$ given by

$$(M *_\Lambda N)_n = \sum_{n=r+s+t} \Lambda^{\otimes r} \otimes M_s \otimes N_t \ . \tag{6.4}$$

In [40] was described a similar tensor product on the $\mathscr{V}$-category $\mathscr{A}^{\mathfrak{S}}$ of *$\mathscr{A}$-valued species*—here $\mathfrak{S}$ is the groupoid of finite sets and bijections—whose unit J and binary tensor $*_\Lambda$ are given by

$$JX = \begin{cases} I & \text{if } X = \emptyset; \\ 0 & \text{otherwise;} \end{cases} \qquad \text{and} \qquad (M *_\Lambda N)X = \sum_{\substack{U,V \subset X \\ X = U \cup V}} \Lambda^{|U \cap V|} \otimes MU \otimes NV \ . \tag{6.5}$$

As previously, these monoidal structures restrict back to the respective subcategories $\mathscr{A}^\ell \subset \mathscr{A}^{\mathbb{N}}$ and $\mathscr{A}^{\mathfrak{S}_\ell} \subset \mathscr{A}^{\mathfrak{S}}$ for any $\ell \in \mathbb{N}$; here $\mathfrak{S}_\ell$ is the full subcategory of $\mathfrak{S}$ on sets of cardinality $< \ell$. Our next result will reconstruct these monoidal structures through an argument like that of Section 5.

We exploit the symmetric monoidal bicategory $\mathscr{V}$-**Vect** of $\mathscr{V}$-vector spaces, wherein 1-cells are *$\mathscr{V}$-linear* $\mathscr{V}$-functors—ones preserving finite coproducts and $\mathscr{V}$-tensors—2-cells are $\mathscr{V}$-natural transformations, and the tensor product $\otimes$ classifies *$\mathscr{V}$-bilinear* $\mathscr{V}$-bifunctors—ones which preserve finite coproducts and $\mathscr{V}$-tensors in each variable separately. The unit object is $\mathscr{V}$ itself. This monoidal bicategory is biclosed in the sense of [13], with the internal hom $[\mathscr{A}, \mathscr{B}]$ being the functor $\mathscr{V}$-category of $\mathscr{V}$-linear $\mathscr{V}$-functors. There is a free-forgetful biadjunction $\mathscr{V}\text{-}\mathbf{Vect} \leftrightarrows \mathbf{Cat}$ whose left adjoint is *strong monoidal* with respect to the tensor product of $\mathscr{V}$-vector spaces and the cartesian product of categories. For a given category $\mathscr{A}$, we write $\langle \mathscr{A} \rangle$ for the free $\mathscr{V}$-vector space thereon.

A (symmetric) monoidale[3] in $\mathscr{V}$-**Vect** is precisely a (symmetric) $\mathscr{V}$-algebra in the sense described above; correspondingly, we refer to comonoidales in $\mathscr{V}$-**Vect** as *$\mathscr{V}$-coalgebras*. By convolution as in [13], the internal hom from a comonoidale to a monoidale in a biclosed monoidal bicategory is again a monoidale; so if $\mathscr{C}$ is a $\mathscr{V}$-coalgebra and $\mathscr{A}$ is a $\mathscr{V}$-algebra, then the $\mathscr{V}$-linear hom $[\mathscr{C}, \mathscr{A}]_{\mathscr{V}}$ is a $\mathscr{V}$-algebra.

Consider now the $\mathscr{V}$-coalgebra $\mathscr{C}(\Lambda)$ with underlying $\mathscr{V}$-vector space $\langle E, D \rangle$—so that $\mathscr{C}(\Lambda) \otimes \mathscr{C}(\Lambda) \simeq \langle E \otimes E, E \otimes D, D \otimes E, D \otimes D \rangle$—with counit $\varepsilon \colon \mathscr{C}(\Lambda) \to \mathscr{V}$ and

[2] Although now this is extra structure not just a condition

[3] Also called a *pseudomonoid*.

comultiplication $\delta\colon \mathscr{C}(\Lambda) \to \mathscr{C}(\Lambda) \otimes \mathscr{C}(\Lambda)$ given on generators by

$$\varepsilon(E) = I\ ,\ \varepsilon(D) = 0\ ,\ \delta(E) = E \otimes E\ ,\ \delta(D) = D \otimes E + E \otimes D + \Lambda \otimes D \otimes D. \tag{6.6}$$

The coassociativity and counit coherences are given on generators in the obvious way. There is a comonoidale morphism $\mathscr{V} \to \mathscr{C}(\Lambda)$ sending V to $V \otimes E$ so making $\mathscr{C}(\Lambda)$ into a *pointed* $\mathscr{V}$-coalgebra. Convolving with a $\mathscr{V}$-algebra $\mathscr{A}$ gives monoidal structures on the $\mathscr{V}$-linear hom $[\mathscr{C}(\Lambda), \mathscr{A}]_{\mathscr{V}} \simeq \mathscr{A}^2$; the unit object is $J = (I, 0)$, while the binary tensor is given by

$$(M_0, M_1) *_\Lambda (N_0, N_1) = (M_0 \otimes N_0, M_0 \otimes N_1 + M_1 \otimes N_0 + \Lambda \otimes M_1 \otimes N_1)$$

so that we re-find the case $\ell = 2$ of the Λ-weighted tensor product (6.4), which is also the case $\ell = 2$ of (6.5). We now show how to obtain the corresponding tensor products on $\mathscr{A}^{\mathbb{N}}$ or $[\mathfrak{S}, \mathscr{A}]$ from this by arguing as in Section 5.

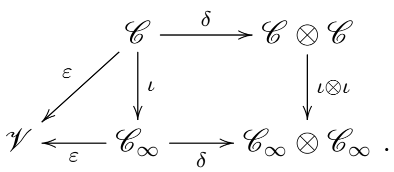
First, since the free $\mathscr{V}$-vector space 2-functor **Cat** $\to \mathscr{V}$-**Vect** is strong symmetric monoidal and cocontinuous, it preserves the construction of free (symmetric) monoidales. As the free monoidal category and the free symmetric monoidal category on the pointed category $\{E\} \to \{E, D\}$ are $(\mathbb{N}, +, 0)$ and $(\mathfrak{S}, +, 0)$ respectively, we conclude that:

Proposition 6.1. *The free $\mathscr{V}$-algebra and free symmetric $\mathscr{V}$-algebra on the pointed $\mathscr{V}$-vector space $\langle E\rangle \to \langle E, D\rangle$ are respectively $\langle\mathbb{N}\rangle$ and $\langle\mathfrak{S}\rangle$ under the monoidal structure given on basis elements by disjoint union.*

The symmetric monoidal structure on the bicategory $\mathscr{V}$-**Vect** lifts to the bicategories $\mathscr{V}$-**Alg** and $\mathscr{V}/\mathscr{V}$-**Vect** of $\mathscr{V}$-algebras and of pointed $\mathscr{V}$-vector spaces; when endowed with these monoidal structures, the forgetful $\mathscr{V}$-**Alg** $\to \mathscr{V}/\mathscr{V}$-**Vect** is thus strict monoidal, so that its left biadjoint is an opmonoidal homomorphism. It follows that the biadjunction passes to the respective bicategories of comonoidales:

$$\mathscr{V}\text{-}\mathbf{Bialg} \leftrightarrows \mathscr{V}/\mathscr{V}\text{-}\mathbf{Coalg}\ .$$

Explicitly, this means that, if $Z\colon \mathscr{V} \to \mathscr{C}$ is a pointed $\mathscr{V}$-coalgebra, and the map $\iota\colon \mathscr{C} \to \mathscr{C}_\infty$ of pointed objects exhibits $\mathscr{C}_\infty$ as the free $\mathscr{V}$-algebra on $Z\colon \mathscr{V} \to \mathscr{C}$ seen as a pointed $\mathscr{V}$-vector space, then $\mathscr{C}_\infty$ bears a coalgebra structure making it into the free $\mathscr{V}$-bialgebra on $\mathscr{V} \to \mathscr{C}$, with counit and comultiplication $\mathscr{V}$-functors obtained as the essentially-unique homomorphisms of $\mathscr{V}$-algebras rendering the following diagram commutative to within natural isomorphisms:



(6.7)

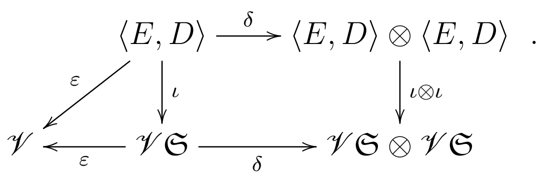
Of course, if we construct instead the free *symmetric* $\mathscr{V}$-algebra on $\mathscr{C}$, then the above argument show that we in fact obtain the free *symmetric* $\mathscr{V}$-bialgebra on $\mathscr{C}$. Applying these two constructions to the pointed $\mathscr{V}$-coalgebra $\mathscr{C}(\Lambda)$ and using Proposition 6.1, we induce $\mathscr{V}$-bialgebra structures on $\langle\mathbb{N}\rangle$ and on $\langle\mathfrak{S}\rangle$; we will show that these convolve to give the weighted tensor products of (6.4) and (6.5). The argument in the two cases is similar, and that for $\langle\mathbb{N}\rangle$ is exactly like that in Section 5 above; so we go through the details only for $\langle\mathfrak{S}\rangle$.

Theorem 6.2. *The free symmetric $\mathscr{V}$-bialgebra $\mathscr{C}(\Lambda)_\infty$ on the pointed $\mathscr{V}$-coalgebra $\mathscr{C}(\Lambda)$ is $\langle \mathfrak{S} \rangle$ with as algebra structure the $\mathscr{V}$-linear extension of $(\mathfrak{S}, +, 0)$, and coalgebra structure determined on basis elements $X \in \mathfrak{S}$ by:*

$$\varepsilon(X) = \begin{cases} I & \text{if } X = 0; \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \delta(X) = \sum_{X = U \cup V} \Lambda^{\otimes |U \cap V|} \otimes U \otimes V \ . \tag{6.8}$$

For any $\mathscr{V}$-algebra $\mathscr{A}$, the convolution algebra structure on $[\mathscr{C}(\Lambda)_\infty, \mathscr{A}]_{\mathscr{V}} \simeq \mathscr{A}^{\mathfrak{S}}$ is given by the Λ-weighted tensor product of (6.5).

Proof. [40, Proposition 23] shows that the above data do indeed define a $\mathscr{V}$-bialgebra structure on $\langle \mathfrak{S} \rangle$. Let $\iota \colon \langle E, D \rangle \to \langle \mathfrak{S} \rangle$ be given on generators by $E \mapsto 0$ and $D \mapsto 1$. By Proposition 6.1, this map exhibits $\langle \mathfrak{S} \rangle$ as the free symmetric $\mathscr{V}$-algebra on the pointed object $\langle E \rangle \to \langle E, D \rangle$; now by comparing (6.9) with (6.6), we see that the following diagram commutes to within isomorphism since it does so on generators:



By the argument following Proposition 6.1, we conclude that $\langle \mathfrak{S} \rangle$, equipped with the given algebra and coalgebra structures, is the free symmetric $\mathscr{V}$-bialgebra $\mathscr{C}(\Lambda)_\infty$ on $\mathscr{C}(\Lambda)$. The final claim follows immediately by comparing (6.9) with (6.5). ☐

We now relate the Λ-weighted tensor products on $\mathscr{A}^{\mathbb{N}}$ and $\mathscr{A}^{\mathfrak{S}}$ to the pointwise ones, following the pattern set out in Section 5 above. To this end, consider the pointed $\mathscr{V}$-coalgebra $\mathscr{D}$ with the same underlying $\mathscr{V}$-vector space $\langle E, D \rangle$ as $\mathscr{C}(\Lambda)$ and the same pointing, but the diagonal comonoidale structure:

$$\varepsilon(E) = I \ , \ \varepsilon(D) = I \ , \ \delta(E) = E \otimes E \ , \ \delta(D) = D \otimes D \ .$$

For any $\mathscr{V}$-algebra $\mathscr{A}$, convolution with $\mathscr{D}$ induces the pointwise $\mathscr{V}$-algebra structure on the $\mathscr{V}$-linear hom $[\mathscr{D}, \mathscr{A}] \simeq \mathscr{A}^2$. Arguing as previously, we now have that:

Theorem 6.3. *The free symmetric $\mathscr{V}$-bialgebra $\mathscr{D}_\infty$ on the pointed $\mathscr{V}$-coalgebra $\mathscr{D}$ is $\langle \mathfrak{S} \rangle$ with as algebra structure the $\mathscr{V}$-linear extension of $(\mathfrak{S}, +, \emptyset)$, and coalgebra structure given on homogeneous elements $X \in \mathfrak{S}$ by:*

$$\varepsilon(X) = I \quad \text{and} \quad \delta(X) = X \otimes X \ . \tag{6.9}$$

For any $\mathscr{V}$-algebra $\mathscr{A}$, the convolution algebra structure on $[\mathscr{D}_\infty, \mathscr{A}]_{\mathscr{V}} \simeq \mathscr{A}^{\mathfrak{S}}$ is the pointwise algebra structure.

To compare the pointwise and the Λ-weighted monoidal structures on $\mathscr{A}^{\mathfrak{S}}$, it will thus suffice to compare their restrictions to $\mathscr{A}^2$. So consider the $\mathscr{V}$-linear $\mathscr{V}$-functor $\Theta \colon \mathscr{D} \to \mathscr{C}(\Lambda)$ defined on generators by $\Theta(E) = E$ and $\Theta(D) = \Lambda \otimes D + E$; we may without difficulty equip this with the structure of a strong morphism of pointed comonoidales in $\mathscr{V}$-**Vect**. The pointings on $\mathscr{C}(\Lambda)$ and $\mathscr{D}$ correspond to the $\mathscr{V}$-algebra morphisms given by the first projection, while $\Theta \colon \mathscr{D} \to \mathscr{C}(\Lambda)$ transports under convolution to yield $\hat{\Theta} \colon \mathscr{A}^2 \to \mathscr{A}^2$ sending (M_0, M_1) to $(M_0, M_0 + \Lambda \otimes M_1)$. Since Θ is a strong morphism of pointed comonoidales, this yields:

Proposition 6.4. $\widehat{\Theta}\colon (\mathscr{A}^2, *_\Lambda, J) \to (\mathscr{A}^2, pointwise)$ *is strong monoidal.*

The map Θ induces a map $\Theta_\infty\colon \mathscr{D}_\infty \to \mathscr{C}(\Lambda)_\infty$ of symmetric bimonoidales which is determined in an essentially-unique manner by the requirement that the square

$$\begin{array}{ccc} \mathscr{D} & \xrightarrow{\Theta} & \mathscr{C}(\Lambda) \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\ \mathscr{D}_\infty & \xrightarrow{\Theta_\infty} & \mathscr{C}(\Lambda)_\infty \end{array} \tag{6.10}$$

should commute to within isomorphism. It is easy to see that these requirements are satisfied by taking Θ_∞ to be defined on basis elements $X \in \mathfrak{S}$ by:

$$\Theta_\infty(X) = \sum_{W \subseteq X} \Lambda^{\otimes |W|} \otimes W \ .$$

The proof that this is a map of bimonoidales uses the equation $U+V = U\cup V + U\cap V$ for $U, V \subseteq X$. For any $\mathscr{V}$-algebra $\mathscr{A}$, convolution with Θ_∞ yields a $\mathscr{V}$-functor $\widehat{\Theta}_\infty\colon \mathscr{A}^{\mathfrak{S}} \to \mathscr{A}^{\mathfrak{S}}$ defined by $\bar{\Theta}_\infty(M)X = \sum_{W\subseteq X} \Lambda^{\otimes|W|} \otimes MW$, and since Θ_∞ is a strong map of bimonoidales, we conclude that:

Proposition 6.5. $\widehat{\Theta}_\infty\colon (\mathscr{A}^{\mathfrak{S}}, *_\Lambda, J) \to (\mathscr{A}^{\mathfrak{S}}, pointwise)$ *is strong monoidal.*

Note that the preceding two results are further examples of a transformation converting a convolution product into pointwise product as promoted in [12].

7. Correspondences of Dold–Kan type

In these final two sections, we consider a different categorification of Sections 3–5. This time, we replace modules over a commutative ring k by *additive Karoubian* categories—**Ab**-enriched categories which admit finite biproducts and splittings of idempotents; and we replace (commutative) k-algebras by (symmetric) **Ab**-*algebras*: (symmetric) monoidal additive Karoubian categories.

In [33] is described a general theory for establishing equivalences of additive Karoubian categories. From an ordinary category $\mathscr{P}$ equipped with a subcategory of monomorphisms $\mathscr{M}$ satisfying some axiomatic assumptions is constructed a category $\mathscr{D}$ enriched over the category of pointed sets together with an equivalence

$$\Gamma\colon [\mathscr{D}, \mathscr{X}]_{\mathrm{pt}} \to [\mathscr{P}, \mathscr{X}] \tag{7.11}$$

for each additive Karoubian $\mathscr{X}$; here, on the right we have the ordinary functor category, and on the left the category of zero-map preserving functors. For suitable choices of $\mathscr{P}$ and $\mathscr{M}$, equivalences obtained in this way include the classical Dold–Puppe–Kan correspondence [14] between simplicial abelian groups and chain complexes; the correspondence between cubical and semi-simplicial abelian groups; and the equivalence between linear species and the FI♯-*modules* of [8].

In the situation of (7.11), if the additive Karoubian $\mathscr{X}$ is an **Ab**-algebra, then so too is $[\mathscr{P}, \mathscr{X}]$ under the pointwise tensor product. Transporting across the equivalence yields an **Ab**-algebra structure also on $[\mathscr{D}, \mathscr{X}]_{\mathrm{pt}}$; in fact, this turns out to be a Hurwitz-style ($\lambda = 1$) tensor product, and the comparison functor (7.11) yet another example of a transform taking convolution to pointwise product.

We show this in detail for a special but useful case of (7.11). Let $\mathscr{C}$ be a category equipped with an orthogonal factorisation system $(\mathscr{E}, \mathscr{M})$ in the sense of [17] such that all $\mathscr{M}$-maps are monomorphisms, pullbacks of $\mathscr{M}$-maps along arbitrary morphisms exist, and each $A \in \mathscr{C}$ has only a finite set of distinct $\mathscr{M}$-subobjects. We take $\mathscr{P} = \mathbf{Par}(\mathscr{C}, \mathscr{M})$, whose objects are those of $\mathscr{C}$, whose maps from A to B are isomorphism-classes of spans $m\colon A \leftarrowtail R \to B\colon f$ in $\mathscr{C}$ with $m \in \mathscr{M}$, and whose composition is by pullback. We write $\mathscr{R}$ for the category whose objects are those of $\mathscr{C}$ and whose maps are the $\mathscr{E}$-maps, and we define $\mathscr{D}$ to be the free category with zero maps on $\mathscr{R}$. With this choice of $\mathscr{P}$ and $\mathscr{D}$ we obtain by [33, Example 3.1] our first instance of an equivalence (7.11), which, since $\mathscr{D}$ is free on $\mathscr{R}$, may be written more simply as

$$\Gamma\colon [\mathscr{R}, \mathscr{X}] \to [\mathscr{P}, \mathscr{X}] \ . \tag{7.12}$$

In order to see how the pointwise tensor product on $[\mathscr{P}, \mathscr{X}]$ transports under this equivalence, we will need an explicit description of both Γ and its pseudoinverse. Choose for each $A \in \mathscr{C}$ a representing set $\mathrm{Sub}(A)$ of $\mathscr{M}$-subobjects, and write $B \leqslant_n A$ to mean that $n\colon B \rightarrowtail A$ is in $\mathrm{Sub}(A)$ and $B <_n A$ to mean that $n \in \mathrm{Sub}(A)$ is *proper*: that is, non-invertible. For $F \in [\mathscr{R}, \mathscr{X}]$, we now take $\Gamma F\colon \mathscr{P} \to \mathscr{X}$ to be given on objects by the (finite) direct sum

$$(\Gamma F)A = \bigoplus_{B \leqslant_n A} FB \ .$$

To specify ΓF at a map $m\colon A \leftarrowtail R \to A'\colon f$ in $\mathscr{P}$, suppose that $B \leqslant_n A$ and that $B' \leqslant_{n'} A'$, and define $\xi(m, f)_{nn'}\colon FB \to FB'$ to be Fe if there is a (necessarily unique) diagram of the form

$$\begin{array}{ccccc} B & = & B & \xrightarrow{e \in \mathscr{E}} & B' \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle n'} \\ A & \xleftarrow{\ m\ } & R & \xrightarrow{\ f\ } & A' \end{array} \tag{7.13}$$

and define $\xi(m, f)_{nn'} = 0$ otherwise. We now take $(\Gamma F)(m, f)\colon (\Gamma F)A \to (\Gamma F)A'$ to be the matrix of size $\mathrm{Sub}(A) \times \mathrm{Sub}(A')$ with entries $\xi(m, f)_{nn'}$. Note in particular that, if $f = m$ in (7.13), then e must be invertible, whence $n = n'$ in $\mathrm{Sub}(A)$; thus $\xi(m, m)_{nn'}$ is the identity if $n = n'$ and n factors through m, and is zero otherwise. This shows that $(\Gamma F)(m, m)$ is the idempotent on $\bigoplus_{B \leqslant_n A} FB$ which projects onto those summands $n \in \mathrm{Sub}(A)$ which factor through m.

The inverse equivalence $N\colon [\mathscr{P}, \mathscr{X}] \to [\mathscr{R}, \mathscr{X}]$ to (7.11) sends $H \in [\mathscr{P}, \mathscr{X}]$ to $NH\colon \mathscr{R} \to \mathscr{X}$ defined on objects by

$$(NH)(A) = \bigcap_{R <_m A} \ker(H(m, m)\colon HA \to HA) \ ; \tag{7.14}$$

note that this is well-defined in the Cauchy-complete $\mathscr{X}$, since the limit involved may be constructed by splitting the idempotent $\prod_{R <_m A}(1 - H(m, m))$ of the ring $\mathscr{X}(HA, HA)$. The action of NH on a morphism $e\colon A \to A'$ is the unique factorisation of the composite $(NH)(A) \rightarrowtail HA \to HA'$ through $(NH)(A') \rightarrowtail HA'$; the existence of such a factorisation is verified in [33, Theorem 4.1], while the fact that Γ and N are indeed pseudoinverse is proved by Theorem 6.7 of *ibid.*

Using the above formulae, we may now derive the existence of a Hurwitz-style tensor product on $[\mathscr{R}, \mathscr{X}]$ for any **Ab**-algebra $\mathscr{X}$ which transports the pointwise one on $[\mathscr{P}, \mathscr{X}]$. In the statement of the following result, we call a pair of subobjects $n, n' \in \mathrm{Sub}(A)$ *covering* if any $\mathscr{M}$-map through which both n and n' factor is invertible.

Proposition 7.1. *Let $\mathscr{C}$ be a category equipped with an $(\mathscr{E}, \mathscr{M})$-factorisation system, let all pullbacks along $\mathscr{M}$-maps exist and let each $A \in \mathscr{C}$ have but a finite set of $\mathscr{M}$-subobjects. Writing as above $\mathscr{P} = \mathbf{Par}(\mathscr{C}, \mathscr{M})$ and $\mathscr{R}$ for the category of $\mathscr{E}$-maps, there is for any (symmetric)* **Ab***-algebra $\mathscr{X}$ a (symmetric)* **Ab***-algebra structure $(*, J)$ on $[\mathscr{R}, \mathscr{X}]$ whose unit J and binary tensor $*$ are given by*

$$J(A) = \begin{cases} I & \text{if } |\mathrm{Sub}(A)| = 1; \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad (F * G)(A) = \bigoplus_{\substack{B \leqslant_n A,\ B' \leqslant_{n'} A \\ n,n' \text{ are covering}}} FB \otimes GB' \tag{7.15}$$

Moreover, (7.12) is strong monoidal as a functor $([\mathscr{R}, \mathscr{X}], *) \to ([\mathscr{P}, \mathscr{X}], \text{pointwise})$.

Proof. It suffices to show that the pointwise tensor product on $[\mathscr{P}, \mathscr{X}]$ transports to the given structure on $[\mathscr{R}, \mathscr{X}]$. For the unit this is immediate from (7.14). For the binary tensor, given $F, G \in [\mathscr{R}, \mathscr{X}]$, we form the pointwise tensor $H = \Gamma F \otimes \Gamma G$ with

$$HA = (\Gamma F)A \otimes (\Gamma G)A = \bigoplus_{B \leqslant_n A} FB \otimes \bigoplus_{B' \leqslant_{n'} A} GB' \cong \bigoplus_{\substack{B \leqslant_n A \\ B' \leqslant_{n'} A}} FB \otimes GB'$$

on objects; we will not need the full definition on morphisms, but we see by the observations above that, for any $\mathscr{M}$-map $m \colon R \rightarrowtail A$, the idempotent $H(m, m) \colon HA \to HA$ is given by projection onto those summands $(n, n') \in \mathrm{Sub}(A)^2$ for which both n and n' factor through m. Comparing with (7.14), we conclude that $(NH)(A)$ is given by restricting HA to those direct summands (n, n') which do *not* have this property for any proper $m \in \mathrm{Sub}(A)$: which gives the formula for $F * G$ displayed above. ☐

Examples 7.2. (i) Let $\mathscr{C}$ be the category of finite sets and injections, and $(\mathscr{E}, \mathscr{M})$ the (isomorphisms, all maps) factorisation system. Then $\mathscr{R}$ is the groupoid $\mathfrak{S}$ of finite sets and bijections and $\mathscr{P}$ is the category of finite sets and partial injections, denoted by FI♯ in [8]; the equivalence $[\mathfrak{S}, \mathscr{X}] \to [\mathrm{FI}\sharp, \mathscr{X}]$ for any additive Karoubian $\mathscr{X}$ is now that of Theorem 4.1.5 of *ibid.* When $\mathscr{X}$ is an **Ab**-algebra, the induced tensor product on $[\mathfrak{S}, \mathscr{X}]$ is the case $\Lambda = \mathbb{Z}$ of (6.5), so that this tensor product corresponds to the pointwise tensor product of FI♯-modules.

(ii) In a similar way, when $\mathscr{C}$ is Δ^+_{inj}, the category of finite ordinals $[n]$ and monotone injections, and $(\mathscr{E}, \mathscr{M}) = (\text{identities}, \text{all maps})$, we have $\mathscr{R} = \mathbb{N}$ and $\mathscr{P}$ the category FO♯ of finite ordinals and partial monotone injections; in this way, we re-find for any **Ab**-algebra $\mathscr{X}$ the case $\Lambda = \mathbb{Z}$ of the tensor product (6.4) on $\mathscr{X}^{\mathbb{N}}$, but now with the additional information that it is monoidally equivalent to the pointwise monoidal structure on $[\mathrm{FO}\sharp, \mathscr{X}]$.

(iii) Take $\mathscr{C}$ itself to be FO♯, and let $\mathscr{E}$ and $\mathscr{M}$ comprise the maps therein with entire codomain and domain respectively. Then $\mathscr{R}$ is isomorphic to $(\Delta^+_{\mathrm{inj}})^{\mathrm{op}}$, while $\mathscr{P}$ is the cube category $\mathbb{I}$ of [9], diagrams on which are the cubical sets[4] of [28]. For any **Ab**-algebra $\mathscr{X}$, the pointwise product on cubical objects in $\mathscr{X}$ thus

[4]With degeneracies but without symmetries or connections.

transports to a Hurwitz-style tensor product on the category $[(\Delta^+_{\mathrm{inj}})^{\mathrm{op}}, \mathscr{X}]$ of augmented semi-simplicial objects in $\mathscr{X}$.

(iv) Let $\mathscr{C}$ be the category of finite sets and *all* maps, and $(\mathscr{E}, \mathscr{M})$ the (epi, mono) factorisation system. In this case, $\mathscr{P}$, the category of finite sets and partial maps, is isomorphic to Segal's category Γ of finite pointed sets, while $\mathscr{R}$ is the category Ω of finite sets and epimorphisms. The equivalence $[\Omega, \mathscr{X}] \to [\Gamma, \mathscr{X}]$ for any additive Karoubian $\mathscr{X}$ was described in [35, Theorem 3.1], and our construction yields a Hurwitz-style tensor product on $[\Omega, \mathscr{X}]$ for any **Ab**-algebra $\mathscr{X}$.

We now summarise the analogue of the preceding results for the general case of (7.11) as described in [33]. The basic data are a category $\mathscr{P}$; a subcategory $\mathscr{M}$ containing all the isomorphisms; and an identity-on-objects functor $(-)^*\colon \mathscr{M}^{\mathrm{op}} \to \mathscr{P}$ such that $m^* \circ m = 1$ for every $m \in \mathscr{M}$ (so in particular, each m is a split monomorphism in $\mathscr{C}$). The class of morphisms $\mathscr{R}$ is defined to comprise those $r \in \mathscr{P}$ such that, if $r = m \circ x \circ n^*$ for any $m, n \in \mathscr{M}$, then m and n are invertible. The category $\mathscr{D}$ has the same objects as $\mathscr{P}$, and as morphisms the maps in $\mathscr{R}$ with a zero morphism freely adjoined between any pair of objects; if $r, s \in \mathscr{R}$ are composable morphisms in $\mathscr{D}$, then their composite is $s \circ r$ if this also lies in $\mathscr{R}$, and is zero otherwise.

These data are required to satisfy various Assumptions which are listed in [33, Section 2]; one of these is that every morphism $f \in \mathscr{P}$ factors as $f = n \circ r \circ m^*$, uniquely up to isomorphism, for $m, n \in \mathscr{M}$ and $r \in \mathscr{R}$, while another is that each set $\mathrm{Sub}(A)$ of $\mathscr{M}$-subobjects is finite. Given these data, Theorem 6.7 of *ibid.* defines the equivalence Γ of (7.11) as follows. For $F \in [\mathscr{D}, \mathscr{X}]_{\mathrm{pt}}$, we take $\Gamma F\colon \mathscr{P} \to \mathscr{X}$ to be given on objects as before by

$$(\Gamma F)A = \bigoplus_{B \leqslant_n A} FB \ ,$$

where the sum is again over the (finite) set of $\mathscr{M}$-subobjects of A. For its value at a map $f\colon A \to A'$ in $\mathscr{P}$, suppose that $B \leqslant_n A$ and that $B' \leqslant_{n'} A'$, and define $\xi(f)_{nn'}\colon FB \to FB'$ to be Fr if there is a (necessarily unique) diagram of the form

$$\begin{array}{ccc} B & \xrightarrow{r \in \mathscr{R}} & B' \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle n'} \\ A & \xrightarrow{f} & A' \end{array}$$

and to be zero otherwise; we now take $(\Gamma F)(f)$ to be the matrix with entries $\xi(f)_{nn'}$. It follows from the assumptions that, for any $m \in \mathscr{M}$, the map $(\Gamma F)(mm^*)$ is the idempotent on $\bigoplus_{B \leqslant_n A} FB$ projecting onto those summands $n \in \mathrm{Sub}(A)$ for which $m^* n \in \mathscr{M}$. The pseudoinverse N to Γ is once again defined by the formula (7.14) (with $H(mm^*)$ replacing $H(m,m)$) and now tracing through the remainder of the argument given above yields the following more general version of Proposition 7.3. When interpreting (7.15) in this context, we say that $n, n' \in \mathrm{Sub}(A)$ are *covering* if whenever $m \in \mathscr{M}$ is such that $m^* n$ and $m^* n'$ are both in $\mathscr{M}$, then m is invertible.

Proposition 7.3. *Suppose given a category $\mathscr{P}$, subcategory $\mathscr{M}$, and identity-on-objects $(-)^*\colon \mathscr{M}^{\mathrm{op}} \to \mathscr{P}$ satisfying the Assumptions of [33], and let $\mathscr{D}$ be the associated category with zero maps as above. For any (symmetric)* **Ab**-*algebra $\mathscr{X}$ there is a* *(symmetric)* **Ab***-algebra structure* $(*, J)$ *on* $[\mathscr{D}, \mathscr{X}]_{\mathrm{pt}}$ *defined as in* (7.15)*. Moreover,* (7.11) *is strong monoidal as a functor* $\Gamma\colon ([\mathscr{D}, \mathscr{X}]_{\mathrm{pt}}, *, J) \to ([\mathscr{P}, \mathscr{X}], \textit{pointwise})$*.*

The main additional example that this more general case allows is the following one. If we take $\mathscr{P}$ to be Δ^{op}, for Δ the category of non-empty finite ordinals and monotone maps, take $\mathscr{M}$ to comprise the surjective monotone maps, and for each $m \in \mathscr{M}$ take m^* to be its right adjoint, then the assumptions of [33] may be verified to hold as in Example 3.3 of *ibid.* In this case, the category $\mathscr{D}$ turns out to be the indexing category for chain complexes, and the equivalence (7.11) the classical Dold–Kan equivalence between simplicial objects and chain complexes in any additive Karoubian $\mathscr{X}$. The preceding proposition now describes for any **Ab**-algebra $\mathscr{X}$ a tensor product of chain complexes in $\mathscr{X}$ given by

$$(A * B)(n) = \bigoplus_{\substack{\sigma\colon [k] \twoheadleftarrow [n] \twoheadrightarrow [\ell]\colon \tau \\ \sigma,\tau \text{ jointly monic}}} Ak \otimes B\ell \ .$$

This tensor product was calculated explicitly in unpublished work [**?**] of Lack and Hess; the key point is that $A * B$ is a well-behaved retract of the usual tensor product $A \otimes B$ of chain complexes. This can be understood as part of the fact that the equivalence $N\colon [\Delta^{\mathrm{op}}, \mathscr{X}] \to \mathbf{Ch}(\mathscr{X})$ is a Frobenius monoidal functor, as explained in [1, Chapter 5].

8. Dold–Kan equivalences from small coalgebras

We conclude this paper by discussing coalgebraic aspects of the equivalences described in the previous section. To this end, we consider the 2-category $\mathbf{Ab}\text{-}\mathbf{Cat}_{cc}$ whose objects are additive Karoubian categories, and whose 1- and 2-cells are **Ab**-enriched functors and transformations. $\mathbf{Ab}\text{-}\mathbf{Cat}_{cc}$ is a symmetric monoidal biclosed bicategory: the tensor product is obtained by Cauchy completing the usual **Ab**-categorical tensor product, while the internal hom is the standard **Ab**-enriched functor category. We can thus talk about (symmetric) monoidales and comonoidales in $\mathbf{Ab}\text{-}\mathbf{Cat}_{cc}$; the monoidales are the **Ab**-algebras considered previously, while the comonoidales we refer to as **Ab**-*coalgebras*. Much as in Section 6, we have a free–forgetful biadjunction $\mathbf{Ab}\text{-}\mathbf{Cat}_{cc} \leftrightarrows \mathbf{Cat}$ whose left biadjoint is a strong monoidal homomorphism, and we reuse the notation $\langle \mathscr{A} \rangle$ for the free additive Karoubian category on $\mathscr{A} \in \mathbf{Cat}$. If $\mathscr{A}$ is a category with zero morphisms, then we write $\langle \mathscr{A} \rangle_{\mathrm{pt}}$ for the free additive Karoubian category on $\mathscr{A}$ *qua* category with zero morphisms.

Suppose now that we are given a category $\mathscr{P}$, a subcategory $\mathscr{M}$ and a functor $(-)^*\colon \mathscr{M}^{\mathrm{op}} \to \mathscr{P}$ satisfying the assumptions of [33], and let $\mathscr{D}$ be as before the associated category with zero maps. We may see the equivalences $\Gamma\colon [\mathscr{D}, \mathscr{X}]_{\mathrm{pt}} \to [\mathscr{P}, \mathscr{X}]$ of (7.11) for any additive Karoubian $\mathscr{X}$ as induced by precomposition with an equivalence $\langle \mathscr{P} \rangle \to \langle \mathscr{D} \rangle_{\mathrm{pt}}$ of additive Karoubian categories, defined on generating objects by $A \mapsto \bigoplus_{B \leqslant_n A} B$. The point we wish to make is that, for several of our examples above, the **Ab**-categories $\langle \mathscr{P} \rangle$ and $\langle \mathscr{D} \rangle_{\mathrm{pt}}$ are **Ab**-algebras which are free on pointed additive Karoubian categories, and the equivalence between them generated by an equivalence at this more primitive level. For our first example of this, we consider the equivalence $[\mathfrak{S}, \mathscr{X}] \simeq [\mathrm{FI}\sharp, \mathscr{X}]$ of Examples 7.2(i). We write **SE** for the *free split* *epimorphism* as displayed in:

$$\begin{array}{ccc} E & \xrightarrow{m} & D \\ & \searrow^{1} & \downarrow e \\ & & E\ . \end{array}$$

Proposition 8.1. *The free symmetric* **Ab**-*algebras on the pointed additive Karoubian categories* $\langle E\rangle \to \langle E, D\rangle$ *and* $\langle E\rangle \to \langle \mathbf{SE}\rangle$ *are respectively* $\langle \mathfrak{S}\rangle$ *and* $\langle \mathrm{FI}\sharp\rangle$ *under the monoidal structures given on basis elements by disjoint union.*

Proof. The first claim follows as in Proposition 6.1 above, and the second does too once we observe that FI♯ is the free symmetric monoidal category on the pointed category $\{E\} \to \mathbf{SE}$, with the generators E and D corresponding to the empty and singleton sets 0 and 1. □

It is straightforward to see that $\langle E, D\rangle$ and $\langle \mathbf{SE}\rangle$ are equivalent as pointed **Ab**-categories. In one direction, we have the **Ab**-functor $\Theta\colon \langle \mathbf{SE}\rangle \to \langle E, D\rangle$ classifying the split epimorphism $\pi_1\colon E\oplus D \to E$ with section $\iota_1\colon E \to E\oplus D$. In the other, we have the **Ab**-functor $\langle E, D\rangle \to \langle \mathbf{SE}\rangle$ picking out the pair of objects E and $\ker(e\colon D \to E)$ (here, as before, this kernel can be constructed as a splitting of the idempotent $1-me$ in $\langle \mathbf{SE}\rangle$). Applying Proposition 8.1, we deduce the existence of a strong monoidal equivalence Θ_∞ fitting into a pseudocommutative square

$$\begin{array}{ccc} \langle \mathbf{SE}\rangle & \xrightarrow{\Theta} & \langle E, D\rangle \\ \downarrow & & \downarrow \\ \langle \mathrm{FI}\sharp\rangle & \xrightarrow{\Theta_\infty} & \langle \mathfrak{S}\rangle\ , \end{array}$$

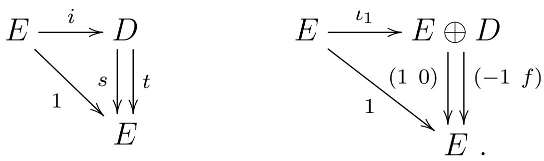
and so by composing with Θ_∞ an equivalence $[\mathfrak{S}, \mathscr{X}] \simeq [\mathrm{FI}\sharp, \mathscr{X}]$, which a direct analysis shows is precisely the equivalence of (7.12).

Moreover, the induced (symmetric) **Ab**-algebra structures on $[\mathfrak{S}, \mathscr{X}]_{\mathrm{pt}}$ and $[\mathrm{FI}\sharp, \mathscr{X}]$ when $\mathscr{X}$ is a (symmetric) **Ab**-algebra may be seen as induced by convolution with symmetric **Ab**-coalgebra structures on $\langle \mathfrak{S}\rangle$ and $\langle \mathrm{FI}\sharp\rangle$. As before, these structures make $\langle \mathfrak{S}\rangle$ and $\langle \mathrm{FI}\sharp\rangle$ into symmetric **Ab**-*bialgebras*; and as before, these symmetric bialgebra structures are in fact freely generated from pointed **Ab**-coalgebra structures on $\langle E, D\rangle$ and $\langle \mathbf{SE}\rangle$ respectively. Explicitly, we equip $\langle E, D\rangle$ with the diagonal coalgebra structure—so $\varepsilon(E) = \varepsilon(D) = \mathbb{Z}$ and $\delta(E) = E\otimes E$, $\delta(D) = D\otimes D$—and endow $\langle \mathbf{SE}\rangle$ with the coalgebra structure given on generating objects by

$$\varepsilon(E) = \mathbb{Z}\ ,\ \varepsilon(D) = 0\ ,\ \delta(E) = E\otimes E\ ,\ \delta(D) = D\otimes E \oplus E\otimes D \oplus D\otimes D$$

and on generating morphisms in the unique possible manner. It is quite straightforward to see that, with respect to these structures, Θ becomes a strong morphism of **Ab**-coalgebras, and so that, by the argument of Section 6 above, Θ_∞ is an equivalence of symmetric **Ab**-bialgebras. As in Theorem 6.3 above, the induced coalgebra structure on $\langle \mathfrak{S}\rangle$ is the diagonal one, so that the induced monoidal structure on $[\mathfrak{S}, \mathscr{X}]$ is pointwise. By uniqueness of transport of monoidal structure, it follows that the coalgebra structure on $\langle \mathrm{FI}\sharp\rangle$ must be the one which, under convolution, induces the Hurwitz product on each $[\mathrm{FI}\sharp, \mathscr{X}]$. In the situation just discussed, we may also take free *non-symmetric* monoidales; whereupon the equivalence $\Theta\colon \langle E, D\rangle \to \langle \mathbf{SE}\rangle$ of pointed **Ab**-coalgebras yields an equivalence of **Ab**-bialgebras $\langle \mathbb{N} \rangle \to \langle \mathrm{FO}\sharp \rangle$, inducing the (monoidal) equivalences of Examples 7.2(ii).

Finally, let us show how Examples 7.2(iii) may be obtained in a corresponding manner. If we consider the arrow category $\mathbf{2} = \{f : D \to E\}$ and the free category $\mathbf{G}$ on a reflexive graph, as to the left in:



As is well-known, there is an equivalence of additive Karoubian categories $\Theta : \langle \mathbf{G} \rangle \to \langle \mathbf{2} \rangle$ whose action on generators picks out the reflexive graph as right above. With each category pointed by the object E, it is easy to see that the free monoidal categories thereon are given by $(\Delta^{+}_{\mathrm{inj}})^{\mathrm{op}}$ and the cube category $\mathbb{I}$ respectively, and so we obtain an equivalence of **Ab**-algebras $\langle (\Delta^{+}_{\mathrm{inj}})^{\mathrm{op}} \rangle \simeq \langle \mathbb{I} \rangle$ inducing the equivalences of Examples 7.2(iii) above. Once again, the equivalence between $\langle \mathbf{G} \rangle$ and $\langle \mathbf{2} \rangle$ may be made into one of pointed **Ab**-coalgebras in such a way that, on passing to the associated free **Ab**-bialgebras, we reconstruct the monoidal equivalence $[\mathbb{I}, \mathscr{X}] \simeq [(\Delta^{+}_{\mathrm{inj}})^{\mathrm{op}}, \mathscr{X}]$ for any **Ab**-algebra $\mathscr{X}$.

References

[1] Marcelo Aguiar and Swapneel Mahajan, *Monoidal functors, species and Hopf algebras*, CRM Monograph Series **29** (Amer. Math. Soc., 2010). 2, 18

[2] Michael Barr, *Coequalizers and free triples*, Math. Zeitschrift **116** (1970) 307–322.

[3] Glen Earl Baxter, *An analytic problem whose solution follows from a simple algebraic identity*, Pacific Journal Math. **10** (1960) 731–742. 1

[4] Robert Blackwell, *Some existence theorems in the theory of doctrines*, (Univ. New South Wales, Ph.D. Thesis,1976).

[5] Thomas Booker and Ross Street, *Tannaka duality and convolution for duoidal categories*, Theory and Applications of Categories **28** (2013)166–205.

[6] Francis Borceux, *Handbook of categorical algebra 2*, Encyclopedia of Mathematics and its Applications **51** (Cambridge University Press, 1994). 8

[7] Pierre Cartier, *On the Structure of Free Baxter Algebras*, Advances in Math. **9** (1972) 253–265.

[8] T. Church, J. S. Ellenberg, and B. Farb, *FI-modules and stability for representations of symmetric groups*, Duke Mathematical Journal, **164(9)** (2015) 1833–1910. 2, 14, 16

[9] Sjoerd Crans, *Pasting schemes for the monoidal biclosed structure on ω-Cat*, PhD thesis, Utrecht University, 1995. 16

[10] Brian J. Day, Construction of Biclosed Categories (PhD Thesis, UNSW, 1970) `http://www.math.mq.edu.au/~street/DayPhD.pdf`.

[11] Brian J. Day, *On closed categories of functors*, Lecture Notes in Mathematics **137** (Springer-Verlag, 1970) 1–38.

[12] Brian J. Day and Ross Street, *Kan extensions along promonoidal functors*, Theory and Applications of Categories **1(4)** (1995) 72–78. 14

[13] Brian J. Day and Ross Street, *Monoidal bicategories and Hopf algebroids*, Advances in Math. **129** (1997) 99–157. 11

[14] Albrecht Dold and Dieter Puppe, *Homologie nicht-additiver Funktoren. Anwendungen*, Université de Grenoble. Annales de l'Institut Fourier. **11** (1961) 201–312. 14

[15] Eduardo Dubuc, *Free monoids*, Journal of Algebra, **29** (1974) 208–228. 9

[16] Samuel Eilenberg and G. Max Kelly, *Closed categories*, Proceedings of the Conference on Categorical Algebra (La Jolla, 1965), (Springer-Verlag,1966) 421–562.
[17] Peter J. Freyd and G. Max Kelly, *Categories of continuous functors I*, J. Pure and Applied Algebra **2** (1972) 169–191; Erratum Ibid. **4** (1974) 121. 15
[18] Marco Grandis and Robert Paré, Limits in double categories, Cahiers de Topologie et Géométrie Différentielle Catégoriques **40** (1999) 162–220.
[19] Li Guo, William Keigher and Shilong Zhang, *Baxter algebras and shuffle products*, Advances in Math. **150** (2000) 117–149.
[20] Li Guo and William Keigher, *On differential Rota-Baxter algebras*, Journal of Pure and Applied Algebra **212** (2008) 522–540. 1, 2, 4, 7
[21] Li Guo, William Keigher and Shilong Zhang, *Monads and distributive laws for Rota-Baxter and differential algebras*, arXiv:1412.8058v2. 1, 2, 6, 7
[22] Nathan Jacobson, "Basic Algebra II" (Dover Books on Mathematics, Second Edition, reprinted 2012). 6
[23] André Joyal, *Une théorie combinatoire des séries formelles*, Advances in Mathematics **42** (1981) 1–82.
[24] André Joyal, *Foncteurs analytiques et espèces de structures*, Lecture Notes in Mathematics **1234** (Springer 1986) 126–159.
[25] André Joyal and Ross Street, *Tortile Yang-Baxter operators in tensor categories*, Journal of Pure and Applied Algebra **71** (1991) 43–51.
[26] André Joyal and Ross Street, *Braided tensor categories*, Advances in Mathematics **102** (1993) 20–78.
[27] André Joyal and Ross Street, *The category of representations of the general linear groups over a finite field*, Journal of Algebra **176** (1995) 908–946.
[28] Daniel Kan, *Abstract homotopy. I*, Proceedings of the National Academy of Sciences of the United States of America, **41** (1955) 1092–1096. 16
[29] Mikhail M. Kapranov, *Analogies between the Langlands correspondence and topological quantum field theory*, Progress in Mathematics **131** (Birkhäuser Boston, Massachusetts, 1995) 119–151.
[30] G. Max Kelly, *A unified treatment of transfinite construction for free algebras, free monoids, colimits, associated sheaves, and so on*, Bulletin Austral. Math. Soc. **22** (1980) 1–83; **26** (1982) 221–237. 8
[31] G. Max Kelly, *Basic concepts of enriched category theory*, London Mathematical Society Lecture Note Series **64** (Cambridge University Press, Cambridge, 1982). 11
[32] G. Max Kelly and Ross Street, *Review of the elements of 2-categories*, Lecture Notes in Mathematics **420** (Springer-Verlag, 1974) 75–103.
[33] Stephen Lack, *The Dold-Kan correspondence and tensor products of chain complexes*, Talk at the Australian Category Seminar (Macquarie University, 27 October 2010).
[34] Stephen Lack and Ross Street, *Combinatorial categorical equivalences of Dold-Kan type*, Journal of Pure and Applied Algebra, **219(10)** (2015) 4343–4367. 2, 14, 15, 17, 18
[35] Saunders Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics **5** (Springer-Verlag, 1971).
[36] Teimuraz Pirashvili, *Dold-Kan type theorem for Γ-groups*, Mathematische Annalen, **318(2)** (2000) 277–298. 2, 17
[37] Gian-Carlo Rota, *Baxter algebras and combinatorial identities I*, Bulletin Amer. Math. Soc. **5** (1969) 325–329. 1, 4
[38] Jolanta Słomińska, *Dold-Kan type theorems and Morita equivalences of functor categories*, Journal of Algebra **274** (2004) 118–137. 2
[39] Ross Street, *Monoidal categories in, and linking, geometry and algebra*, Bulletin of the Belgian Mathematical Society – Simon Stevin **19(5)** (2012) 769–821.
[40] Ross Street, *Skew-closed categories*, Journal of Pure and Applied Algebra **217(6)** (2013) 973–988.
[41] Ross Street, *Weighted tensor products of Joyal species, graphs, and charades*, http://arxiv.org/abs/1503.02783. 1, 2, 11, 13

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