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Generalising the étale groupoid–complete pseudogroup correspondence [☆]

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ABSTRACT

We prove a generalisation of the correspondence, due to Rensede and Lawson–Lenz, between étale groupoids—which are topological groupoids whose source map is a local homeomorphisms—and complete pseudogroups—which are inverse monoids equipped with a particularly nice representation on a topological space.

Our generalisation improves on the existing functorial correspondence in four ways, and subsumes various other generalisations existing in the literature. Firstly, we enlarge the classes of maps appearing to each side. Secondly, we generalise on one side from inverse monoids to inverse categories, and on the other side, from étale groupoids to what we call *partite* étale groupoids. Thirdly, we generalise from étale groupoids to source-étale categories, and on the other side, from inverse monoids to restriction monoids. Fourthly, and most far-reachingly, we generalise from topological étale groupoids to étale groupoids internal to any *join restriction category* \mathcal{C} with local glueings; and on the other side, from complete pseudogroups to “complete \mathcal{C} -pseudogroups”, i.e., inverse monoids with a nice representation on an object of \mathcal{C} . Taken together, our results yield an equivalence, for a join restriction category \mathcal{C} with local glueings, between join restriction categories with a well-behaved functor to \mathcal{C} , and partite source-étale inter-

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nal categories in \mathcal{C} . In fact, we obtain this by cutting down a larger adjunction between arbitrary restriction categories over \mathcal{C} , and partite internal categories in \mathcal{C} .

Beyond proving this main result, numerous applications are given, which reconstruct and extend existing correspondences in the literature, and provide general formulations of completion processes.

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1. Introduction

It is a well-known and classical fact [11, Theorem II.1.2.1] that sheaves on a topological space X can be presented in two ways. On the one hand, they can be seen as functors $\mathcal{O}(X)^{\text{op}} \rightarrow \text{Set}$ defined on the poset of open subsets of X satisfying a glueing condition.

On the other hand, they can be seen as local homeomorphisms $Y \rightarrow X$ over X . The functor $\mathcal{O}(X)^{\text{op}} \rightarrow \text{Set}$ associated to a local homeomorphism $p: Y \rightarrow X$ has its value at $U \in \mathcal{O}(X)$ given by the set of partial sections of p defined on the open set U ; while the local homeomorphism $p: Y \rightarrow X$ associated to a functor $F: \mathcal{O}(X)^{\text{op}} \rightarrow \text{Set}$ is found by glueing together copies of open sets of X , taking one copy of $U \rightarrow X$ for each element of FU .

These different views on sheaves underlie a richer correspondence between *étale groupoids* and *complete pseudogroups*, whose basic idea dates back to work of Ehresmann [9] and Haefliger [15]. On the one hand, an *étale groupoid* is a topological groupoid whose source map (and hence also target map) is a local homeomorphism. On the other hand, a *complete pseudogroup* is a certain kind of *inverse monoid*, i.e., a monoid S such that for every $s \in S$ there is a unique $s^* \in S$ with $ss^*s = s$ and $s^*s^*s^* = s^*$. The key example of a complete pseudogroup is $\mathcal{J}(X)$, the monoid of partial self-homeomorphisms of a space X ; a general complete pseudogroup is an inverse monoid S “modelled on some $\mathcal{J}(X)$ ”, meaning that it is equipped with a monoid homomorphism $\theta: S \rightarrow \mathcal{J}(X)$ such that:

- (i) S has all joins of compatible families of elements, in a sense to be made precise in Section 2.4 below; we call such an S a *join inverse monoid*.
- (ii) θ restricts to an isomorphism $E(S) \rightarrow E(\mathcal{J}(X))$ on sets of idempotents; we call such a θ *hyperconnected*.

The étale groupoid–complete pseudogroup correspondence builds on the sheaf correspondence as follows. On the one hand, if \mathbb{A} is an étale groupoid, then the corresponding complete pseudogroup is of the form $\theta: \Phi(\mathbb{A}) \rightarrow \mathcal{J}(\mathbb{A}_0)$, where $\Phi(\mathbb{A})$ given by the set of *partial bisections* of \mathbb{A} , i.e., partial sections $s: U \rightarrow \mathbb{A}_1$ of the source map $\sigma: \mathbb{A}_1 \rightarrow \mathbb{A}_0$ whose composite with the target map $\tau: \mathbb{A}_1 \rightarrow \mathbb{A}_0$ is a partial bijection. Now θ is the function $s \mapsto \tau \circ s$, while the inverse monoid structure of $\Phi(\mathbb{A})$ is derived from the groupoid structure on \mathbb{A} .

On the other hand, if $\theta: S \rightarrow \mathcal{J}(X)$ is a complete pseudogroup, then the corresponding étale groupoid $\Psi(S)$ is of the form $\sigma: X \leftarrow Y \rightarrow X: \tau$, where Y is given by glueing together copies of open sets in X , taking one copy of U for each $s \in S$ with $\text{dom}(\theta(s)) = U$; where σ is the induced map from this glueing into X ; where τ acts as $\theta(s)$ on the patch corresponding to $s \in S$; and where the groupoid structure comes from the inverse monoid structure of S .

At this level of generality, the étale groupoid–pseudogroup correspondence was first established in [28, Theorem I.2.15]. However, it was not until [25] that the correspondence was made *functorial*, by equating not only the objects but also suitable classes of morphisms to each side of the correspondence. The choices of these morphism are non-obvious: between étale groupoids, [25] takes them to be the *covering functors* (also

known as *discrete opfibrations*), while between complete pseudogroups, they are the class of *callitic* morphisms.¹

There is a variant of this correspondence dealing with *localic*, rather than topological, étale groupoids, and with *localic* complete pseudogroups; this is, again, originally due again to Resende [28]. The localic correspondence is perhaps a little smoother, and has subsequently been extended in two directions. Firstly, Resende in [30] enhanced its functoriality in a manner we will describe shortly; and secondly, Kudryavtseva and Lawson in [23] extended the class of objects to each side, from inverse semigroups to “restriction semigroups”, and from étale groupoids to étale *categories*.

The goal of this article is to unify and extend the various correspondences described above: we will describe natural generalisations of the basic case [25] along four distinct axes. While the first two of these go in the same direction as [30,23], the last two are novel—as too is the possibility of doing all four simultaneously. Taken together, these extensions allow a range of practically useful applications, and shed light on what makes the construction work.

Richer functoriality. Our first axis of generalisation enlarges the classes of maps found in [25] to match those described in [30]. To the one side, the maps of complete pseudogroups we consider are much simpler: a map from $\theta: S \rightarrow \mathcal{J}(X)$ to $\chi: T \rightarrow \mathcal{J}(Y)$ is just a monoid homomorphism $\alpha: S \rightarrow T$ and a continuous function $\beta: Y \rightarrow X$, such that, for all $s \in S$, we have $\beta \circ \chi(\alpha(s)) = \theta(s) \circ \beta$ as partial homeomorphisms from Y to X . (In fact, these conditions also imply that α preserves joins; see Lemma 5.2.) The corresponding maps of étale groupoids are less obvious: rather than functors, they are the so-called *cofunctors*. A cofunctor $G \rightsquigarrow H$ between étale groupoids involves a mapping *forward* on morphisms, but a mapping *backward* on objects; see Definition 5.4 below.

Our nomenclature comes from Aguiar [1]; he traces the notion of cofunctor to Higgins and Mackenzie [18], who identified cofunctors—or as they term them, *comorphisms*—as the most general class of morphism with respect to which the passage from a Lie groupoid to its associated Lie algebroid is functorial. However, cofunctors between groupoids have also been studied in the pseudogroup literature under the name “algebraic morphisms” [2,3,30], where they have been identified as the class of maps with respect to which the process of taking groupoid bisections is functorial; the idea, in this case, seems to have originated in the work of Zakrzewski [32]. In particular, this is the nomenclature adopted by Resende [30] in establishing the extended functoriality for his correspondence between étale localic groupoids and complete localic pseudogroups; see Section 8.2.

Drop invertibility. Our three other axes of generalisation enlarge the classes of objects to each side of the correspondence. For the first of these we replace, to the one side, étale groupoids by *source-étale categories*; these are topological categories whose source map is a local homeomorphism. On the other side we replace complete pseudogroups

¹ Let us note also that the definition of “complete pseudogroup” in [25] is—besides dropping the adjective “complete”—slightly different to the above, involving a join inverse monoid S *without* an explicit representation in some $\mathcal{J}(X)$; we will return to this point later.

$\theta: S \rightarrow \mathcal{J}(X)$ by “complete pseudomonoids” $\theta: S \rightarrow \mathcal{M}(X)$, which rather than being inverse monoids, are examples of (right) *restriction monoids* [20,5]. These are monoids S endowed with an operation $(\bar{}): S \rightarrow S$ assigning to each $s \in S$ an idempotent, called a *restriction idempotent*, measuring its “domain of definition”. The key example of a complete pseudomonoid is the monoid $\mathcal{M}(X)$ of partial continuous endofunctions of a space X ; while the general example takes the form $\theta: S \rightarrow \mathcal{M}(X)$, where S is a restriction monoid with joins and θ is *hyperconnected*, meaning that it induces an isomorphism on restriction idempotents.

For the localic case, this direction of generalisation has also been explored in [23], but there are important differences. While we consider *source-étale* categories, [23] looks at the more restrictive class of *étale* categories, whose source and target maps are local homeomorphisms, and whose unit and composition maps are open; while on the pseudogroup side, they consider *two-sided*, rather than *right* restriction monoids, which are a correspondingly smaller class than ours. So even with respect to [23], our generalisation is broader in scope.

Many objects. For our next axis of generalisation, we replace, on the one side, complete pseudogroups with their many-object generalisations, “complete pseudogroupoids”. These are particular kinds of *inverse categories*, i.e., categories \mathcal{C} such that for each $s \in \mathcal{C}(x, y)$ there is a unique $s^* \in \mathcal{C}(y, x)$ satisfying $ss^*s = s$ and $s^*ss^* = s^*$. The key example of a complete pseudogroupoid is the inverse category $\mathcal{J}op_{ph}$ of topological spaces and partial homeomorphisms; the general example takes the form $P: \mathcal{A} \rightarrow \mathcal{J}op_{ph}$ where \mathcal{A} is an inverse category \mathcal{A} with joins, and P is a *hyperconnected* functor (i.e., inducing isomorphisms on sets of idempotent arrows).

On the other side, we replace étale groupoids by what we call *partite étale groupoids*. These involve a set I of “components”; for each $i \in I$, a space of objects A_i ; for each $i, j \in I$, a space of morphisms A_{ij} ; and continuous maps

$$A_i \xleftarrow{\sigma_{ij}} A_{ij} \xrightarrow{\tau_{ij}} A_j \quad \eta_i: A_i \rightarrow A_{ii} \quad \mu_{ijk}: A_{jk} \times_{A_j} A_{ij} \rightarrow A_{ik} \quad \nu_{ij}: A_{ij} \rightarrow A_{ji}$$

with each σ_{ij} a local homeomorphism, satisfying the obvious analogues of the groupoid axioms.

Arbitrary base. Our final axis of generalisation is the most far-reaching: we replace étale *topological* groupoids with étale groupoids living in some other world. By a “world”, we here mean a *join restriction category* [4,14]; these are the common generalisation of join restriction monoids and join inverse categories, and provide a purely algebraic setting for discussing notions of partiality and glueing such as arise in sheaf theory. A basic example is the category $\mathcal{J}op_p$ of topological spaces and partial continuous maps, but other important examples include $Smooth_p$ (smooth manifolds and partial smooth maps) and Sch_p (schemes and partial scheme morphisms).

In any join restriction category, we can define what it means for a map to be a *partial isomorphism* or a *local homeomorphism*, and when \mathcal{C} satisfies an additional condition of *having local glueings* (defined in Section 3.1 below), we can recreate the correspondence

between local homeomorphisms and sheaves within the \mathcal{C} -world. We can then build on this, like before, to establish the correspondence between *étale groupoids internal to \mathcal{C}* and *complete \mathcal{C} -pseudogroups*. An étale groupoid internal to \mathcal{C} is simply an internal groupoid whose source map is a local homeomorphism; while a complete \mathcal{C} -pseudogroup comprises a join inverse monoid S , an object $X \in \mathcal{C}$, and a hyperconnected map $S \rightarrow \mathbb{J}(X)$ into the join inverse monoid of all partial automorphisms of $X \in \mathcal{C}$.

The main result of this paper will arise from performing all four of the above generalisations simultaneously; we can state it as follows:

Theorem. *Let \mathcal{C} be a join restriction category with local glueings. There is an equivalence*

$$\text{peCat}_c(\mathcal{C}) \simeq \text{jrCat} // {}_h\mathcal{C} \quad (1.1)$$

between the category of source-étale partite internal categories in \mathcal{C} , with cofunctors as morphisms, and the category of join restriction categories with a hyperconnected functor to \mathcal{C} , with lax-commutative triangles as morphisms.

In fact, we will derive this theorem from a more general one. Rather than constructing the stated equivalence directly, we will construct a larger adjunction, and then cut down the objects on each side.

Theorem. *Let \mathcal{C} be a join restriction category with local glueings. There is an adjunction*

$$\text{pCat}_c(\mathcal{C}) \begin{array}{c} \xleftarrow{\Psi} \\ \perp \\ \xrightarrow{\Phi} \end{array} \text{rCat} // \mathcal{C} \quad (1.2)$$

between $\text{pCat}_c(\mathcal{C})$, the category of partite internal categories in \mathcal{C} , with cofunctors as morphisms, and $\text{rCat} // \mathcal{C}$, the category of restriction categories with a restriction functor to \mathcal{C} , with lax-commutative triangles as morphisms.

The left adjoint Ψ of this adjunction *internalises* a restriction category over the base \mathcal{C} to a source étale partite category in \mathcal{C} ; while the right adjoint Φ *externalises* a partite category in \mathcal{C} to a join restriction category sitting above \mathcal{C} via a hyperconnected join restriction functor. These two processes are evidently not inverse to each other, but constitute a so-called *Galois adjunction*; this means that applying either Φ or Ψ yields a *fixpoint*, that is, an object at which the counit $\Psi(\Phi(\mathbb{A})) \rightarrow \mathbb{A}$ or unit $P \rightarrow \Phi(\Psi(P))$, as appropriate, is invertible. These fixpoints turn out to be precisely the source-étale partite internal categories, respectively the join restriction functors hyperconnected over \mathcal{C} , and so by restricting the adjunction to these, we reconstruct our main equivalence. On the other hand, we see that the process $P \mapsto \Phi(\Psi(P))$ is a universal way of turning an arbitrary restriction functor into a hyperconnected join restriction functor; while $\mathbb{A} \mapsto \Psi(\Phi(\mathbb{A}))$ is a universal way of turning an arbitrary internal partite category into a source étale one.

We now give a more detailed overview of the contents of the paper which, beyond proving our main theorem, also develops a body of supporting theory, and gives a range of applications to problems of practical interest. We begin in Section 2 by recalling necessary background on restriction categories and inverse categories, along with a range of running examples. The basic ideas here have been developed both by semigroup theorists [20,19] and by category theorists [13,5], but we tend to follow the latter—in particular because the further developments around *joins* in restriction categories are due to this school [4, 14].

In Section 3, we develop the theory of *local homeomorphisms* in a join restriction category \mathcal{C} . This builds on a particular characterisation of local homeomorphisms of topological spaces: they are precisely the total maps $p: Y \rightarrow X$ which can be written as a join (with respect to the inclusion ordering on partial maps) of partial isomorphisms $Y \rightarrow X$. This definition carries over unchanged to any join restriction category; we show that it inherits many of the desirable properties of the classical notion so long as the join restriction category \mathcal{C} has *local glueings*—an abstract analogue of the property of being able to build local homeomorphisms over X by glueing together open subsets of X .

Section 4 exploits the preceding material to explain how the correspondence between sheaves as functors, and sheaves as local homeomorphisms, generalises to any join restriction category \mathcal{C} with local glueings. We show that, for any object $X \in \mathcal{C}$, there is an equivalence as to the left in:

$$\mathcal{C}/_{\ell h} X \xleftarrow[\Gamma]{\sim} \text{Sh}(X) \qquad \mathcal{C}/_t X \xleftarrow[\Gamma]{\perp} \mathcal{P}\text{sh}(X)$$

between local homeomorphisms over X in \mathcal{C} , and sheaves on $X \in \mathcal{C}$. Here, a *sheaf* on $X \in \mathcal{C}$ can be defined as a functor $\mathcal{O}(X)^{\text{op}} \rightarrow \text{Set}$ satisfying a glueing condition, where here $\mathcal{O}(X)$ is the complete lattice of restriction idempotents on $X \in \mathcal{C}$; in fact, we prefer to use an equivalent formulation due to Fourman and Scott [10]. Just as in the classical case (and as in our main theorem), we obtain the desired equivalence by cutting down a Galois adjunction, as to the right above, between *total* maps over X and *presheaves* on X .

In Section 5, we are finally ready to prove our main result: first constructing the Galois adjunction (1.2), and then restricting it back to the equivalence (1.1)—exploiting along the way the correspondence between local homeomorphisms and sheaves of the preceding section. Having established this theorem, the remainder of the paper is used to illustrate some of its consequences.

In Section 6, we roll back some generality by adapting our main results to the groupoid case. An *internal partite groupoid* is an internal partite category equipped with inverse operations on arrows satisfying the expected identities; and one can ask to what these correspond under the equivalence (1.1). We provide two different—albeit equivalent—answers. The first and most immediate answer is that they correspond to *étale* join restriction categories hyperconnected over the base. A join restriction category is *étale*

(see Definition 2.23) if every map is a join of partial isomorphisms. The second answer, which corresponds more closely to that provided in the theory of pseudogroups, is that they correspond to join inverse categories (see Definition 2.17) which are hyperconnected over the base. The answers are equivalent as the categories of join inverse categories and the category of étale categories are equivalent (see Corollary 2.24). Finally, by restricting the generality even further back, one obtains the correspondence between (non-partite) internal étale groupoids and one-object join inverse categories hyperconnected over the base \mathcal{C} —in other words, *complete \mathcal{C} -pseudogroups*.

When $\mathcal{C} = \mathcal{T}op_p$, this last result goes beyond the existing [25] by virtue of its richer functoriality. In Section 7, we consider how this too may be rolled back. As we have mentioned, in [25] the morphisms considered between étale topological groupoids are the so-called *covering functors*; we characterise these as the partite internal cofunctors which are bijective on components and arrows, and show that under externalization, these correspond to what we call *localic* join restriction functors (Definition 7.4)—our formulation of the *callitic* morphisms of [25]. The names *localic* and *hyperconnected* for classes of functors originate in topos theory, where they provide a fundamental factorization system on geometric morphisms [22, §A4.6]. For mere restriction categories, a (localic, hyperconnected) factorisation system was described and used in [6]; the corresponding factorisation system for *join* restriction categories is, not surprisingly, fundamental to our development here. The factorization of a join restriction functor into localic and hyperconnected parts is achieved as a direct application of our main theorem: by internalising and then externalising the given join restriction functor over its codomain, we reflect it into a *hyperconnected* join restriction functor, which is the second part of the desired (localic, hyperconnected) factorisation.

In Section 8 we turn to applications. First we consider the analogues of *Haefliger groupoids* in our setting. These originate in the observation of Ehresmann [9] that the space of germs of partial isomorphisms between any two spaces is itself a space $\Pi(A, B)$: Haefliger [16] considered smooth manifolds and the spaces $\Pi(A) = \Pi(A, A)$, which form Lie groupoids with object space A . (Haefliger's groupoid on \mathbb{R}^n is often denoted Γ^n .) Generalising this, we define the *Haefliger category* of a join restriction category \mathcal{C} with local glueings as the source-étale partite internal category $\mathcal{H}(\mathcal{C})$ in \mathcal{C} obtained by internalising the *identity* functor $\mathcal{C} \rightarrow \mathcal{C}$, and the *Haefliger groupoid* $\Pi(\mathcal{C})$ of \mathcal{C} as the internalisation of the inclusion $\text{Piso}(\mathcal{C}) \subseteq \mathcal{C}$ of the category of partial isomorphisms in \mathcal{C} . Because our Haefliger groupoid is *partite*, it includes in the topological case *all* of the $\Pi(A, B)$'s as its spaces of arrows. As an application, we use these ideas to show that the category $\text{Lh}(\mathcal{C})$ of local homeomorphisms in \mathcal{C} has binary products; this generalises a result of Selinger [31] for spaces.

In Sections 8.2 and 8.3, we relate our main results to the motivating ones in the literature. We first consider the correspondence, due to Resende [29], between étale groupoids in the category of locales $\mathcal{L}oc_p$ and what he calls *abstract complete pseudogroups*; in our nomenclature, these are simply join inverse monoids. *A priori*, our results yield a correspondence between étale localic groupoids and complete $\mathcal{L}oc_p$ -pseudogroups; however,

$\mathcal{L}oc_p$ has the special property that any join restriction category \mathcal{A} has an essentially unique hyperconnected map $\mathcal{O}: \mathcal{A} \rightarrow \mathcal{L}oc_p$ to $\mathcal{L}oc_p$ called the *fundamental functor*—and this allows us to identify complete $\mathcal{L}oc_p$ -pseudogroups with abstract complete pseudogroups, so re-finding Resende’s result.

We then turn to the Lawson–Lenz correspondence [25] for étale topological groupoids. As in the Resende correspondence, the structures to which such groupoids are related are not the expected complete $\mathcal{T}op_p$ -pseudogroups, but rather abstract complete pseudogroups; this time, however, the correspondence is only a Galois adjunction, restricting to an equivalence only for *sober étale* groupoids and *spatial* complete pseudogroups. We explain this in terms of the Galois adjunction between $\mathcal{T}op_p$ and $\mathcal{L}oc_p$, induced by the fundamental functor $\mathcal{O}: \mathcal{T}op_p \rightarrow \mathcal{L}oc_p$ and its right adjoint.

In Section 8.4, we describe how the Ehresmann–Schein–Nambooripad correspondence, as generalized by DeWolf and Pronk [8] can be explained in terms of our correspondence; and finally in Section 8.5 we exploit the power of having the adjunction (1.2), rather than merely the equivalence (1.1), to describe the construction of *full monoids* and *relative join completions*.

2. Background

2.1. Restriction categories

We begin by recalling the notion of *restriction category* which will be central to our investigations.

Definition 2.1. [5] A *restriction category* is a category \mathcal{C} equipped with an operation assigning to each map $f: A \rightarrow B$ in \mathcal{C} a map $\bar{f}: A \rightarrow A$, called its *restriction*, subject to the four axioms:

- (i) $f\bar{f} = f$ for all $f: A \rightarrow B$;
- (ii) $\bar{f}\bar{g} = \bar{g}\bar{f}$ for all $f: A \rightarrow B$ and $g: A \rightarrow C$;
- (iii) $g\bar{f} = \bar{g}\bar{f}$ for all $f: A \rightarrow B$ and $g: A \rightarrow C$;
- (iv) $\bar{g}f = f\bar{g}\bar{f}$ for all $f: A \rightarrow B$ and $g: B \rightarrow C$.

Just as an inverse monoid is an abstract monoid of partial automorphisms, so a restriction category is an abstract category of partial maps. We now illustrate this with a range of examples of restriction categories.

Examples 2.2.

- The category $\mathcal{S}et_p$ of sets and partial functions, where the restriction of a partial function $f: A \rightarrow B$ is taken as the partial function $\bar{f}: A \rightarrow A$ with $\bar{f}(a)$ taken to be a if $f(a)$ is defined, and to be undefined otherwise.

- The category Top_p of topological spaces and partial continuous maps defined on an open subset of the domain. In this example, and in most of the others which follow, the restriction is defined just as in Set_p .
- The category Pos_p of partially ordered sets and partial monotone maps defined on a down-closed subset of the domain.
- The category Loc_p of locales and partial locale maps. A *locale* is a complete lattice A satisfying the infinite distributive law $a \wedge \bigvee_i b_i = \bigvee_i a \wedge b_i$; a partial locale map $f: A \rightarrow B$ is a monotone function $f^*: B \rightarrow A$ preserving binary meets and arbitrary joins; we call f^* the *inverse image map* of f . The restriction $\bar{f}: A \rightarrow A$ of f has inverse image map $a \mapsto f^*(\top) \wedge a$.
- The category Man_p of topological manifolds and partial continuous maps defined on an open sub-manifold. Here, by a topological manifold, we simply mean a topological space which is locally homeomorphic to a Euclidean space; we do not require conditions such as being Hausdorff or paracompact.
- The category Smooth_p of smooth manifolds (again, with no further topological conditions) and partial smooth maps defined on an open sub-manifold.
- The category $\mathcal{LR}\mathsf{Top}_p$ of locally ringed topological spaces and partial continuous maps. Objects are pairs (X, \mathcal{O}_X) of a space X and a sheaf of local rings \mathcal{O}_X on X (cf. [17, Chapter II.2]); a map $(f, \varphi): (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is a map $f: X \rightarrow Y$ of Top_p and a local homomorphism $\varphi: f^*(\mathcal{O}_Y) \rightarrow \mathcal{O}_X|_{\text{dom}(f)}$ of sheaves of rings on $\text{dom}(f) \subseteq X$. The restriction of (f, φ) is (\bar{f}, id) where the first component is restriction in Top_p .
- The full subcategory Sch_p of $\mathcal{LR}\mathsf{Top}_p$ whose objects are schemes, i.e., locally ringed spaces (X, \mathcal{O}_X) for which there is an open cover $X = \bigcup_i U_i$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to an affine scheme $(\text{Spec } R, \mathcal{O}_{\text{Spec } R})$.
- The *category of partial maps* $\mathsf{Par}(\mathcal{C}, \mathcal{M})$, where \mathcal{C} is a category and \mathcal{M} a pullback-stable, composition-closed class of monics in \mathcal{C} . Its objects are those of \mathcal{C} ; maps $X \rightarrow Y$ are isomorphism-classes of spans $m: X \leftarrow X' \rightarrow Y: f$ with $m \in \mathcal{M}$; and the restriction of $[m, f]: X \rightarrow Y$ is $[m, m]: X \rightarrow X$.
- The category $\mathsf{Bun}(\mathcal{C})$ of bundles in a restriction category \mathcal{C} . Objects of $\mathsf{Bun}(\mathcal{C})$ are *total* maps (as defined below) $x: X' \rightarrow X$ in \mathcal{C} , while morphisms are commuting squares. Restriction is given componentwise: $\overline{(f, g)} = (\bar{f}, \bar{g})$.
- Any meet-semilattice M can be seen as a one-object restriction category ΣM , where maps are the elements of M , composition is given by binary meet, and every map is its own restriction.

It follows from the axioms that the operation $f \mapsto \bar{f}$ in a restriction category is idempotent, and that maps of the form \bar{f} are themselves idempotent; we call them *restriction idempotents*, and write $\mathcal{O}(A)$ for the set of restriction idempotents of $A \in \mathcal{C}$. As presaged by the last example above, $\mathcal{O}(A)$ is a meet-semilattice under the operation of composition. We call a map $f \in \mathcal{C}(A, B)$ *total* if $\bar{f} = 1_A$. Each hom-set $\mathcal{C}(A, B)$ in

a restriction category comes endowed with two relations: the *canonical partial order* \leq , and the *compatibility relation* \smile , defined by

$$f \leq g \text{ iff } f = g\overline{f} \quad \text{and} \quad f \smile g \text{ iff } f\overline{g} = g\overline{f} .$$

When $\mathcal{C} = \text{Set}_p$, the natural partial order is given by inclusion of graphs, while $f \smile g$ precisely when f and g agree on their common domain of definition.

Definition 2.3. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between restriction categories is a *restriction functor* if $F\overline{f} = \overline{Ff}$ for all $f \in \mathcal{C}(A, B)$. A restriction functor is *hyperconnected* if each of the induced functions $\mathcal{O}(A) \rightarrow \mathcal{O}(FA)$ is an isomorphism. We write rCat for the 2-category of restriction categories, restriction functors, and natural transformations with total components.

2.2. *Join restriction categories*

We now turn our attention to restriction categories in which compatible families of parallel maps can be glued together.

Definition 2.4. A restriction category \mathcal{C} is a *join restriction category* [4,14] if every pairwise-compatible family of maps in $\mathcal{C}(A, B)$ admits a join $\bigvee_i f_i$ with respect to the natural partial order, and these joins are preserved by precomposition, i.e.,

$$(\bigvee_i f_i)g = \bigvee_i (f_i g) \text{ for all } g \in \mathcal{C}(A', A). \tag{2.1}$$

The restriction to pairwise-compatible families in this definition is necessary; indeed, if the parallel maps $\{f_i\}$ have *any* upper bound h , then $f_i\overline{f_j} = h\overline{f_i} \overline{f_j} = h\overline{f_j} \overline{f_i} = f_j\overline{f_i}$ so that $f_i \smile f_j$ for each i, j .

Lemma 2.5. *In a join restriction category \mathcal{C} we have that:*

- (i) $\overline{\bigvee_i f_i} = \bigvee_i \overline{f_i}$ for all compatible families $\{f_i: A \rightarrow B\}$;
- (ii) $h(\bigvee_i f_i) = \bigvee_i h f_i$ for all compatible $\{f_i: A \rightarrow B\}$ and all $h: B \rightarrow B'$.

Proof. See [4, Proposition 2.14]. \square

Examples 2.6.

- Each of $\text{Set}_p, \text{Top}_p, \text{Pos}_p, \text{Man}_p$ and Smooth_p is a join restriction category. In Set_p , a family of partial functions $f_i: A \rightarrow B$ is pairwise-compatible just when the union of the graphs of the f_i 's is again the graph of a partial function—which is then the join $\bigvee_i f_i$. Much the same thing happens in $\text{Top}_p, \text{Pos}_p, \text{Man}_p$ and Smooth_p .

- Loc_p is a join restriction category. A family of partial locale maps $f_i: A \rightarrow B$ is compatible when $f_i^*(b) \wedge f_j^*(\top) \leq f_j^*(b)$ for all i, j , and the corresponding join has inverse image map $(\bigvee f_i)^*(b) = \bigvee_i (f_i^*b)$.
- $\mathcal{LR}\text{Top}_p$ is a join restriction category. Given a compatible family of maps $(f_i, \varphi_i): (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$, where each f_i is defined on $U_i \subseteq X$, we first glue the compatible maps f_i to a continuous map $f: X \rightarrow Y$ defined on $U = \bigcup_i U_i$. Now on U we may consider the sheaf $\text{Hom}_\ell(f^*\mathcal{O}_Y, \mathcal{O}_X|_U)$ of local homomorphisms of rings; the family of maps φ_i is a matching family of local sections, and so can be glued to a global section $\varphi: f^*\mathcal{O}_Y \rightarrow \mathcal{O}_X|_U$. The map (f, φ) so obtained is the desired join of the (f_i, φ_i) 's. It follows that the full subcategory Sch_p of schemes is also a join restriction category.
- A restriction category of the form $\text{Par}(\mathcal{C}, \mathcal{M})$ is a join restriction category precisely when \mathcal{C} admits pullback-stable unions of \mathcal{M} -subobjects which are computed as the pushout over the pairwise intersections; see [27, Theorem 11]. In particular, if \mathcal{E} is a Grothendieck topos, and \mathcal{M} comprises all the monomorphisms, then $\text{Par}(\mathcal{E}, \mathcal{M})$ is a join restriction category.
- If \mathcal{C} is a join restriction category, then so too is the restriction category of bundles $\text{Bun}(\mathcal{C})$, with joins computed pointwise.
- If M is a meet-semilattice, then the corresponding one-object restriction category ΣM is a join restriction category just when M is a locale.

In general, if A is an object of a join restriction category, then any subset of the set of restriction idempotents $\mathcal{O}(A)$ is compatible; whence each $\mathcal{O}(A)$ is a locale. The joins of this locale endow each hom-set $\mathcal{C}(A, B)$ with an “ $\mathcal{O}(A)$ -valued equality”, which will be important in what follows.

Definition 2.7. For maps $f, g: A \rightrightarrows B$ in a join restriction category \mathcal{C} we define

$$\llbracket f = g \rrbracket = \bigvee \{e \in \mathcal{O}(A) : e \leq \bar{f}\bar{g} \text{ and } fe = ge\} .$$

Lemma 2.8. For any $f, g, h: A \rightarrow B$ in a join restriction category \mathcal{C} :

- (i) If $f \smile g$ then $\llbracket f = g \rrbracket = \bar{f}\bar{g}$; in particular, if $f \leq g$ then $\llbracket f = g \rrbracket = \bar{f}$.
- (ii) $\llbracket f = g \rrbracket = \llbracket g = f \rrbracket$.
- (iii) $\llbracket g = h \rrbracket \llbracket f = g \rrbracket \leq \llbracket f = h \rrbracket$.
- (iv) $f\llbracket f = g \rrbracket = g\llbracket f = g \rrbracket$, and this is the meet $f \wedge g$ in $(\mathcal{C}(A, B), \leq)$.

(Note that the meets described in (iv) do not necessarily make \mathcal{C} into a *meet restriction category* in the sense of [14, Chapter 4]; one might call the present meet an “interior” meet which arises purely from the join structure.)

Proof. (i) and (ii) are clear. For (iii), by distributivity of joins we have

$$\llbracket g = h \rrbracket \llbracket f = g \rrbracket = \bigvee \{ ed : d \leq \overline{f\bar{g}}, e \leq \overline{g\bar{h}}, fd = gd, ge = he \},$$

so it suffices to show each ed as to the right is an element of the join defining $\llbracket f = h \rrbracket$. For this we observe that $ed \leq \overline{g\bar{h}}\overline{f\bar{g}} \leq \overline{f\bar{h}}$ and $fed = fde = gde = ged = hed$, as desired. Finally, for (iv), we have by distributivity that

$$f \llbracket f = g \rrbracket = \bigvee \{ fe : e \leq \overline{f\bar{g}}, fe = ge \} = \bigvee \{ ge : e \leq \overline{f\bar{g}}, fe = ge \} = g \llbracket f = g \rrbracket .$$

To prove this element is $f \wedge g$, observe that if $h \leq f$ and $h \leq g$ then $h = f\bar{h} = g\bar{h}$, so that $\bar{h} \leq \llbracket f = g \rrbracket$ and so $h = f\bar{h} \leq f \llbracket f = g \rrbracket$ as desired. \square

Just as before, join restriction categories assemble into a 2-category.

Definition 2.9. A *join restriction functor* between join restriction categories is a restriction functor $F: \mathcal{C} \rightarrow \mathcal{D}$ which preserves joins in that $F(\bigvee_i f_i) = \bigvee_i Ff_i$. We write jrCat for the 2-category of join restriction categories, join restriction functors and total transformations.

Lemma 2.10. A restriction functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between join restriction categories preserves joins whenever it preserves joins of restriction idempotents—in particular, whenever F is hyperconnected.

Proof. We always have $\bigvee_i Ff_i \leq F(\bigvee_i f_i)$. If F preserves joins of restriction idempotents, then $\overline{\bigvee_i Ff_i} = \bigvee_i \overline{Ff_i} = F \bigvee_i \overline{f_i} = F \bigvee_i f_i$ whence $\bigvee_i Ff_i = F(\bigvee_i f_i)$. \square

2.3. Inverse categories

In the introduction, we defined an *inverse category* to be a category \mathcal{J} such that for each $s \in \mathcal{J}(A, B)$, there is a unique $s^* \in \mathcal{J}(B, A)$ such that $ss^*s = s$ and $s^*ss^* = s^*$. By a standard argument [24, Theorem 5], it is equivalent to ask that there exists *some* s^* with these properties, and that additionally all idempotents in \mathcal{C} commute. Since inverse structure is equationally defined, it is preserved by any functor; it thus makes sense to give:

Definition 2.11. iCat is the 2-category of inverse categories, arbitrary functors, and arbitrary natural isomorphisms.

Importantly, any inverse category \mathcal{J} is a restriction category on defining the restriction of $s: A \rightarrow B$ to be $\bar{s} = s^*s: A \rightarrow A$; see [5, Theorem 2.20]. In this way, we obtain a full embedding of 2-categories $\text{iCat} \rightarrow \text{rCat}$. The image of this embedding can be characterised in terms of the following notion.

Definition 2.12. A *partial isomorphism* in a restriction category \mathcal{C} is a map $s: A \rightarrow B$ with a *partial inverse*: a map $s^*: B \rightarrow A$ with $s^*s = \bar{s}$ and $ss^* = \overline{s^*}$.

Examples 2.13.

- A partial isomorphism $s: X \rightarrow Y$ in Set_p is one whose graph describes a bijection between a subset $A \subseteq X$ and a subset $B \subseteq Y$. The same holds in $\text{Top}_p, \text{Pos}_p, \text{Man}_p$ and Smooth_p , where, for example, in Top_p this involves a bijection between *open* subsets of X and Y .
- A partial locale map $f: A \rightarrow B$ is a partial isomorphism when there exist $a \in A$ and $b \in B$ such that f^* maps $\downarrow b$ bijectively onto $\downarrow a$ and we have a factorisation

$$f^* = B \xrightarrow{(-)\wedge b} \downarrow b \xrightarrow{f^*} \downarrow a \xleftarrow{\text{include}} A.$$

- Partial isomorphisms in $\mathcal{P}\text{ar}(\mathcal{C}, \mathcal{M})$ are spans of the form $m: X \leftarrow Z \rightarrow Y: n$ for which both m and n are in \mathcal{M} .

The following lemma describes the key properties of partial isomorphisms.

Lemma 2.14.

- (i) If s^* is partial inverse to s , then $ss^*s = s$ and $s^*ss^* = s^*$.
- (ii) Partial inverses are unique when they exist.
- (iii) Partial isomorphisms are composition-closed with $(st)^* = t^*s^*$.
- (iv) The idempotent partial isomorphisms are exactly the restriction idempotents.
- (v) If $s, t: A \rightarrow B$ are partial isomorphisms, then $s \smile t$ just when st^* is a restriction idempotent.

Proof. For (i), we have $s = s\bar{s} = ss^*s$ and $s^* = s^*\overline{s^*} = s^*ss^*$. For (ii), if t and u are partial inverse to s then $t = tst = \bar{s}t = ust = u\bar{t}$. So $t \leq u$; dually $u \leq t$ and so $t = u$. For (iii), we calculate that $s^*t^*ts = s^*\bar{t}s = s^*s\bar{t}s = \bar{s}\bar{t}s = \bar{t}s$ and dually $tss^*t^* = \overline{s^*t^*}$; whence $(ts)^* = s^*t^*$ by (ii).

For (iv), each restriction idempotent is clearly its own partial inverse; conversely, if e is an idempotent partial isomorphism then $\bar{e} = e^*e = e^*ee = \bar{e}e = \bar{e}\bar{e} = e\bar{e} = e$, so that e is a restriction idempotent. Finally, for (v), if $s \smile t$ then $st^*t = ts^*s$, whence $st^* = st^*tt^* = ts^*st^* = \overline{st^*}$ is a restriction idempotent. Conversely, if st^* is a restriction idempotent, then it is its own partial inverse, so $st^* = ts^*$ and

$$s\bar{t} = st^*t = st^*st^*t = ts^*st^*t = t\bar{s}\bar{t} = t\bar{s}. \quad \square$$

Using this, we thus obtain the following, which is again [5, Theorem 2.20]:

Proposition 2.15. *A restriction category \mathcal{C} is in the essential image of $\text{iCat} \rightarrow \text{rCat}$ just when every morphism in \mathcal{C} is a partial isomorphism.*

We are thus justified in confusing an inverse category with the corresponding restriction category. We also obtain Proposition 2.24 of [5].

Corollary 2.16. *The full inclusion 2-functor $\text{iCat} \rightarrow \text{rCat}$ has a right 2-adjoint $\text{PIso}: \text{rCat} \rightarrow \text{iCat}$ sending each restriction category \mathcal{C} to its subcategory $\text{PIso}(\mathcal{C})$ comprising all objects and all partial isomorphisms between them.*

For example, $\text{PIso}(\text{Set}_p)$ is the inverse category $\mathcal{P}\text{Inj}$ of sets and partial isomorphisms, $\text{PIso}(\text{Top}_p)$ is the category of spaces and partial homeomorphisms, and so on.

2.4. Join inverse categories

Joins in inverse categories are slightly different from joins in restriction categories. We say that parallel maps $s, t: A \rightarrow B$ in an inverse category are *bicompatible*, written $s \asymp t$, if both $s \smile t$ and $s^* \smile t^*$.

Definition 2.17. An inverse category \mathcal{J} is a *join inverse category* if every pairwise-bicompatible family of maps $\{s_i\} \subseteq \mathcal{J}(A, B)$ admits a join $\bigvee_i s_i$ with respect to the natural partial order, and these joins are preserved by precomposition. We write jiCat for the 2-category of join inverse categories, functors preserving joins, and natural isomorphisms.

This time, it is pairwise-bicompatibility that is a necessary condition for a join to exist in an inverse category. The gap between compatibility and bicompatibility complicates the relation between join inverse and join restriction categories. The more straightforward direction is from join restriction to join inverse.

Lemma 2.18. *Let $\{s_i\}$ be a compatible family of partial isomorphisms in a join restriction category. The join $\bigvee_i s_i$ is a partial isomorphism if and only if $\{s_i\}$ is a bicompatible family.*

Proof. If $\{s_i\}$ is bicompatible, then $\{s_j^*\}$ is compatible, and so $\bigvee_j s_j^*$ exists. Now compatibility of $\{s_i\}$ shows by Lemma 2.14(v) that each $s_i s_j^*$ is a restriction idempotent, whence $s_i s_j^* = \overline{s_i s_j^*} \leq \overline{s_j^*} = s_j s_j^*$. This gives the starred equality in

$$(\bigvee_i s_i)(\bigvee_j s_j^*) = \bigvee_{i,j} s_i s_j^* \stackrel{(*)}{=} \bigvee_j s_j s_j^* = \bigvee_j \overline{s_j^*} = \overline{\bigvee_j s_j^*}.$$

This, together with the dual calculation (swapping the s_i 's and s_i^* 's) shows that $\bigvee_i s_i$ and $\bigvee_j s_j^*$ are partial inverse. Suppose conversely that $s = \bigvee_i s_i$ is a partial isomorphism.

Since $s_i \leq s$ for each i , also $s_i^* \leq s^*$ by Lemma 2.14(iii), and so both $\{s_i\}$ and $\{s_i^*\}$ are pairwise-compatible, i.e., $\{s_i\}$ is pairwise-bicompatible. \square

It follows from this that:

Proposition 2.19. *If \mathcal{C} is a join restriction category, then $\text{PIso}(\mathcal{C})$ is a join inverse category; this assignment yields a 2-functor $\text{PIso}: \text{jrCat} \rightarrow \text{jiCat}$.*

In the other direction, however, a join inverse category *qua* restriction category is *not* typically a join restriction category, since it has only joins of bicompatible families and not compatible ones. However, we can freely adjoin these:

Definition 2.20. Let \mathcal{J} be a join inverse category. Its *join restriction completion* $\text{jr}(\mathcal{J})$ has the same objects as \mathcal{J} , while a morphism $A \rightarrow B$ is a compatible subset $S \subseteq \mathcal{J}(A, B)$ which is closed downwards and under joins of bicompatible families. The identity on A is the downset $\downarrow 1_A$; the composite of $S: A \rightarrow B$ and $T: B \rightarrow C$ is the closure of $\{ts : t \in T, s \in S\}$ downwards and under bicompatible joins; and the restriction $\bar{S}: A \rightarrow A$ is the downset of $\bigvee_{s \in S} \bar{s}$.

Proposition 2.21. *$\text{jr}(\mathcal{J})$ is a join restriction category for any join inverse category \mathcal{J} . It provides the value at \mathcal{J} of a left 2-adjoint to $\text{PIso}: \text{jrCat} \rightarrow \text{jiCat}$.*

Proof. See [14, Theorem 3.1.33]. \square

The unit of the 2-adjunction $\text{jr} \dashv \text{PIso}$ at $\mathcal{J} \in \text{jiCat}$ is the join-preserving functor $\eta_{\mathcal{J}}: \mathcal{J} \rightarrow \text{PIso}(\text{jr}(\mathcal{J}))$ sending f to $\downarrow f$; while the counit at $\mathcal{C} \in \text{jrCat}$ is the join restriction functor $\varepsilon_{\mathcal{C}}: \text{jr}(\text{PIso}(\mathcal{C})) \rightarrow \mathcal{C}$ sending $S: A \rightarrow B$ to $\bigvee S: A \rightarrow B$. What was not observed in [14] is that:

Proposition 2.22. *The unit η of the 2-adjunction $\text{jr} \dashv \text{PIso}$ is an isomorphism. Each counit component $\varepsilon_{\mathcal{C}}$ is bijective on objects and faithful, and is full precisely when each map in \mathcal{C} is a join of partial isomorphisms.*

Proof. Clearly each $\eta_{\mathcal{J}}: \mathcal{J} \rightarrow \text{PIso}(\text{jr}(\mathcal{J}))$ is bijective on objects and faithful. To show it is full, consider a map $S: A \rightarrow B$ in $\text{jr}(\mathcal{J})$ which is a partial isomorphism. Note that $S = \bigvee_{s \in S} \eta_{\mathcal{J}}(s)$ in $\text{jr}(\mathcal{J})$, so that by Proposition 3.12, $\{\eta_{\mathcal{J}}(s) : s \in S\}$ is a bicompatible family. By fidelity of $\eta_{\mathcal{J}}$, also S is a bicompatible family in \mathcal{J} so that $\bigvee S$ exists. But now $\eta_{\mathcal{J}}(\bigvee S) = \bigvee_{s \in S} \eta_{\mathcal{J}}(s) = S$ so that $\eta_{\mathcal{J}}$ is full.

Turning now to $\varepsilon_{\mathcal{C}}: \text{jr}(\text{PIso}(\mathcal{C})) \rightarrow \mathcal{C}$, clearly it is bijective on objects. For fidelity, we claim that any $S: A \rightarrow B$ in $\text{jr}(\text{PIso}(\mathcal{C}))$ is determined by $\varepsilon_{\mathcal{C}}(S)$ as $S = \{t \in \mathcal{C}(A, B) : t \text{ a partial isomorphism and } t \leq \varepsilon_{\mathcal{C}}(S)\}$. For the non-trivial inclusion, suppose that $t \leq \varepsilon_{\mathcal{C}}(S) = \bigvee S$ is a partial isomorphism; then we have $t = (\bigvee S)\bar{t} = \bigvee_{s \in S} s\bar{t}$. The family $\{s\bar{t} : s \in S\}$ is contained in S , since S is down-closed; it is also bicompatible, since

it is bounded above by the partial isomorphism t . Since S is closed under bicompatible joins, it follows that $t = \bigvee_s s\bar{t} \in S$ as required.

Finally, we consider when $\varepsilon_{\mathcal{C}}$ is full. If $p = \bigvee_i p_i: A \rightarrow B$ is any join of partial isomorphisms in \mathcal{C} , then on taking $S \subseteq \mathcal{C}(A, B)$ to be the closure of $\{p_i\}$ downwards and under bicompatible joins, we have $p = \bigvee_i p_i = \bigvee S = \varepsilon_{\mathcal{C}}(S)$. So the maps in the image of $\varepsilon_{\mathcal{C}}$ are precisely the joins of partial isomorphisms, and $\varepsilon_{\mathcal{C}}$ is full just when every map in \mathcal{C} is of this form. \square

This leads us to make:

Definition 2.23. An *étale* map in a join restriction category \mathcal{C} is a join of partial isomorphisms. We call \mathcal{C} an *étale join restriction category* if all its maps are étale.

And we thus obtain:

Corollary 2.24. *The 2-functor $\text{jr}: \text{jiCat} \rightarrow \text{jrCat}$ is a full coreflective embedding of 2-categories; its essential image comprises the étale join restriction categories. The induced coreflector $\text{jr} \circ \text{PIso}$ on jrCat sends a join restriction category to its subcategory $\text{Et}(\mathcal{C})$ of étale maps.*

3. The theory of local homeomorphisms

The previous result highlights the importance of étale maps in join restriction categories. Of particular relevance to us will be the total étale maps, which we call *local homeomorphisms*.

Examples 3.1.

- Any map in Set_p is a local homeomorphism.
- The local homeomorphisms in Top_p or Man_p are the local homeomorphisms in the usual sense, while in Smooth_p they are the local diffeomorphisms.
- A total map $f: P \rightarrow Q$ in Pos_p is a local homeomorphism if and only if it is a *discrete fibration*: that is, for any $p \in P$ and $q \leq f(p)$ in Q , there exists a unique $p' \leq p$ in P with $f(p') = q$.
- A total map $f: L \rightarrow M$ in Loc_p is a local homeomorphism just when there is a decomposition $\top_L = \bigvee_i \varphi_i$ and corresponding elements $\psi_i \in M$ such that for each i , the map $m \mapsto f^*(m) \wedge \varphi_i$ gives a bijection $\downarrow \psi_i \rightarrow \downarrow \varphi_i$.
- A total map $(f, \varphi): (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ in $\mathcal{LR}\text{Top}_p$ is a local homeomorphism just when $f: X \rightarrow Y$ is a local homeomorphism of spaces and φ is invertible; and similarly in the full subcategory of schemes (such maps are usually called *local isomorphisms* of schemes).

- In a join restriction category $\mathcal{P}ar(\mathcal{C}, \mathcal{M})$, the total map specified by $f: X \rightarrow Y$ in \mathcal{C} is a local homeomorphism just when there is a jointly surjective family of \mathcal{M} -maps $(m_i: X_i \rightarrow X)_{i \in I}$ such that each $f m_i: X_i \rightarrow Y$ is in \mathcal{M} .
- If \mathcal{C} is a join restriction category, then a local homeomorphism in $\mathcal{B}un(\mathcal{C})$ over an object $x: X' \rightarrow X$ can be shown to be a *pullback square*

$$\begin{array}{ccc}
 A' & \xrightarrow{a} & A \\
 p' \downarrow & & \downarrow p \\
 X' & \xrightarrow{x} & X
 \end{array} \tag{3.1}$$

where both p and p' are local homeomorphisms.

In this section, we redevelop the classical theory of local homeomorphisms between spaces in the context of a general join restriction category \mathcal{C} .

3.1. Local atlases and local glueings

If $A \rightarrow X$ is a local homeomorphism in a join restriction category, then we may think of A as being built by patching together local fragments of X . The following structure specifies the data for such a patching; in the nomenclature of [10], it would be called an $\mathcal{O}(X)$ -set.

Definition 3.2. A *local atlas* on an object X of a join restriction category comprises a set I and a matrix of restriction idempotents $(\varphi_{ij} \in \mathcal{O}(X) : i, j \in I)$ such that

$$\varphi_{ij} = \varphi_{ji} \quad \text{and} \quad \varphi_{jk} \varphi_{ij} \leq \varphi_{ik}.$$

The manner in which a local homeomorphism induces a local atlas is given by:

Lemma 3.3. Let $p = \bigvee_i p_i: A \rightarrow X$ be a local homeomorphism, and let each p_i have the partial inverse $s_i: X \rightarrow A$. The family $\varphi_{ij} = p_j s_i$ is a local atlas on X .

Proof. By compatibility of the p_i 's and Lemma 2.14(v), each $p_j s_i$ is a restriction idempotent. Since a restriction idempotent is its own restriction inverse, it follows that $p_j s_i = p_i s_j$. Finally, we calculate that $p_k s_j p_j s_i = p_k \overline{p_j} s_i \leq p_k s_i$. \square

In fact, the local atlas associated to a local homeomorphism p determines it up to unique isomorphism over X . This follows *a fortiori* from the following result, which states that maps out of the domain of a local homeomorphism $A \rightarrow X$ are determined by their action on each of the patches of X from which A is built. (In fact, maps into A are also so determined, but we will not need this).

Lemma 3.4. *Let $p = \bigvee_i p_i : A \rightarrow X$ be a local homeomorphism where each p_i has partial inverse s_i . Let $\varphi_{ij} = p_j s_i$ be the associated local atlas. Precomposition with the s_i 's induces a bijection between maps $f : A \rightarrow B$ and families of maps $(f_i : X \rightarrow B)_{i \in I}$ such that*

$$f_i \varphi_{ii} = f_i \quad \text{and} \quad f_j \varphi_{ij} \leq f_i \quad \text{for all } i, j \in I. \tag{3.2}$$

Under this bijection, f is total if and only if $\overline{f}_i = \varphi_{ii}$ for each $i \in I$.

Proof. Since $s_i \varphi_{ii} = s_i p_i s_i = s_i$ and $s_j \varphi_{ij} = s_j p_j s_i = \overline{p_j} s_i \leq s_i$, any family of maps $(f s_i)$ satisfies (3.2). Conversely, for a family (f_i) satisfying (3.2), we have $f_j p_j \overline{f_i p_i} \leq f_j p_j \overline{p_i} = f_j p_j s_i p_i = f_j \varphi_{ij} p_i \leq f_i p_i$ so that the family $(f_i p_i : A \rightarrow B)$ is compatible. Let $f = \bigvee_i f_i p_i$ and note $f s_i = \bigvee_j f_j p_j s_i = f_i \varphi_{ii} \vee \bigvee_{j \neq i} f_j \varphi_{ij} = f_i$, where the last equality uses both clauses in (3.2). On the other hand, if $g : A \rightarrow B$ satisfies $g s_i = f_i$ for all i , then $g = \bigvee_i g \overline{p_i} = \bigvee_i g s_i p_i = \bigvee_i f_i s_i p_i = f$. Finally, if f is total, then $\overline{f s_i} = \overline{s_i} = p_i s_i = \varphi_{ii}$. Conversely, if $\overline{f}_i = \varphi_{ii}$ for each i , then $\overline{f} = \bigvee_i \overline{f_i p_i} = \bigvee_i \overline{f_i} \overline{p_i} = \bigvee_i \varphi_{ii} \overline{p_i} = \bigvee_i \overline{p_i s_i p_i} = \bigvee_i \overline{p_i} = 1_A$. \square

The preceding result implies that local homeomorphisms are completely determined by their local atlases. However, some local atlases in \mathcal{C} may not be induced by any local homeomorphism; the following definition rectifies this.

Definition 3.5. A join restriction category \mathcal{C} has *local glueings* if every local atlas φ on every $X \in \mathcal{C}$ is induced by some local homeomorphism $p : A \rightarrow X$. We call p (or sometimes merely the object A) a *glueing* of the local atlas φ .

Example 3.6. The join restriction category $\mathcal{J}op_p$ has local glueings. For any $X \in \mathcal{J}op_p$, we can identify $\mathcal{O}(X)$ with the lattice of open subsets of X . Thus an I -object local atlas on a space X comprises a family of open subsets $(U_{ij})_{i,j \in I}$ satisfying $U_{ij} = U_{ji}$ and $U_{jk} \cap U_{ij} \subseteq U_{ik}$ for all $i, j, k \in I$.

In this situation, for each $x \in X$ we may define on the set $\{i \in I : x \in U_{ii}\}$ an equivalence relation \sim_x with $i \sim_x j$ just when $x \in U_{ij}$. If $i \in I$ with $x \in U_{ii}$, then we write i_x for its \sim_x -equivalence class, and call it the *germ of i at x* . We write U_x for the set of all germs at x . Now the glueing $p : A \rightarrow X$ of the given local atlas has $A := \sum_{x \in X} U_x$ the disjoint union of the sets of U -germs, with topology generated by the basic open sets

$$\langle i, V \rangle = \{i_x : x \in V\} \quad \text{for all } i \in I \text{ and open } V \subseteq U_{ii}.$$

The projection $p : A \rightarrow X$ sends i_x to x ; it is a local homeomorphism since $p = \bigvee_i p_i$, where $p_i : A \rightarrow X$ is the partial isomorphism obtained by restricting p to the open set $\langle i, U_{ii} \rangle$, with partial inverse $s_i : X \rightarrow A$ given by $x \mapsto i_x$ on the open set U_{ii} . For any $i, j \in I$, the partial identity $p_j s_i : X \rightarrow X$ is defined at x just when i_x and j_x are defined and equal—that is, just when $x \in U_{ij}$. So $p : A \rightarrow X$ induces the local atlas U , as desired.

Example 3.7. In the situation of the previous example, it is easy to see that the glueing of a local atlas on a discrete space is again discrete. Therefore Set_p , identified with the full sub-restriction category of Top_p on the discrete spaces, also admits local glueings. Similarly, the glueing of a local atlas on an *Alexandroff* space is again Alexandroff, so that Pos_p , identified with the full subcategory of Top_p on the Alexandroff spaces, has all local glueings.

In a similar manner, the glueing of a local atlas on a topological manifold is again a topological manifold: indeed, if X is locally homeomorphic to a Euclidean space, and $p: A \rightarrow X$ is a local homeomorphism, then A is again locally homeomorphic to Euclidean space. So Man_p has local glueings. The property of smoothness is easily seen to be preserved in this situation so that Smooth_p also has local glueings.

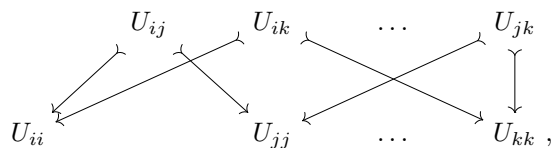
Example 3.8. The join restriction category Loc_p has local glueings. In this case, for any $X \in \text{Loc}_p$ we can identify $\mathcal{O}(X)$ with X itself, so that an I -object atlas comprises a family of elements $(\varphi_{ij} \in X)$ such that $\varphi_{ij} = \varphi_{ji}$ and $\varphi_{jk} \wedge \varphi_{ij} \leq \varphi_{ik}$. The corresponding glueing is the locale

$$A = \{(\theta_i \in X : i \in I) \mid \theta_i \leq \varphi_{ii} \text{ and } \theta_j \wedge \varphi_{ij} \leq \theta_i \text{ for all } i, j \in I\}$$

under the pointwise ordering. Joins and binary meets are computed pointwise, while the top element is $(\varphi_{ii} : i \in I)$. The total projection map $p: A \rightarrow X$ is defined by $p^*(\psi) = (\psi \wedge \varphi_{ii} : i \in I)$, and this is the join of the partial isomorphisms $p_i: A \rightarrow X$ defined by $p_i^*(\psi) = (\psi \wedge \varphi_{ij} : j \in I)$, with corresponding partial inverses $s_i: X \rightarrow A$ given by $s_i^*(\theta) = \theta_i$. Since $s_j p_i: X \rightarrow X$ is the partial identity $\psi \mapsto \psi \wedge \varphi_{ij}$, we see that $p: A \rightarrow X$ induces the local atlas φ , as desired.

Example 3.9. The join restriction category $\mathcal{LR}\text{Top}_p$ has local glueings. In this case, an atlas on (X, \mathcal{O}_X) is simply an atlas on X , for which we have the associated glueing $p: A \rightarrow X$ in Top_p ; the corresponding glueing in $\mathcal{LR}\text{Top}_p$ can be taken to be $(p, \text{id}): (A, p^*\mathcal{O}_X) \rightarrow (X, \mathcal{O}_X)$. Just as in the case of manifolds, the glueing of any local atlas on a scheme will itself be a scheme, so that Sch_p also has all local glueings.

Example 3.10. In a join restriction category of the form $\mathcal{P}\text{ar}(\mathcal{C}, \mathcal{M})$, an I -object local atlas on the object X can be identified with a family of \mathcal{M} -subobjects $(\varphi_{ij}: U_{ij} \rightarrow X)$ in \mathcal{C} subject to the usual axioms. In this case, a local glueing can be obtained as the colimit A of a diagram of the form



whenever this colimit exists. The local homeomorphism $p: A \rightarrow X$ is induced by the cocone of maps $\varphi_{ij}: U_{ij} \rightarrow X$ under this diagram.

Example 3.11. If \mathcal{C} is a join restriction category with local glueings, then so too is $\text{Bun}(\mathcal{C})$, with glueings computed componentwise. Indeed, a restriction idempotent on $x: X' \rightarrow X$ in $\text{Bun}(\mathcal{C})$ is a map of the form (\overline{ex}, e) for some $e \in \mathcal{O}(X)$; it follows that an I -object local atlas on x comprises I -object atlases φ on X and φ' on X' related by $\varphi'_{ij} = \overline{\varphi_{ij}x}$. If $p': A' \rightarrow X'$ and $p: A \rightarrow X$ are glueings for these two atlases, then by applying the argument of Section 3.3 below, we obtain a pullback square of the required form (3.1).

As these examples show, many join restriction categories of practical interest already admit local glueings. We now show that, if we do not have local glueings, then we can always adjoin them in a straightforward manner. This is a special case of Grandis' *manifold completion* [13], and so we only sketch the details.

Proposition 3.12. *The inclusion of the full sub-2-category $\text{jrCat}_{lg} \rightarrow \text{jrCat}$ of join restriction categories with local glueings has a left biadjoint Gl ; each unit component $\mathcal{C} \rightarrow \text{Gl}(\mathcal{C})$ of the biadjunction is injective on objects and fully faithful, and is an equivalence whenever \mathcal{C} has local glueings.*

Proof (sketch). $\text{Gl}(\mathcal{C})$ has as objects, pairs (X, φ) where $X \in \mathcal{C}$ and φ is a local atlas on X , and as maps $(X, \varphi) \rightarrow (Y, \psi)$, families of maps $(f_{ik}: X \rightarrow Y)$ in \mathcal{C} with

$$\psi_{k\ell}f_{ik} = f_{i\ell}\overline{f_{ik}} \quad \text{and} \quad f_{ik}\varphi_{ii} = f_{ik} \quad \text{and} \quad f_{jk}\varphi_{ij} \leq f_{ik} .$$

The identity on (X, φ) has components $\varphi_{ij}: X \rightarrow X$; the composite of maps $f: (X, \varphi) \rightarrow (Y, \psi)$ and $g: (Y, \psi) \rightarrow (Z, \rho)$ has

$$(gf)_{im} := (\bigvee_k g_{km}f_{ik})_{i \in I, m \in M} ;$$

while the restriction of $f: (X, \varphi) \rightarrow (Y, \psi)$ is $(\overline{f})_{ij} = \varphi_{ij}(\bigvee_k \overline{f_{ik}})$.

With extensive verification, we can check that these data are well-defined, and make $\text{Gl}(\mathcal{C})$ into a join restriction category. To show it has local glueings, consider a K -object atlas $(\psi_{k\ell})$ on (X, φ) ; this is given by a family of restriction idempotents $(\psi_{k\ell})_{ij}$ on X satisfying various conditions. From these data, we obtain a new $I \times K$ -object atlas on X with $\gamma_{(i,k),(j,\ell)} = (\psi_{k\ell})_{ij}$, and obtain for each $\ell \in K$ a partial isomorphism $p_\ell: (X, \gamma) \rightarrow (X, \varphi)$ in $\text{Gl}(\mathcal{C})$ with components $(p_\ell)_{(i,k),j} = (\psi_{k\ell})_{ij}$. Taking joins yields a local homeomorphism $p = \bigvee_\ell p_\ell: (X, \gamma) \rightarrow (X, \varphi)$ with associated local atlas ψ , as desired.

This shows $\text{Gl}(\mathcal{C}) \in \text{jrCat}_{lg}$. There is a fully faithful join restriction functor $\eta_{\mathcal{C}}: \mathcal{C} \rightarrow \text{Gl}(\mathcal{C})$ with $\eta_{\mathcal{C}}(X) = (X, \{1_X\})$, which we claim exhibits $\text{Gl}(\mathcal{C})$ as a bireflection of \mathcal{C} into jrCat_{lg} . This means showing that, for each join restriction category \mathcal{D} with local glueings, the functor

$$(-) \circ \eta_{\mathcal{C}} : \text{jrCat}(\mathfrak{Gl}(\mathcal{C}), \mathcal{D}) \rightarrow \text{jrCat}(\mathcal{C}, \mathcal{D}) \tag{3.3}$$

is an equivalence of categories. A suitable pseudoinverse functor takes a join restriction functor $F : \mathcal{C} \rightarrow \mathcal{D}$ to the join restriction functor $\tilde{F} : \mathfrak{Gl}(\mathcal{C}) \rightarrow \mathcal{D}$ whose value at (X, φ) is some chosen glueing of the atlas $F\varphi$ in \mathcal{D} . Finally, if \mathcal{C} itself has glueings, then the extension $\widetilde{\text{id}}_{\mathcal{C}} : \mathfrak{Gl}(\mathcal{C}) \rightarrow \mathcal{C}$ provides the desired pseudo-inverse to the functor $\eta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathfrak{Gl}(\mathcal{C})$. \square

3.2. Partial sections

Classically, a local homeomorphism $A \rightarrow X$ can be analysed in terms of its “partial sections” $U \rightarrow A$ defined on some open subset $U \subseteq X$. In this section, we reconstruct this analysis in the join restriction context.

Definition 3.13. A *partial section* of a map $p : A \rightarrow X$ in a restriction category is a map $s : X \rightarrow A$ such that $ps = \bar{s}$.

Clearly, any partial section satisfies $ps \leq 1$; conversely, if $p : A \rightarrow X$ is a total map, then any s with $ps \leq 1$ will satisfy $ps = \overline{ps} = \bar{s}$ by totality of p .

Lemma 3.14. Let $p = \bigvee_i p_i : A \rightarrow X$ be a local homeomorphism, and let each p_i have partial inverse s_i .

- (i) Each $s_i : X \rightarrow A$ is a partial section of p .
- (ii) If s is a partial section of p , then $\llbracket s = s_i \rrbracket = p_i s$; in particular, $\llbracket s_i = s_j \rrbracket = p_j s_i$.
- (iii) Any partial section s of p can be written as a join

$$s = \bigvee_i s_i \llbracket s = s_i \rrbracket . \tag{3.4}$$

- (iv) Each partial section s of p is a partial isomorphism with $s^* \leq p$.

Proof. For (i), we have $ps_i = \bigvee_j p_j s_i = p_i s_i \vee \bigvee_{j \neq i} p_j s_i = p_i s_i \vee \bigvee_{j \neq i} p_j s_i \varphi_{ii} = p_i s_i = \bar{s}_i$. For (ii), note that $p_i s \leq ps = \bar{s}$ is a restriction idempotent; and as $\overline{s_i} p_i s = p_i s_i p_i s = p_i s$, we have also $p_i s \leq \overline{s}_i$. Since $s_i p_i s = \overline{p_i} s = \overline{sp_i s} = sp_i s$, we thus have $p_i s \leq \llbracket s = s_i \rrbracket$. On the other hand, by Lemma 2.8(iv), we have $p_i s \llbracket s = s_i \rrbracket = p_i s_i \llbracket s = s_i \rrbracket = \overline{s}_i \llbracket s = s_i \rrbracket = \llbracket s = s_i \rrbracket$ so that $\llbracket s = s_i \rrbracket \leq p_i s$.

For (iii) we have $\bigvee_i s_i \llbracket s = s_i \rrbracket \leq s$ again by Lemma 2.8(iv); but since by (ii) $\overline{\bigvee_i s_i \llbracket s = s_i \rrbracket} = \bigvee_i p_i s = ps = \bar{s}$ we have $s = \bigvee_i s_i \llbracket s = s_i \rrbracket$. Finally, for (iv), note that the compatible family of partial isomorphisms $\{s_i \llbracket s = s_i \rrbracket\}$ in (3.4) has family of partial inverses $\{\llbracket s = s_i \rrbracket p_i\}$ —and this is another compatible family since $\{p_i\}$ is so. So $\{s_i \llbracket s = s_i \rrbracket\}$ is bicompatible, whence by Lemma 2.18 its join s is a partial isomorphism with $s^* = \bigvee_i \llbracket s = s_i \rrbracket p_i \leq p$. \square

The following results explore the property in part (iii) of this lemma further.

Lemma 3.15. *Let $p: A \rightarrow X$ be a local homeomorphism, and let $(s_i: X \rightarrow A)_{i \in I}$ be a family of partial sections of p . The following are equivalent:*

- (i) $p = \bigvee_i s_i^*$;
- (ii) Any partial section s of p has the form $s = \bigvee_i s_i \llbracket s = s_i \rrbracket$;
- (iii) Any partial section s of p has the form $s = \bigvee_i s_i \theta_i$ for suitable $\theta_i \in \mathcal{O}(A)$;
- (iv) The s_i 's are jointly epimorphic.

Proof. We have (i) \Rightarrow (ii) by Lemma 3.14(iii) and clearly (ii) \Rightarrow (iii). For (iii) \Rightarrow (i), note each $s_i^* \leq p$ by Lemma 3.14(iv), whence $\bigvee_i s_i^*$ exists and is $\leq p$; it remains to show $p \leq \bigvee_i s_i^*$. Write $p = \bigvee_j p_j$ as a join of partial isomorphisms. Using (ii), we can write each p_j^* as a join $\bigvee_i s_i \theta_{ij}$ for suitable $\theta_{ij} \in \mathcal{O}(X)$; taking partial inverses, we have $p_j = \bigvee_i \theta_{ij} s_i^*$ and so $p = \bigvee_{i,j} \theta_{ij} s_i^* \leq \bigvee_i s_i^*$, as desired.

Now (i) \Rightarrow (iv) by Lemma 3.4, and it remains to show (iv) \Rightarrow (i). For all $i, j \in I$ we have $s_j^* s_i \leq p s_i = \overline{s_i}$ and $s_i^* s_i = \overline{s_i}$, and so $(\bigvee_i s_i^*) s_j = \overline{s_j}$ for each j ; since also $p s_j = \overline{s_j}$, we must have $\bigvee_i s_i^* = p$ since the s_j 's are jointly epimorphic. \square

In the sequel, we will call any family of partial sections $\{s_i\}$ as in the preceding Lemma a *basis* for the local homeomorphism p . The following result gives a precise formulation of what we mean by “suitable” θ_i 's in (iii) above; in the terminology of [10], these families would be called *singletons*.

Lemma 3.16. *Let $\{s_i\}$ be a basis for the local homeomorphism $p: A \rightarrow X$. The assignation $s \mapsto (\llbracket s = s_i \rrbracket : i \in I)$ sets up a bijection between partial sections of p , and families of restriction idempotents $(\theta_i \in \mathcal{O}(X) : i \in I)$ satisfying*

$$\theta_j \theta_i \leq \llbracket s_i = s_j \rrbracket \quad \text{and} \quad \theta_j \llbracket s_i = s_j \rrbracket \leq \theta_i \quad \text{for all } i, j \in I . \tag{3.5}$$

Proof. If s is a partial section of p , then by Lemma 2.8 the family $\theta_i = \llbracket s = s_i \rrbracket$ satisfies (3.5), and the assignation $s \mapsto (\llbracket s = s_i \rrbracket : i \in I)$ is injective by Lemma 3.14(iii). It remains to show surjectivity. Given a family (θ_i) satisfying (3.5), the first condition implies that $\theta_i \leq \overline{s_i}$ and so that

$$s_j \theta_j \overline{s_i \theta_i} = s_j \theta_j \theta_i \leq s_j \llbracket s_i = s_j \rrbracket \leq s_i .$$

Thus the family $\{s_i \theta_i\}$ is compatible. Let s be its join; by Lemma 3.14(ii) and the second condition in (3.5) we have $\llbracket s = s_i \rrbracket = p_i s = \bigvee_j p_i s_j \theta_j = \bigvee_j \llbracket s_j = s_i \rrbracket \theta_j = \theta_i$, so that $(\theta_i : i \in I)$ is in the image of $s \mapsto (\llbracket s = s_i \rrbracket : i \in I)$ as required. \square

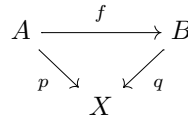
3.3. Composition and pullback

We will later need some understanding of how to compose and pull back local homeomorphisms in a join restriction category. The astute reader will notice that here we should be talking about *restriction* pullbacks and limits, as described in [7]. However, restriction limits behave on total maps exactly as ordinary limits behave and almost all the limits we will actually need use total maps, so we have elided this elaboration in order to simplify the exposition.

We first consider these operations on partial sections.

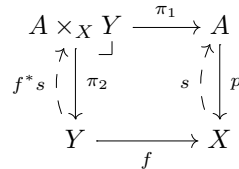
Lemma 3.17. *Let \mathcal{C} be a restriction category.*

- (i) *If $s: X \rightarrow A$ and $t: A \rightarrow A'$ are partial sections of the total maps $p: A \rightarrow X$ and $q: A' \rightarrow A$ in \mathcal{C} , then $ts: X \rightarrow A'$ is a partial section of $pq: A' \rightarrow X$.*
- (ii) *Given a commutative triangle of total maps*



and a partial section s of p , the composite fs is a partial section of q .

- (iii) *Given a pullback square in \mathcal{C} of total maps, as in the solid part of:*



*and a partial section s of p , the map $f^*s = (sf, \overline{sf}): Y \rightarrow A \times_X Y$ is the unique partial section of π_2 making the upwards-pointing square commute.*

Proof. For (i), we have $pqts = p\overline{ts} = p\overline{st\overline{s}} = \overline{st\overline{s}} = \overline{ts}$ as desired. For (ii), we have $qfs = ps = \overline{s} = \overline{fs}$ since f is total. For (iii), if f^*s is to be a partial section of π_2 making the upwards square commute, we must have

$$\pi_1 \circ f^*s = sf \quad \text{and} \quad \pi_2 \circ f^*s = \overline{f^*s} = \overline{\pi_1 \circ f^*s} = \overline{sf}$$

using that π_1 is total. So the given definition of f^* is forced; note it is well-defined since $psf = \overline{s}f = \overline{fsf}$, and is a partial section of π_2 since $\pi_2 \circ f^*s = \overline{sf} \leq 1_Y$. \square

We now consider composition and pullback of local homeomorphisms.

Lemma 3.18.

(i) If $p: A \rightarrow X$ and $q: A' \rightarrow A$ are local homeomorphisms, then so is pq . If $\{s_i\}_{i \in I}$ and $\{t_k\}_{k \in K}$ are bases of partial sections for p and q , with corresponding local atlases φ and ψ , then $\{t_k s_i\}_{(i,k) \in I \times K}$ is a basis of partial sections for pq with corresponding local atlas

$$\theta_{(i,k),(j,\ell)} = \varphi_{ij} \circ \overline{\psi_{k\ell} s_i} . \tag{3.6}$$

(ii) In a join restriction category with local glueings, the pullback of a local homeomorphism $p: A \rightarrow X$ along any total $f: X' \rightarrow X$ exists, and is a local homeomorphism $q: A' \rightarrow X'$. If p has a basis of partial sections $\{s_i\}$ inducing the local atlas φ , then q has basis $\{f^*(s_i)\}$ inducing the local atlas

$$\psi_{ij} = \overline{\varphi_{ij} f} . \tag{3.7}$$

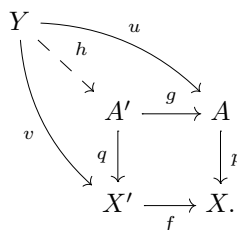
Proof. For (i), write $p_i = s_i^*$ and $q_i = t_i^*$ so that $p = \bigvee_i p_i$ and $q = \bigvee_i q_i$. Now by distributivity of joins over composition $pq = \bigvee_{i,k} p_i q_k$, where each $p_i q_k$ is a partial isomorphism by Lemma 2.14; now the corresponding partial inverses $t_k s_i$ are the desired basis for pq . Finally, we calculate the induced local atlas to be

$$\theta_{(i,k),(j,\ell)} = p_j q_\ell t_k s_i = p_j \psi_{k\ell} s_i = p_j s_i \overline{\psi_{k\ell} s_i} = \varphi_{ij} \circ \overline{\psi_{k\ell} s_i} .$$

We now turn to (ii). Like before, write $p_i = s_i^*$ so that $p = \bigvee_i p_i$. Now the idempotents in (3.7) clearly satisfy $\psi_{ij} = \psi_{ji}$, while $\psi_{jk} \psi_{ij} = \overline{\varphi_{jk} f} \overline{\varphi_{ij} f} = \overline{\varphi_{jk} f \varphi_{ij} f} = \overline{\varphi_{jk} \varphi_{ij} f} \leq \overline{\varphi_{ik} f} = \psi_{ik}$, and so we have a local atlas on X' ; we note for future use that $\varphi_{ij} f = f \overline{\varphi_{ij} f} = f \psi_{ij}$. Let $q = \bigvee_i q_i: A' \rightarrow X'$ be a glueing for this local atlas, with corresponding basis of partial sections $\{t_i\}$. Now observe that the family of maps $s_i f: X' \rightarrow A$ satisfy

$$s_i f \psi_{ij} = s_i \varphi_{ij} f \leq s_j f \quad \text{and} \quad \overline{s_i f} = \overline{s_i} \overline{f} = \overline{\varphi_{ii} f} = \psi_{ii} ,$$

so that by Lemma 3.4, there is a unique total map $g: A' \rightarrow A$ with $g t_i = s_i f$ for each i . Moreover, since $p_j g t_i = p_j s_i f = \varphi_{ij} f = f \psi_{ij} = f q_j t_i$ for each i, j , we conclude by Lemma 3.4 that $p_j g = f q_j$ for each j . Taking joins over j , we have that $pg = fq$, yielding the commuting square bottom right in:



We claim this square is a pullback in \mathcal{C} . Indeed, given maps u, v with $fv = pu$ as in the outside above, we may consider the maps $t_i v \overline{p_i u}: Y \rightarrow A'$. These are compatible as a consequence of the calculation

$$\begin{aligned} t_j v \overline{p_j u} \overline{t_i v \overline{p_i u}} &\leq t_j v \overline{p_j u} \overline{p_i u} = t_j v \overline{p_j u \overline{p_i u}} = t_j v \overline{p_j \overline{p_i u}} = t_j v \overline{p_j s_i \overline{p_i u}} \\ &= t_j v \overline{\varphi_{ij} \overline{p_i u}} \leq t_j v \overline{\varphi_{ij} \overline{p u}} = t_j v \overline{\varphi_{ij} f v} = t_j v \overline{\psi_{ij} v} = t_j \psi_{ij} v \leq t_i v, \end{aligned}$$

and so we can form the join $h = \bigvee_i t_i v \overline{p_i u}: Y \rightarrow A'$. Noting that

$$g t_i v \overline{p_i u} = s_i f v \overline{p_i u} = s_i p u \overline{p_i u} = s_i p_i u = \overline{p_i u}$$

we obtain on taking joins that $gh = \bigvee_i g t_i v \overline{p_i u} = \bigvee_i \overline{p_i u} = u$. Moreover, we have

$$q t_i v \overline{p_i u} = \overline{t_i v \overline{p_i u}} = \overline{v \overline{t_i v \overline{p_i u}}} = \overline{v g t_i v \overline{p_i u}} = \overline{v \overline{p_i u}}$$

and so, on taking joins, $qh = \bigvee_i q t_i v \overline{p_i u} = v \bigvee_i \overline{p_i u} = \overline{v p u} = \overline{v f v} = \overline{f v} = v$. So h makes both triangles above commute; it remains to show that it is unique such. But if k is another such map, then we have, since $f q = g p$ is total that $\overline{q_i} = \overline{f q} \overline{q_i} = \overline{f q} \overline{q_i} = \overline{f q_i}$ and so

$$q_i k = q k \overline{q_i k} = \overline{v q_i k} = \overline{v f q_i k} = \overline{v p_i g k} = \overline{v \overline{p_i u}}.$$

It follows that $k = \bigvee_i \overline{q_i k} = \bigvee_i t_i q_i k = \bigvee_i t_i v \overline{p_i u} = h$ as desired. Finally, note that since for each i we have $g t_i = s_i f$, we must have $t_i = f^*(s_i)$ by unicity in Lemma 3.17(iii). \square

4. Local homeomorphisms and sheaves

In this section, as a warm-up for our main result, we explain how the classical correspondence between local homeomorphisms over a space X and sheaves on X generalises to objects X of a suitable join restriction category \mathcal{C} . Much as in the classical case, the correspondence sought will arise as part of a larger adjunction.

4.1. The categories of interest

One side of our correspondence will involve local homeomomorphisms over an object $X \in \mathcal{C}$, or more generally, total maps over X .

Definition 4.1. Let \mathcal{C} be a join restriction category and $X \in \mathcal{C}$. We write $\mathcal{C}/_t X$ for the category with total maps $A \rightarrow X$ as objects, and commuting triangles

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow p & \swarrow q \\ & & X \end{array} \tag{4.1}$$

as morphisms. We write $\mathcal{C}/_{\ell h}X$ for the full subcategory of $\mathcal{C}/_tX$ whose objects are the local homeomorphisms.

It is perhaps worth noting:

Lemma 4.2. *If (4.1) is a map in $\mathcal{C}/_tX$ then f , as well as p and q , is total; if it is a map in $\mathcal{C}/_{\ell h}X$ then f , as well as p and q , is a local homeomorphism.*

Proof. If p and q are total, then $\overline{f} = \overline{qf} = \overline{p} = 1$ so that f is also total. Now let p and q be local homeomorphisms, and write $p = \bigvee_i p_i$ with corresponding partial sections s_i . By Lemma 3.17(ii), each $f s_i$ is a partial section of q , whence a partial isomorphism by Lemma 3.14(iv); and so each $f s_i p_i$ is a partial isomorphism. Now $f = f\overline{p} = \bigvee_i f\overline{p}_i = \bigvee_i f s_i p_i$ so that f is a local homeomorphism as desired. \square

The other side of the correspondence involves *presheaves* and *sheaves* on $X \in \mathcal{C}$; these are presheaves and sheaves in the classical sense on the locale of restriction idempotents $\mathcal{O}(X)$. Our preferred formulation will follow Fourman and Scott [10].

Definition 4.3. A *presheaf* on an object X of a join restriction category is a set A with an associative, unital right action $A \times \mathcal{O}(X) \rightarrow A$, written $a, e \mapsto ae$, and an *extent* operation $A \rightarrow \mathcal{O}(X)$, written $a \mapsto \overline{a}$, obeying the axioms:

- (i) $a\overline{a} = a$ for all $a \in A$;
- (ii) $\overline{ae} = \overline{a}e$ for all $a \in A$ and $e \in \mathcal{O}(A)$.

Just as in a restriction category, we can define a partial order \leq and compatibility relation \smile on a presheaf A by

$$a \leq b \text{ iff } a = b\overline{a} \quad \text{and} \quad a \smile b \text{ iff } a\overline{b} = b\overline{a};$$

and we call A a *sheaf* if every compatible family of elements has a join with respect to \leq . We write $\mathcal{Psh}(X)$ for the category whose objects are presheaves, and whose maps $A \rightarrow B$ are functions which preserve the right action and the extent, and write $\mathcal{Sh}(X)$ for the full subcategory of sheaves.

Example 4.4. If \mathcal{C} is a join restriction category then each hom-set $\mathcal{C}(X, Y)$ is an $\mathcal{O}(X)$ -sheaf, with the right $\mathcal{O}(X)$ -action given by composition and the extent operation given by restriction. More generally, if $F: \mathcal{B} \rightarrow \mathcal{C}$ is a hyperconnected join restriction functor, then each hom-set $\mathcal{B}(X, Y)$ becomes an $\mathcal{O}(FX)$ -sheaf by transporting the $\mathcal{O}(X)$ -sheaf structure along the isomorphism $\mathcal{O}(X) \rightarrow \mathcal{O}(FX)$.

By analogy with Lemma 2.5, we have the following standard result [10]:

Lemma 4.5. *If X is an object of a join restriction category and $A \in \text{Sh}(X)$, then:*

- (i) $\overline{\bigvee_i a_i} = \bigvee_i \overline{a_i}$ for all compatible families $\{a_i\} \subseteq A$;
- (ii) $(\bigvee_i a_i)e = \bigvee_i(a_i e)$ for all compatible families $\{a_i\} \subseteq A$ and all $e \in \mathcal{O}(X)$;
- (iii) $a(\bigvee_i e_i) = \bigvee_i(ae_i)$ for all $a \in A$ and all families $\{e_i\} \subseteq \mathcal{O}(X)$.

Moreover, if $f: A \rightarrow B$ in $\text{Sh}(X)$, then $f(\bigvee_i a_i) = \bigvee_i f(a_i)$ for all compatible families $\{a_i\} \subseteq A$.

Proof. (i) is as in Lemma 2.5. For (ii), we easily have $\bigvee_i(a_i e) \leq (\bigvee_i a_i)e$, but using (i) and (2.1) in \mathcal{C} , we also have $\overline{\bigvee_i(a_i e)} = \bigvee_i \overline{a_i e} = \bigvee_i(\overline{a_i} e) = (\bigvee_i \overline{a_i})e = \overline{(\bigvee_i a_i)}e = \overline{(\bigvee_i a_i)}e$; whence equality. (iii) follows like before from (i) and (ii) using [14, Lemma 3.1.8(iii)]. For the final claim, we easily have $\bigvee_i f(a_i) \leq f(\bigvee_i a_i)$, but also that $\overline{\bigvee_i f(a_i)} = \bigvee_i \overline{f(a_i)} = \bigvee_i \overline{a_i} = \overline{\bigvee_i a_i} = \overline{f(\bigvee_i a_i)}$ using (i) and the fact that f preserves restriction. Thus $f(\bigvee_i a_i) = \bigvee_i f(a_i)$ as desired. \square

The $\mathcal{O}(X)$ -valued equality of Definition 2.7 also makes sense for presheaves:

Definition 4.6. ([10]) For any $X \in \mathcal{C}$, any $A \in \mathcal{Psh}(X)$ and any $a, b \in A$, we define

$$\llbracket a = b \rrbracket := \bigvee \{e \in \mathcal{O}(X) : e \leq \overline{ab} \text{ and } ae = be\} .$$

And we have the following standard result, which can be proved exactly as in Lemma 2.8 above, using the previous lemma for part (iv).

Lemma 4.7. ([10]) For any $X \in \mathcal{C}$, any $A \in \mathcal{Psh}(X)$ and any $a, b, c \in A$ we have

- (i) If $a \leq b$ then $\llbracket a = b \rrbracket = \overline{b}$;
- (ii) $\llbracket a = b \rrbracket = \llbracket b = a \rrbracket$;
- (iii) $\llbracket b = c \rrbracket \llbracket a = b \rrbracket \leq \llbracket a = c \rrbracket$;
- (iv) If A is a sheaf, then $a \llbracket a = b \rrbracket = b \llbracket a = b \rrbracket$.

4.2. The adjunction

Our objective in this section is to construct, for any object X of a join restriction category with local glueings, an adjunction

$$\mathcal{C}/_t X \begin{matrix} \xleftarrow{\Delta} \\ \perp \\ \xrightarrow{\Gamma} \end{matrix} \mathcal{Psh}(X) . \tag{4.2}$$

We begin with the right adjoint Γ , which is given by taking partial sections.

Definition 4.8. Let $p: A \rightarrow X$ be a total map in a join restriction category. We write $\Gamma(p)$ for the X -presheaf of partial sections of p , with right $\mathcal{O}(X)$ -action $s, e \mapsto s \circ e$ and extent operation given by restriction $s \mapsto \bar{s}$.

Proposition 4.9. Let X be an object of a join restriction category \mathcal{C} . The assignment $p \mapsto \Gamma(p)$ underlies a functor $\Gamma: \mathcal{C}/_t X \rightarrow \mathcal{Psh}(X)$ whose image lands inside the subcategory $\mathcal{Sh}(X)$.

Proof. Each $f: p \rightarrow q$ of $\mathcal{C}/_t X$, as in (4.1), induces the map $\Gamma(f): \Gamma(p) \rightarrow \Gamma(q)$ given by $s \mapsto fs$, which is well-defined by Lemma 3.17(ii). It is clear that $\Gamma(f)$ commutes with the right $\mathcal{O}(X)$ -action and the extent operations, and that $f \mapsto \Gamma(f)$ is itself functorial. It remains to show that the image of Γ lands inside $\mathcal{Sh}(X)$, i.e., that each $\Gamma(p)$ is a sheaf. The relations \leq and \smile on $\Gamma(p)$ are the same as those on $\mathcal{C}(X, A)$, so that any compatible family $\{s_i: X \rightarrow A\} \subseteq \Gamma(p)$ admits a join $\bigvee_i s_i: X \rightarrow A$ *qua* family in \mathcal{C} . This join is again a partial section since $p \bigvee_i s_i = \bigvee_i ps_i = \bigvee_i \bar{s}_i = \overline{\bigvee_i s_i}$, and as such is the join in $\Gamma(p)$. \square

We now show that, in the presence of local glueings, the functor $\Gamma: \mathcal{C}/_t X \rightarrow \mathcal{Psh}(X)$ of Proposition 4.9 has a left adjoint Δ , defined on objects as follows.

Definition 4.10. Let \mathcal{C} be a join restriction category with local glueings. For any $X \in \mathcal{C}$ and $A \in \mathcal{Psh}(X)$, we have by Lemma 4.7(ii) and (iii) an A -object local atlas on X with $\varphi_{ab} = \llbracket a = b \rrbracket$. We denote by $p_A: \Delta A \rightarrow X$ a glueing of this atlas, with basis of partial sections $\{s_a\}_{a \in A}$ and corresponding partial inverses p_a .

We now define the unit of the desired adjunction and verify the adjointness property; functoriality of Δ will follow from this.

Lemma 4.11. The assignment $a \mapsto s_a$ gives a presheaf map $\eta_A: A \rightarrow \Gamma \Delta A$.

Proof. η_A preserves extents since $\overline{s_a} = p_a s_a = \llbracket a = a \rrbracket = \bar{a}$. To see it preserves the right $\mathcal{O}(X)$ -action, note that for any $e \in \mathcal{O}(X)$, we have $s_a e = s_a \bar{a} e = s_a \llbracket a e = a \rrbracket = s_a p_a s_{a e} \leq s_{a e}$; but since $\overline{s_a e} = \overline{a e} = \overline{s_{a e}}$ we have $s_a e = s_{a e}$ as desired. \square

Lemma 4.12. For each $A \in \mathcal{Psh}(X)$, the morphism $\eta_A: A \rightarrow \Gamma \Delta A$ exhibits $p_A: \Delta A \rightarrow X$ as the value at A of a left adjoint Δ to Γ .

Proof. Let $q: B \rightarrow X$ in $\mathcal{C}/_t X$. We must show each presheaf map $f: A \rightarrow \Gamma q$ is a composite $\Gamma g \circ \eta_A: A \rightarrow \Gamma \Delta A \rightarrow \Gamma q$ for a unique g in a commuting triangle

$$\begin{array}{ccc}
 \Delta A & \xrightarrow{g} & B \\
 \searrow p_A & & \swarrow q \\
 & X &
 \end{array}$$

Now if $\Gamma(g) \circ \eta_A = \Gamma(g') \circ \eta_A$, then $gs_a = g's_a$ for each $a \in A$, whence $g = g'$ since the s_a 's are a basis. For existence of g , consider the partial sections $f_a : X \rightarrow B$ giving the values of the presheaf map $f : A \rightarrow \Gamma q$. Since f is a presheaf map, we have $\overline{f_a} = \overline{a}$ and $f_a e = f_{ae}$ for any $e \in \mathcal{O}(X)$; we thus have $\overline{f_a} = \overline{a} = \llbracket a = a \rrbracket$ and

$$\begin{aligned} f_b \llbracket a = b \rrbracket &= \bigvee \{f_{be} : e \leq \overline{ab}, ae = be\} = \bigvee \{f_{be} : e \leq \overline{ab}, ae = be\} \\ &= \bigvee \{f_{ae} : e \leq \overline{ab}, ae = be\} = \bigvee \{f_a e : e \leq \overline{ab}, ae = be\} \leq f_a . \end{aligned}$$

So by Lemma 3.4, there is a unique total map $g : \Delta A \rightarrow B$ with $gs_a = f_a$ for each $a \in A$; this says exactly that $\Gamma(g) \circ \eta_A = f$. Finally, since $qgs_a = qf_a = \overline{f_a} = \overline{a} = p_A s_a$ for each a , we have since the s_a 's are a basis that $qg = p_A$. \square

This completes the construction of the adjunction (4.2).

4.3. The fixpoints

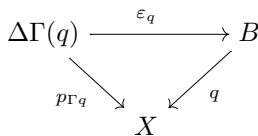
By a *right fixpoint* of an adjunction $F \dashv G : \mathcal{B} \rightarrow \mathcal{A}$, we mean an object $A \in \mathcal{A}$ at which the unit map $\eta_A : A \rightarrow GFA$ is invertible; similarly, a *left fixpoint* is an object $B \in \mathcal{B}$ with ε_B invertible. Each adjunction restricts to an equivalence between the full subcategories on the left fixpoints, and the right fixpoints. In this section, we show that, for the adjunction (4.2), this yields the desired equivalence between local homeomorphisms and sheaves.

Proposition 4.13. *The component $\eta_A : A \rightarrow \Gamma \Delta A$ of the unit of (4.2) is invertible if and only if A is a sheaf.*

Proof. Each $\Gamma(p)$ is a sheaf, so if η_A is invertible then $A \cong \Gamma \Delta A$ is a sheaf. Suppose conversely that A is a sheaf. To see η_A is injective, note that if $s_a = s_b$ then $a = a \llbracket a = a \rrbracket = ap_a s_a = ap_a s_b = a \llbracket a = b \rrbracket = b \llbracket a = b \rrbracket$, using Lemma 4.7(iv) at the last step. So $a \leq b$ and by symmetry $b \leq a$, whence $a = b$.

To show η_A is surjective, we must show each partial section s of $p_A : \Delta A \rightarrow X$ is equal to s_a for some $a \in A$. By Lemma 3.14(ii), we can write $s = \bigvee_b s_b \llbracket s = s_b \rrbracket$. In particular, the family $\{s_b \llbracket s = s_b \rrbracket\}$ is compatible in $\Gamma \Delta A$. Because η_A is an injective presheaf map, it follows that $\{b \llbracket s = s_b \rrbracket\}$ is a compatible family in A . Since A is a sheaf, we can form $a = \bigvee_b b \llbracket s = s_b \rrbracket$, and because any presheaf map preserves joins by Lemma 4.5, we have $s_a = \bigvee_b s_b \llbracket s = s_b \rrbracket = s$ as desired. \square

Proposition 4.14. *The component*



of the counit of (4.2) is invertible if and only if q is a local homeomorphism.

Proof. By construction each $\Delta A \rightarrow X$ is a local homeomorphism; so if ε_q is invertible then q is a local homeomorphism since $\Delta\Gamma(q)$ is. Conversely, let $q = \bigvee_i q_i$ be a local homeomorphism with corresponding basis of partial sections $\{t_i\}$. Since by Proposition 4.13 $\eta_{\Gamma(q)}: \Gamma(q) \rightarrow \Gamma\Delta\Gamma(q)$ is invertible, the partial sections $\{s_{t_i}\}$ are a basis for $p_{\Gamma q}: \Delta\Gamma(q) \rightarrow X$, whence by Lemma 3.15 we have $p_{\Gamma(q)} = \bigvee_i p_{t_i}$. It follows that $p_{\Gamma q}$ is a glueing of the I -object local atlas $p_{t_j} s_{t_i}$. But by definition of $p_{\Gamma q}$ and Lemma 3.14(ii) we have $p_{t_j} s_{t_i} = \llbracket t_i = t_j \rrbracket = q_j t_i$, so that q is a glueing of this same atlas. Thus, by Lemma 3.4, we must have $\Delta\Gamma(q) \cong B$ over X , via the unique map $h: \Delta\Gamma(q) \rightarrow B$ with $h s_{t_i} = t_i$ for all i . Since ε_q satisfies this property, we must have $\varepsilon_q = h$ and so ε_q is invertible. \square

We have thus proven:

Theorem 4.15. *Restricting (4.2) to its fixpoints yields an equivalence*

$$\mathcal{C}/_{\ell h} X \begin{array}{c} \xleftarrow{\Delta} \\ \xrightarrow[\Gamma]{\sim} \end{array} \text{Sh}(X).$$

Just as in the classical case, we can say slightly more. We call an adjunction $F \dashv G: \mathcal{B} \rightarrow \mathcal{A}$ *Galois* if every FA is a left fixpoint, or equivalently, every GB is a right fixpoint. The full subcategory of left fixpoints is then coreflective in \mathcal{B} via FG , while the right fixpoints are reflective in \mathcal{A} via GF . For the adjunction (4.2), each ΔA is a local homeomorphism, hence a left fixpoint, and so:

Corollary 4.16. *The adjunction (4.2) is Galois. In particular, $\text{Sh}(X)$ is reflective in $\text{Psh}(X)$ via $\Gamma\Delta$, while $\mathcal{C}/_{\ell h} X$ is coreflective in $\mathcal{C}/_t X$ via $\Delta\Gamma$.*

The reflector $\Gamma\Delta: \mathcal{Psh}(X) \rightarrow \text{Sh}(X)$ performs *sheafification*, and we can read off an explicit description (in fact that given in [10]). If $A \in \mathcal{Psh}(X)$ then the local homeomorphism $p_A: \Delta A \rightarrow X$ has basis of local sections $(s_a : a \in A)$ and associated local atlas $\llbracket a = b \rrbracket$. So by Lemma 3.16, $\Gamma\Delta A$ is the set of all families

$$(\theta_a \in \mathcal{O}(X) : a \in A) \quad \text{such that } \theta_b \theta_a \leq \llbracket a = b \rrbracket, \theta_b \llbracket a = b \rrbracket \leq \theta_a \text{ for all } a, b \in A$$

made into a sheaf via the pointwise $\mathcal{O}(X)$ -action and extent operation $\bar{\theta} = \bigvee_i \theta_i$. The unit $\eta: A \rightarrow \Gamma\Delta A$ sends b to $(\llbracket a = b \rrbracket : a \in A)$.

Remark 4.17. Note that the category $\text{Sh}(X)$ to the right of our main equivalence does not really depend on $X \in \mathcal{C}$, but only on the locale of restriction idempotents $\mathcal{O}(X)$. In particular, this allows us to identify $\mathcal{C}/_{\ell h} X$ with $\mathcal{D}/_{\ell h} Y$ whenever we have an identification of $\mathcal{O}(X)$ with $\mathcal{O}(Y)$.

A particularly natural example of this is as follows. Suppose that $F: \mathcal{C} \rightarrow \mathcal{D}$ is any join restriction functor between join restriction categories with local glueings. For any $X \in \mathcal{C}$, the action of F induces a functor $\mathcal{C}/_{\ell_h} X \rightarrow \mathcal{D}/_{\ell_h} FX$. Now if F is hyperconnected, then this functor is an equivalence, with an explicit pseudoinverse being given as follows. Given $p: A \rightarrow FX$ a local homeomorphism in \mathcal{D} , we first construct the associated local atlas φ on FX , then use hyperconnectedness to lift this to a local atlas ψ on X , and finally form the glueing $q: B \rightarrow X$ of ψ . Since q has associated atlas ψ , Fq will have associated atlas φ and so $Fq \cong p$ as desired.

5. The main theorem

In this section we prove the main result of the paper, establishing an equivalence between source-étale partite internal categories in \mathcal{C} , and join restriction categories hyperconnected over \mathcal{C} , for any join restriction category \mathcal{C} with local glueings.

5.1. The categories of interest

As in the previous section, the desired equivalence will arise as the fixpoints of a Galois adjunction between larger categories; we begin by introducing these. On one side, we have the categories which will play a role analogous to $\mathcal{Psh}(X)$ and $\mathcal{Sh}(X)$ in the previous section.

Definition 5.1. The category rCat/\mathcal{C} has as objects, pairs of a restriction category \mathcal{A} and a restriction functor $P: \mathcal{A} \rightarrow \mathcal{C}$; and has maps $(F, \alpha): (\mathcal{A}, P) \rightarrow (\mathcal{B}, Q)$ given by a restriction functor F and total natural transformation α as in:

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\
 & \searrow P & \swarrow Q \\
 & & \mathcal{C}
 \end{array}
 \quad \begin{array}{c}
 \alpha \\
 \leftarrow \\
 \leftarrow \\
 \leftarrow
 \end{array}
 \tag{5.1}$$

Composition is given by $(G, \beta) \circ (F, \alpha) = (GF, \beta F \circ \alpha)$ while $1_{(\mathcal{A}, P)} = (1_{\mathcal{A}}, 1_P)$. We write $\text{jrCat}/_h \mathcal{C}$ for the full subcategory of rCat/\mathcal{C} whose objects are those $P: \mathcal{A} \rightarrow \mathcal{C}$ for which \mathcal{A} is a *join* restriction category, and P is a *hyperconnected* restriction functor, necessarily join-preserving by Lemma 2.10.

The sense in which $\text{jrCat}/_h \mathcal{C}$ is analogous to $\mathcal{Sh}(X)$ is found in Example 4.4: for any $P: \mathcal{A} \rightarrow \mathcal{C}$ in $\text{jrCat}/_h \mathcal{C}$, each hom-set $\mathcal{A}(x, y)$ is an $\mathcal{O}(Px)$ -sheaf. Under this analogy, the category corresponding to $\mathcal{Psh}(X)$ would most rightly be $\text{rCat}/_h \mathcal{C}$; but it costs us nothing to consider the more general rCat/\mathcal{C} , and so we do so.

It is reasonable to ask why we do not restrict the *maps* in $\text{jrCat}/_h \mathcal{C}$ to be those (5.1) for which F is a join restriction functor. In fact, this is unnecessary.

Lemma 5.2. *If (5.1) is a map in $\text{jrCat}/\!/_{h}\mathcal{C}$, then F , as well as P and Q , is join-preserving.*

Proof. By Lemma 2.10, it suffices to show that F preserves any join of restriction idempotents $\bigvee_i e_i \in \mathcal{O}(X)$. Since Q is hyperconnected, it suffices for this to show that $QF(\bigvee_i e_i) = \bigvee_i QFe_i$. Clearly $\bigvee_i QFe_i \leq QF(\bigvee_i e_i)$, so it is enough to show both sides have the same restriction. But

$$\begin{aligned} \overline{QF(\bigvee_i e_i)} &= \overline{\alpha_X \circ QF(\bigvee_i e_i)} = \overline{P(\bigvee_i e_i) \circ \alpha_X} = \overline{(\bigvee_i Pe_i) \circ \alpha_X} = \overline{\bigvee_i (Pe_i \circ \alpha_X)} \\ &= \overline{\bigvee_i (\alpha_X \circ QFe_i)} = \overline{\alpha_X \circ \bigvee_i QFe_i} = \overline{\bigvee_i QFe_i} \end{aligned}$$

using totality of α_X and the fact that P preserves joins. \square

We now turn to the categories on the other side of our adjunction, which will play the roles that $\mathcal{C}/_tX$ and $\mathcal{C}/_{\ell h}X$ took in the previous section. The objects of these categories will be what we call *partite* internal categories in \mathcal{C} .

Definition 5.3. Let I be a set (of “components”). An I -partite internal category \mathbb{A} in \mathcal{C} comprises:

- **Objects of objects** A_i for each $i \in I$;
- **Objects of arrows** A_{ij} for each $i, j \in I$, together with a source-target span

$$A_i \xleftarrow{\sigma_{ij}} A_{ij} \xrightarrow{\tau_{ij}} A_j$$

of total maps for which σ_{ij} admits a pullback along any total map in \mathcal{C} ;

- **Identities** maps $\eta_i: A_i \rightarrow A_{ii}$ for each $i \in I$, compatible with source and target in the sense that $\sigma_{ii}\eta_i = 1 = \tau_{ii}\eta_i$; and
- **Composition** maps $\mu_{ijk}: A_{jk} \times_{A_j} A_{ij} \rightarrow A_{ik}$ for each $i, j, k \in I$, compatible with source and target in the sense that $\sigma_{ik}\mu_{ijk} = \sigma_{ij}\pi_2$ and $\tau_{ik}\mu_{ijk} = \tau_{jk}\pi_1$,

such that the following diagrams commute for all $i, j, k, \ell \in I$:

$$\begin{array}{ccc} A_{ij} \times_{A_i} A_{ii} & \begin{array}{c} \xleftarrow{(1, \eta_i \sigma_{ij})} \\ \xrightarrow{(\eta_j \tau_{ij}, 1)} \end{array} & A_{ij} & \xrightarrow{\quad} & A_{jj} \times_{A_j} A_{ij} & \\ & \searrow \mu_{ijj} & \parallel & & \swarrow \mu_{ijj} & \\ & & A_{ij} & & & \end{array} \qquad \begin{array}{ccc} A_{k\ell} \times_{A_k} A_{jk} \times_{A_j} A_{ij} & \xrightarrow{1 \times_{A_k} \mu_{ijk}} & A_{k\ell} \times_{A_k} A_{ik} \\ \mu_{jk\ell} \times_{A_j} 1 \downarrow & & \downarrow \mu_{ik\ell} \\ A_{j\ell} \times_{A_j} A_{ij} & \xrightarrow{\mu_{ij\ell}} & A_{i\ell} \end{array}$$

For example, a 1-partite internal category in \mathcal{C} is simply an internal category. On the other hand, if \mathcal{C} admits a *restriction terminal object* $\mathbf{1}$ —that is, an object to which each other object admits a unique total map—then a partite internal category \mathbb{A} in \mathcal{C} with each $A_i = \mathbf{1}$ is the same as a category *enriched* in \mathcal{C} .

We now describe the relevant morphisms between partite internal categories. As in the introduction, these will not be the obvious internal functors, but rather internal *cofunctors*. Cofunctors first appeared in [18] under the name “comorphism”; the alternative nomenclature we use follows Aguiar [1]. The notion is most likely not a familiar one, so we take a moment to spell it out first in the case of ordinary categories.

Definition 5.4. A cofunctor between ordinary categories $\mathcal{C} \rightsquigarrow \mathcal{D}$ comprises a mapping on objects $F: \text{ob}(\mathcal{D}) \rightarrow \text{ob}(\mathcal{C})$, together with an operation assigning to each $d \in \text{ob}(\mathcal{D})$ and each map $f: Fd \rightarrow c$ in \mathcal{C} , an object $f_*(d)$ and map $F(d, f): d \rightarrow f_*(d)$ in \mathcal{D} satisfying $F(f_*d) = c$, and subject to the two functoriality conditions $F(d, 1_{Fd}) = 1_d$ and $F(d, gf) = F(f_*(d), g) \circ F(d, f)$.

For example, we can obtain a cofunctor from any discrete opfibration. A *discrete opfibration* is a functor $F: \mathcal{D} \rightarrow \mathcal{C}$ with the property that, for any $d \in \mathcal{D}$ and map $f: Fd \rightarrow c$ in \mathcal{C} , there is a unique map $\tilde{f}: d \rightarrow d'$ of \mathcal{D} with $F\tilde{f} = f$ (Lawson in [25] calls these *covering functors*). We can define from this a cofunctor $\mathcal{C} \rightsquigarrow \mathcal{D}$ whose action on objects is that of F , and whose action on arrows takes $F(d, f)$ to be the unique map \tilde{f} described above.

We now describe the appropriate adaption of the notion of cofunctor to the case of partite internal categories.

Definition 5.5. Let \mathbb{A} be an I -partite and \mathbb{B} a J -partite internal category in \mathcal{C} . A *partite cofunctor* $F: \mathbb{A} \rightsquigarrow \mathbb{B}$ comprises:

- A mapping on component-sets $I \rightarrow J$ written $i \mapsto F_i$;
- An action on objects, given by total maps $F_i: B_{Fi} \rightarrow A_i$ for each $i \in I$; and
- An action on arrows, given by total maps $F_{ij}: A_{ij} \times_{A_i} B_{Fi} \rightarrow B_{Fi, F_j}$ for each $i, j \in I$ which are compatible with source in the sense that $\sigma_{F_i, F_j} F_{ij} = \pi_2$,

all subject to the commutativity of the following diagrams (the *target*, *identity* and *composition* axioms).

$$\begin{array}{ccc}
 A_{ij} \times_{A_i} B_{Fi} & \xrightarrow{F_{ij}} & B_{Fi, F_j} \\
 \pi_1 \downarrow & & \downarrow \tau_{F_i, F_j} \\
 A_{ij} & \xrightarrow{\tau_{ij}} A_j \xleftarrow{F_j} & B_{F_j}
 \end{array}
 \qquad
 \begin{array}{ccc}
 B_{Fi} & \xrightarrow{(\eta_i F_i, 1)} & A_{ii} \times_{A_i} B_{Fi} \\
 \eta_{Fi} \searrow & & \swarrow F_{ii} \\
 & B_{Fi, Fi} &
 \end{array}$$

$$\begin{array}{ccccc}
 A_{jk} \times_{A_j} B_{Fi, F_j} & \xrightarrow{\cong} & (A_{jk} \times_{A_j} B_{F_j}) \times_{B_{F_j}} B_{Fi, F_j} & \xrightarrow{F_{jk} \times 1} & B_{F_j, F_k} \times_{B_{F_j}} B_{Fi, F_j} \\
 \uparrow 1 \times F_{ij} & & & & \downarrow \mu_{F_i, F_j, F_k} \\
 A_{jk} \times_{A_j} A_{ij} \times_{A_i} B_{Fi} & \xrightarrow{\mu_{ijk} \times A_i 1} & A_{ik} \times_{A_i} B_{Fi} & \xrightarrow{F_{ik}} & B_{Fi, F_k}
 \end{array}$$

The identity on \mathbb{A} has all data given by identities, while the composition of $F: \mathbb{A} \rightsquigarrow \mathbb{B}$ and $G: \mathbb{B} \rightsquigarrow \mathbb{C}$ has $(GF)(i) = G(F(i))$, $(GF)_i = F_i \circ G_{F_i}: C_{GF_i} \rightarrow A_i$, and $(GF)_{ij}$ given by

$$A_{ij} \times_{A_i} C_{GF_i} \cong (A_{ij} \times_{A_i} B_{F_i}) \times_{B_{F_i}} C_{GF_i} \xrightarrow{F_{ij} \times 1} B_{F_i, F_j} \times_{B_{F_i}} C_{GF_i} \xrightarrow{G_{F_i, F_j}} C_{GF_i, GF_j}.$$

In this way, we obtain a category $\text{pCat}_c(\mathcal{C})$ of partite internal categories and partite cofunctors in \mathcal{C} . We write $\text{peCat}_c(\mathcal{C})$ for the full subcategory whose objects are the *source-étale* partite internal categories—those whose source maps σ_{ij} are all local homeomorphisms.

5.2. The right adjoint

We now start constructing the adjunction

$$\text{pCat}_c(\mathcal{C}) \begin{array}{c} \xleftarrow{\Psi} \\ \perp \\ \xrightarrow{\Phi} \end{array} \text{rCat} // \mathcal{C}$$

whose restriction to fixpoints will yield the desired equivalence $\text{peCat}_c(\mathcal{C}) \simeq \text{jrCat} // \mathcal{C}$. We begin by describing the right adjoint Φ ; much like Γ in the presheaf–total map adjunction, this will be based on the idea of taking partial sections. We first give its construction and then verify well-definedness.

Definition 5.6. Let \mathbb{A} be an I -partite internal category in \mathcal{C} . Its *externalisation* $\pi_{\mathbb{A}}: \Phi \mathbb{A} \rightarrow \mathcal{C}$ is the object of $\text{rCat} // \mathcal{C}$ defined as follows:

- The object-set of $\Phi \mathbb{A}$ is I ;
- $\Phi \mathbb{A}(i, j)$ is the set $\Gamma(\sigma_{ij})$ of partial sections of $\sigma_{ij}: A_{ij} \rightarrow A_i$;
- The identity map $1_i \in \Phi \mathbb{A}(i, i)$ is the (total) section $\eta_i: A_i \rightarrow A_{ii}$;
- The composite $t * s$ of $s \in \Phi \mathbb{A}(i, j)$ and $t \in \Phi \mathbb{A}(j, k)$ is the partial section

$$A_i \xrightarrow{s} A_{ij} \xrightarrow{\tau_{ij}^*(t)} A_{jk} \times_{A_j} A_{ij} \xrightarrow{\mu_{ijk}} A_{ik}$$

of σ_{ik} obtained by applying Lemma 3.17 in the situation:

$$\begin{array}{ccccc}
 A_{jk} & \xleftarrow{\pi_1} & A_{jk} \times_{A_j} A_{ij} & \xrightarrow{\mu_{ijk}} & A_{ik} \\
 \uparrow & \searrow \sigma_{jk} & \downarrow \pi_2 & & \uparrow \\
 & & A_{ij} & \xrightarrow{t * s} & \\
 \uparrow t & & \uparrow \tau_{ij}^* t & & \uparrow \sigma_{ik} \\
 & & A_j & \xleftarrow{\tau_{ij}} & A_{ij} \\
 & & \uparrow s & & \uparrow \sigma_{ij} \\
 & & & & A_i
 \end{array} \tag{5.2}$$

- The restriction of $s: A_i \rightarrow A_{ij}$ in $\Phi\mathbb{A}(i, j)$ is $\eta_i\bar{s}: A_i \rightarrow A_{ii}$ in $\Phi\mathbb{A}(i, i)$;
- The functor $\pi_{\mathbb{A}}: \Phi\mathbb{A} \rightarrow \mathcal{C}$ is given on objects by $i \mapsto A_i$, and on maps by sending $s \in \Phi\mathbb{A}(i, j)$ to $\tau_{ij} \circ s: A_i \rightarrow A_j$.

Definition 5.7. Let $F: \mathbb{A} \rightsquigarrow \mathbb{B}$ be a partite cofunctor in \mathcal{C} . We define a map $(\Phi F, \varpi^F): (\Phi\mathbb{A}, \pi_{\mathbb{A}}) \rightarrow (\Phi\mathbb{B}, \pi_{\mathbb{B}})$ in rCat/\mathcal{C} as follows:

- On objects, ΦF is defined by $i \mapsto F_i$;
- On morphisms, given $s \in \Phi\mathbb{A}(i, j)$ we define $\Phi F(s) \in \Phi\mathbb{B}(F_i, F_j)$ as the partial section

$$B_{F_i} \xrightarrow{F_i^*s} A_{ij} \times_{A_i} B_{F_i} \xrightarrow{F_{ij}} B_{F_i, F_j}$$

of σ_{F_i, F_j} obtained by applying Lemma 3.17 in the situation

$$\begin{array}{ccccc}
 A_{ij} & \xleftarrow{\pi_1} & A_{ij} \times_{A_i} B_{F_i} & \xrightarrow{F_{ij}} & B_{F_i, F_j} \\
 \swarrow s & & \swarrow F_i^*s & & \swarrow \sigma_{F_i, F_j} \\
 & & A_i & \xleftarrow{F_i} & B_{F_i}
 \end{array} \tag{5.3}$$

- $\varpi^F: \pi_{\mathbb{B}} \circ \Phi F \Rightarrow \pi_{\mathbb{A}}$ has components $(\varpi^F)_i = F_i: B_{F_i} \rightarrow A_i$.

Proposition 5.8. The assignments of Definitions 5.6 and 5.7 underlie a well-defined functor $\Phi: \text{pCat}_{\mathcal{C}}(\mathcal{C}) \rightarrow \text{rCat}/\mathcal{C}$ whose image lands inside $\text{jrCat}/_h\mathcal{C}$.

Proof. To begin with, let $\mathbb{A} \in \text{pCat}_{\mathcal{C}}(\mathcal{C})$. We will proceed in stages to verify that $(\Phi_{\mathbb{A}}, \pi_{\mathbb{A}})$ is a well-defined object of rCat/\mathcal{C} , and that it in fact lives in $\text{jrCat}/_h\mathcal{C}$.

$\Phi\mathbb{A}$ is a **category**. For the left unit law, note that $\tau_{jj}^*(\eta_j) = (\eta_j\tau_{jj}, \overline{\eta_j\tau_{jj}}) = (\eta_j\tau_{jj}, 1)$, so that for any $s \in \Phi\mathbb{A}(i, j)$ we have

$$\eta_j * s = \mu_{ijj}(\eta_j\tau_{jj}, 1)s \tag{5.4}$$

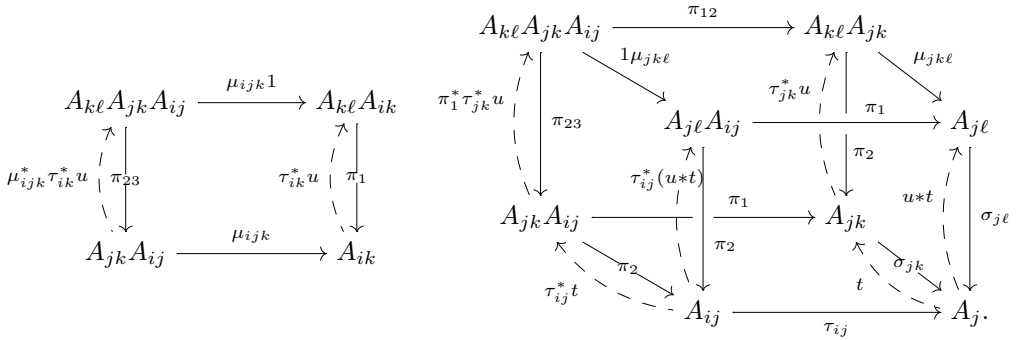
which equals s by the left unit law for \mathbb{A} . For the right unit law, for any $t \in \Phi\mathbb{A}(i, j)$ we have $\tau_{ii}^*(t)\eta_i = (t\tau_{ii}\eta_i, \overline{t\tau_{ii}\eta_i}) = (t, \eta_i\overline{t\tau_{ii}\eta_i}) = (t, \eta_i\bar{t})$, and so

$$t * \eta_i = \mu_{iij}(t, \eta_i\bar{t}) = \mu_{iij}(t, \eta_i\sigma_{ij}t) = \mu_{iij}(1, \eta_i\sigma_{ij}t) \tag{5.5}$$

which equals t by the right unit law for \mathbb{A} . Finally, for associativity, given $s \in \Phi\mathbb{A}(i, j)$ and $t \in \Phi\mathbb{A}(j, k)$ and $u \in \Phi\mathbb{A}(k, \ell)$, we first calculate that

$$\begin{aligned}
 u * (t * s) &= \mu_{ik\ell} \circ \tau_{ik}^* u \circ (t * s) = \mu_{ik\ell} \circ \tau_{ik}^* u \circ \mu_{ijk} \circ \tau_{ij}^* t \circ s \\
 \text{and } (u * t) * s &= \mu_{ij\ell} \circ \tau_{ij}^* (u * t) \circ s .
 \end{aligned}$$

Now consider the following diagrams; here, and subsequently, we may omit fibre product symbols for brevity.



The solid part of the left square is a pullback, and the section to its left obtained by pulling back the one to the right via Lemma 3.17(iii). In particular, the upwards-pointing square commutes, and so we can rewrite $u * (t * s)$ as

$$u * (t * s) = \mu_{ik\ell} \circ (\mu_{ijk} \times 1) \circ \mu_{ijk}^* \tau_{ik}^* u \circ \tau_{ij}^* t \circ s . \tag{5.6}$$

On the other hand, the front, back, top and bottom faces of the cube right above are all pullbacks, and so we can pull back the commuting diagram of sections to the right, which defines $u * t$, to obtain a commuting diagram of sections giving an expression for $\tau_{ij}^*(u * t)$. Using this, we can rewrite $(u * t) * s$ as

$$\begin{aligned} (u * t) * s &= \mu_{ij\ell} \circ (1 \times \mu_{jkl}) \circ \pi_1^* \tau_{jk}^* u \circ \tau_{ij}^* t \circ s \\ &= \mu_{ij\ell} \circ (1 \times \mu_{jkl}) \circ \mu_{ijk}^* \tau_{ik}^* u \circ \tau_{ij}^* t \circ s , \end{aligned} \tag{5.7}$$

where from the first to the second line we have $\pi_1^* \tau_{jk}^* u = \mu_{ijk}^* \tau_{ik}^* u$ by unicity in Lemma 3.17(iii), as both are pullbacks of the partial section u along $\tau_{ik} \mu_{ijk} = \tau_{jk} \pi_1$. Comparing (5.6) and (5.7), we conclude using associativity for \mathbb{A} .

$\Phi\mathbb{A}$ is a restriction category. To avoid confusion we will in this proof write (\sim) for the restriction in $\Phi\mathbb{A}$ and $(\bar{\quad})$ for that in \mathcal{C} . Note first that, for any $s \in \Phi\mathbb{A}(i, j)$ and $t \in \Phi\mathbb{A}(j, k)$, our calculations for the right unit law show that

$$s * \tilde{t} = \bar{s}\bar{t}: A_i \rightarrow A_j \tag{5.8}$$

Given this, the first three restriction axioms $s * \tilde{s} = s$, $\tilde{s} * \tilde{t} = \tilde{t} * \tilde{s}$ and $\widetilde{t * \tilde{s}} = \tilde{t} * \tilde{s}$ each follow immediately from the corresponding axiom in \mathcal{C} . It remains only to verify that $\tilde{t} * s = s * \widetilde{t * s}$ for all $s \in \Phi\mathbb{A}(i, j)$ and $t \in \Phi\mathbb{A}(j, k)$. Now observe that $\tau_{ij}^*(\eta_j \bar{t}) = (\eta_j \bar{t} \tau_{ij}, \overline{\eta_j \bar{t} \tau_{ij}}) = (\eta_j \tau_{ij} \bar{t} \tau_{ij}, \overline{\eta_j \tau_{ij} \bar{t} \tau_{ij}}) = (\eta_j \tau_{ij}, 1) \overline{\bar{t} \tau_{ij}}$, so that

$$\tilde{t} * s = \mu_{ijj}(\eta_j \tau_{ij}, 1) \overline{\bar{t} \tau_{ij}} s = \overline{\bar{t} \tau_{ij}} s = \overline{st \tau_{ij}} s . \tag{5.9}$$

On the other hand, we have by totality of μ_{ijk} and π_1 that

$$s * t * s = s \overline{\mu_{ijk} \tau_{ij}^*(t) s} = s \overline{\tau_{ij}^*(t) s} = s \overline{\pi_1 \tau_{ij}^*(t) s} = s \overline{t \tau_{ij} s} .$$

$\Phi_{\mathbb{A}}$ is a join restriction category. Note that since $t * \tilde{s} = t \bar{s} : A_i \rightarrow A_{ij}$, the compatibility and order relations on $\Phi_{\mathbb{A}}(i, j)$ coincide with those of the sheaf $\Gamma(\sigma_{ij})$; since the latter admits joins, so too does the former. To see that these joins are preserved by precomposition in $\Phi_{\mathbb{A}}$, suppose that $\{t_i\} \subseteq \Phi_{\mathbb{A}}(j, k)$ is a compatible family and $s \in \Phi_{\mathbb{A}}(i, j)$. The unicity in Lemma 3.17(iii) implies that the family $\tau_{ij}^*(t_i)$ is compatible and that $\tau_{ij}^*(\bigvee_i t_i) = \bigvee_i \tau_{ij}^*(t_i)$. We therefore conclude that

$$(\bigvee_i t_i) * s = \mu_{ijk} \tau_{ij}^*(\bigvee_i t_i) s = \mu_{ijk} \bigvee_i \tau_{ij}^*(t_i) s = \bigvee_i \mu_{ijk} \tau_{ij}^*(t_i) s = \bigvee_i (t_i * s) .$$

$\pi_{\mathbb{A}} : \Phi_{\mathbb{A}} \rightarrow \mathcal{C}$ is a hyperconnected restriction functor. For functoriality, note that $\pi_{\mathbb{A}}(1_i) = \tau_{ii} \eta_i = 1_{A_i}$, and that for $s \in \Phi_{\mathbb{A}}(i, j)$ and $t \in \Phi_{\mathbb{A}}(j, k)$ we have

$$\pi_{\mathbb{A}}(t * s) = \tau_{ik} \mu_{ijk} \tau_{ij}^*(t) s = \tau_{jk} \pi_1 \tau_{ij}^*(t) s = \tau_{jk} t \tau_{ij} s = \pi_{\mathbb{A}}(t) \circ \pi_{\mathbb{A}}(s) .$$

Now $\pi_{\mathbb{A}}$ preserves restrictions as $\pi_{\mathbb{A}}(\tilde{s}) = \pi_{\mathbb{A}}(\eta_i \bar{s}) = \tau_{ii} \eta_i \bar{s} = \bar{s} = \overline{\tau_{ij} s} = \overline{\pi_{\mathbb{A}}(s)}$; and is hyperconnected as the mapping $\mathcal{O}_{\Phi_{\mathbb{A}}}(i) \rightarrow \mathcal{O}_{\mathcal{C}}(A_i)$ has inverse $e \mapsto \eta_i e$.

This proves that $(\Phi_{\mathbb{A}}, \pi_{\mathbb{A}})$ is a well-defined object of $\text{rCat} // \mathcal{C}$, and indeed of $\text{jrCat} // {}_h \mathcal{C}$. Consider now $F : \mathbb{A} \rightsquigarrow \mathbb{B}$ a partite cofunctor; we will show $(\Phi F, \varpi^F) : (\Phi_{\mathbb{A}}, \pi_{\mathbb{A}}) \rightarrow (\Phi_{\mathbb{B}}, \pi_{\mathbb{B}})$ is a well-defined map of $\text{jrCat} // {}_h \mathcal{C}$. Again we proceed by stages.

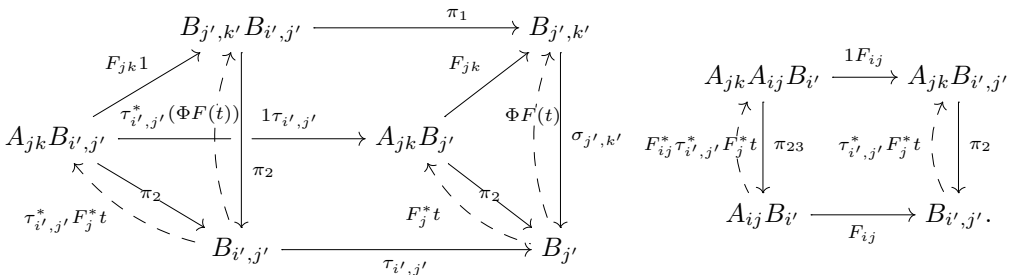
ΦF is functorial. For $i \in \Phi_{\mathbb{A}}$, the identity $\eta_i \in \Phi_{\mathbb{A}}(i, i)$ maps to

$$(\Phi F)(\eta_i) = F_{ii} F_i^*(\eta_i) = F_{ii}(\eta_i F_i, \overline{\eta_i F_i}) = F_{ii}(\eta_i F_i, 1) = \eta_{F_i}$$

as required, using at the last step the unit axiom for F . For binary functoriality, let $s \in \Phi_{\mathbb{A}}(i, j)$ and $t \in \Phi_{\mathbb{A}}(j, k)$. Writing i', j', k' for F_i, F_j, F_k , we have

$$\Phi F(t) * \Phi F(s) = \mu_{i',j',k'} \circ \tau_{i',j'}^*(\Phi F(t)) \circ \Phi F(s) = \mu_{i',j',k'} \circ \tau_{i',j'}^*(\Phi F(t)) \circ F_{ij} \circ F_i^* s .$$

To calculate $\tau_{i',j'}^*(\Phi F(t))$ we consider the diagram left below, wherein $F_{jk} 1$ to the left denotes the top side of the composition axiom in Definition 5.4:

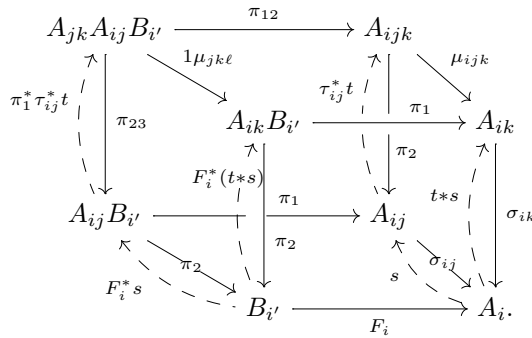


All three horizontal faces are pullbacks, and so we can pull back the commuting diagram of sections to the right to obtain a commuting diagram of sections giving an expression for $\tau_{i',j'}^*(\Phi F(t))$. This gives the first equality in:

$$\begin{aligned} \Phi F(t) * \Phi F(s) &= \mu_{i',j',k'} \circ (F_{jk} \times 1) \circ \tau_{i',j'}^* F_j^* t \circ F_{ij} \circ F_i^* s \\ &= \mu_{i',j',k'} \circ (F_{jk} \times 1) \circ (1 \times F_{ij}) \circ F_{ij}^* \tau_{i',j'}^* F_j^* t \circ F_i^* s \\ &= \mu_{i',j',k'} \circ (F_{jk} \times 1) \circ (1 \times F_{ij}) \circ \pi_1^* \tau_{ij}^* t \circ F_i^* s . \end{aligned} \tag{5.10}$$

The second equality comes from pulling back the section $\tau_{i',j'}^* F_j^* t$ as in the square right above, while the third follows from the target axiom $F_j \tau_{i',j'} F_{ij} = \tau_{ij} \pi_1$.

On the other hand $\Phi F(t * s) = F_{ik} \circ F_i^*(t * s)$, and by considering the diagram



We can expand out the term $F_i^*(t * s)$ and conclude that

$$\Phi F(t * s) = F_{ik} \circ (1 \times \mu_{jkl}) \circ \pi_1^* \tau_{ij}^* t \circ F_i^* s . \tag{5.11}$$

Comparing (5.10) and (5.11) yields the desired equality by rewriting using the composition axiom for the cofunctor F .

ΦF is a restriction functor. We calculate for any $s \in \Phi \mathbb{A}(i, j)$ that

$$\begin{aligned} \Phi F(\bar{s}) &= F_{ii} F_i^*(\eta_i \bar{s}) = F_{ii}(\eta_i \bar{s} F_i, \overline{\eta_i \bar{s} F_i}) = F_{ii}(\eta_i F_i \overline{s F_i}, \overline{\eta_i F_i s F_i}) = F_{ii}(\eta_i F_i, 1) \overline{s F_i} \\ &= \eta_{F_i} \overline{s F_i} = \eta_{F_i} \overline{F_i^* s} = \eta_{F_i} \overline{F_{ii} F_i^* s} = \widetilde{\Phi F(s)} , \end{aligned}$$

where from the first to the second line we use the unit axiom for F .

ϖ^F is a total natural transformation $\pi_{\mathbb{B}} \circ \Phi F \Rightarrow \pi_{\mathbb{A}}$. We compute that

$$\varpi_j^F \circ \pi_{\mathbb{B}}(\Phi F(s)) = F_j \tau_{F_i, F_j} F_{ij} F_i^*(s) = \tau_{ij} \pi_1 F_i^*(s) = \tau_{ij} s F_i = \pi_{\mathbb{A}}(s) \circ \varpi_i^F$$

using the target axiom for F at the second equality.

We have thus verified well-definedness of Φ on morphisms, and it remains only to prove functoriality of Φ itself. Preservation of identities is easy. As for binary composition, given

partite cofunctors $F: \mathbb{A} \rightsquigarrow \mathbb{B}$ and $G: \mathbb{B} \rightsquigarrow \mathbb{C}$ it is clear that $\Phi G \circ \Phi F$ and $\Phi(GF)$ agree on objects. On maps, given $s \in \Phi\mathbb{A}(i, j)$, we have $\Phi G(\Phi F(s))$ and $\Phi(GF)(s)$ given by the upper and lower composites in:

$$\begin{array}{ccc}
 C_{GF_i} & \xrightarrow{G_{F_i}^*(F_{ij} \circ F_i^*(s))} & B_{F_i, F_j} \times_{B_{F_i}} C_{GF_i} & \xrightarrow{G_{F_i, F_j}} & C_{GF_i, GF_j} \\
 (F_i G_{F_i})^*(s) \downarrow & & \uparrow F_{ij} \times 1 & & \\
 A_{ij} \times_{A_i} C_{GF_i} & \xrightarrow{\cong} & (A_{ij} \times_{A_i} B_{F_i}) \times_{B_{F_i}} C_{GF_i} & &
 \end{array}$$

So it suffices to show the left region commutes, which follows much as previously using Lemma 3.17. So $\Phi(GF) = \Phi(G) \circ \Phi(F)$; and finally, it is clear that the 2-cell components ϖ^F and ϖ^G paste together to yield ϖ^{GF} . \square

5.3. The left adjoint

In this section, we show that $\Phi: \text{pCat}_c(\mathcal{C}) \rightarrow \text{rCat}/\mathcal{C}$ has a left adjoint Ψ . Much like Δ in the presheaf–total map adjunction, Ψ will be constructed by glueing suitable local atlases built from $\mathcal{O}(X)$ -valued equality of maps. The equality required is a slight variant of the one in Definition 2.7.

Definition 5.9. Let $P: \mathcal{A} \rightarrow \mathcal{C}$ in rCat/\mathcal{C} . Given maps $f, g: i \rightrightarrows j$ in \mathcal{A} we define the restriction idempotent $\llbracket f =_P g \rrbracket \in \mathcal{O}(Pi)$ by

$$\llbracket f =_P g \rrbracket := \bigvee \{Pe : e \in \mathcal{O}(i), e \leq \bar{f}\bar{g}, fe = ge\} .$$

Remark 5.10. When \mathcal{A} has joins of restriction idempotents which are preserved by P , we can form $\llbracket f = g \rrbracket$ in \mathcal{A} , and by join-preservation have $\llbracket f =_P g \rrbracket = P\llbracket f = g \rrbracket$.

The following lemma is now a slight variant of Lemma 2.8.

Lemma 5.11. Given $P: \mathcal{A} \rightarrow \mathcal{C}$ in rCat/\mathcal{C} and $f, g, h: i \rightarrow j$ in \mathcal{A} , we have that

- (i) $\llbracket f =_P f \rrbracket = P\bar{f}$;
- (ii) $\llbracket f =_P g \rrbracket = \llbracket g =_P f \rrbracket$;
- (iii) $\llbracket g =_P h \rrbracket \circ \llbracket f =_P g \rrbracket \leq \llbracket f =_P h \rrbracket$;
- (iv) $Pf \circ \llbracket f =_P g \rrbracket = Pg \circ \llbracket f =_P g \rrbracket$.

If $f, g: i \rightarrow j$ and $u, v: j \rightarrow k$ then $\llbracket f =_P g \rrbracket \overline{\llbracket u =_P v \rrbracket} \circ Pf \leq \llbracket uf =_P vg \rrbracket$.

Proof. For (i)–(iv) we transcribe the proof of Lemma 2.8. For the final claim, it suffices by distributivity to prove that, if $d \leq \bar{f}\bar{g}$ and $e \leq \bar{u}\bar{v}$ are such that $fd = gd$ and $ue = ve$, then $Pd \circ \overline{Pe \circ Pf} \leq \llbracket uf =_P vg \rrbracket$; and for this, it suffices to show $d\bar{e}\bar{f} \leq \overline{uf\bar{v}\bar{g}}$

and that $ufd\overline{ef} = vgd\overline{ef}$. Since $e \leq \overline{u}$ we have $d\overline{ef} \leq \overline{ef} \leq \overline{uf}$; moreover, we have $d\overline{ef} = \overline{ef}d = \overline{efd} = \overline{egd} \leq \overline{vg}$. Finally, we have $ufd\overline{ef} = uf\overline{efd} = uefd = vefd = vfe\overline{fd} = vfd\overline{ef} = vgd\overline{ef}$, as desired. \square

We are now ready to construct the left adjoint Ψ . We will first define the action of Ψ on objects, and the unit maps for the desired adjunction. We then check the definitions are well-posed, and finally verify adjointness.

Definition 5.12. Let $P: \mathcal{A} \rightarrow \mathcal{C}$ be an object of rCat/\mathcal{C} . Writing $I = \text{ob}(\mathcal{A})$, the *internalisation* of P is the I -partite internal category ΨP in \mathcal{C} for which:

- The object of objects $(\Psi P)_i$ is Pi ;
- The object of arrows with its source map $\sigma_{ij}: \Psi P_{ij} \rightarrow Pi$ is obtained as a glueing of the $\mathcal{A}(i, j)$ -object local atlas on Pi with components $\llbracket f =_P g \rrbracket$; this is a well-defined local atlas by Lemma 5.11. We write $\sigma_{ij} = \bigvee_{f \in \mathcal{A}(i, j)} p_f$ and write s_f for the partial inverse of p_f (recall the notation p_f was introduced in Lemma 3.3).
- $\tau_{ij}: \Psi P_{ij} \rightarrow \Psi P_j$ is the unique map such that

$$\tau_{ij} \circ s_f = Pf: Pi \rightarrow Pj \quad \text{for all } i, j \in \mathcal{A}(i, j). \tag{5.12}$$

- $\eta_i: \Psi P_i \rightarrow \Psi P_{ii}$ is the partial section $s_{1_i}: Pi \rightarrow \Psi P_{ii}$.
- $\mu_{ijk}: \Psi P_{jk} \times_{Pj} \Psi P_{ij} \rightarrow \Psi P_{ik}$ is the unique map such that

$$\mu_{ijk} \circ \tau_{ij}^*(s_g) \circ s_f = s_{gf} \quad \text{for all } f \in \mathcal{A}(i, j), g \in \mathcal{A}(j, k). \tag{5.13}$$

We also define a restriction functor $\eta_P: \mathcal{A} \rightarrow \Phi(\Psi P)$ where:

- On objects, η_P is the identity;
- On maps, $\eta_P(f: i \rightarrow j)$ is the partial section $s_f: Pi \rightarrow \Psi P_{ij}$ of σ_{ij} .

Finally, we define the *unit map at P* to be the following map of rCat/\mathcal{C} :

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{\eta_P} & \Phi(\Psi P). \\
 & \searrow P & \swarrow \pi_{\Psi P} \\
 & \mathcal{C} &
 \end{array}
 \quad \text{with } \eta_P \circ P = \pi_{\Psi P} \tag{5.14}$$

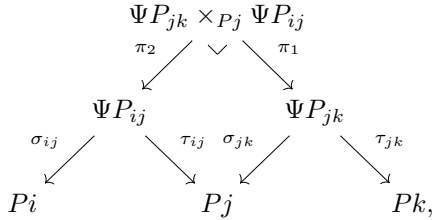
Proposition 5.13. For any $P: \mathcal{A} \rightarrow \mathcal{C}$ in rCat/\mathcal{C} , ΨP is a well-defined object of $\text{peCat}_{\mathcal{C}}(\mathcal{C})$, and (5.14) is a well-defined map of rCat/\mathcal{C} .

Proof. Like before, we check well-definedness in stages.

The data of ΨP are well-defined. For $\Psi P_i, \Psi P_{ij}$ and σ_{ij} there is nothing to do. For τ_{ij} , we observe that the family of maps $(Pf: Pi \rightarrow Pj)_{f \in \mathcal{A}(i, j)}$ satisfies

$$Pf \circ \llbracket f =_P f \rrbracket = Pf \circ P\bar{f} = Pf \quad \text{and} \quad Pg \circ \llbracket f =_P g \rrbracket = Pf \circ \llbracket f =_P g \rrbracket \leq Pf ;$$

so by Lemma 3.4, there is a unique total map τ_{ij} satisfying (5.12). Next, the map $\eta_i = s_{1_i}$ satisfies $\overline{\eta_i} = \overline{s_{1_i}} = \llbracket 1_i =_P 1_i \rrbracket = 1_{P_i}$ and so is total as desired. It remains to show well-definedness of μ_{ijk} . By Lemma 3.18, the following pullback exists:



and the composite $s_{ij}\pi_2$ down the left is a local homeomorphism, with basis of partial sections $\{\tau_{ij}^*(s_g)s_f : f \in \mathcal{A}(i, j), g \in \mathcal{A}(j, k)\}$, and induced local atlas

$$\theta_{(f,g),(h,k)} = \llbracket f =_P h \rrbracket \overline{\llbracket g =_P k \rrbracket \tau_{ij} s_f} = \llbracket f =_P h \rrbracket \overline{\llbracket g =_P k \rrbracket Pf} . \tag{5.15}$$

Now consider the family of maps $s_{gf} : P_i \rightarrow \Psi P_{ik}$ for $f \in \mathcal{A}(i, j)$ and $g \in \mathcal{A}(j, k)$. By Lemma 5.11, these satisfy

$$\begin{aligned}
 \theta_{(f,g),(f,g)} &= \llbracket f =_P f \rrbracket \overline{\llbracket g =_P g \rrbracket Pf} = P\bar{f} \overline{P\bar{g}Pf} = P\bar{gf} = \llbracket gf =_P gf \rrbracket = \overline{s_{gf}} \\
 \text{and } s_{kh}\theta_{(f,g),(h,k)} &= s_{kh} \llbracket f =_P h \rrbracket \overline{\llbracket g =_P k \rrbracket Pf} \leq s_{kh} \llbracket gf =_P kh \rrbracket \leq s_{gf} ;
 \end{aligned}$$

so by Lemma 3.4 there is a total map μ_{ijk} uniquely determined by (5.13).

η and μ are compatible with source and target. For compatibility of η , we note that $\sigma_{ii}\eta_i = \sigma_{ii}s_{1_i} = \overline{s_{1_i}} = 1_{P_i}$ since s_{1_i} is a partial section of σ_{ii} , and that $\tau_{ii}\eta_i = \tau_{ii}s_{1_i} = P(1_i) = 1_{P_i}$ by (5.12). For compatibility of μ , we calculate that

$$\sigma_{ij}\pi_2\tau_{ij}^*(s_g)s_f = \theta_{(f,g),(f,g)} = \overline{s_{gf}} = \sigma_{ik}s_{gf} = \sigma_{ik}\mu_{ijk}\tau_{ij}^*(s_g)s_f$$

for all $f \in \mathcal{A}(i, j)$ and $g \in \mathcal{A}(j, k)$. Since $\{\tau_{ij}^*(s_g)s_f\}$ is a basis for $\sigma_{ij}\pi_2$, we have $\sigma_{ij}\pi_2 = \sigma_{ik}\mu_{ijk}$ by Lemma 3.15. Similarly, $\tau_{ik}\mu_{ijk} = \tau_{jk}\pi_1$ follows since

$$\tau_{jk}\pi_1\tau_{ij}^*(s_g)s_f = \tau_{jk}s_g\tau_{ij} s_f = Pg \circ Pf = P(gf) = \tau_{ik}s_{gf} = \tau_{ik}\mu_{ijk}\tau_{ij}^*(s_g)s_f .$$

ΦP verifies the category axioms. For this, it will be convenient to borrow the notation of Definition 5.6 and write $t * s$ for a composite of the form $\mu_{ijk} \circ \tau_{ij}^*(t) \circ s$. In these terms, (5.13) states that $s_g * s_f = s_{gf}$. Now, to verify the left unit law for ΨP , we calculate that, for all $f \in \mathcal{A}(i, j)$ we have

$$s_f = s_{1_j}f = s_{1_j} * s_f = \eta_j * s_f = \mu_{ijj}(\eta_j\tau_{ij}, 1)s_f$$

where the last equality is (5.4). It follows that $\mu_{ijj}(\eta_j\tau_{ij}, 1) = 1$ by Lemma 3.4. For the right unit law, we calculate similarly that, for all $f \in \mathcal{A}(i, j)$, we have

$$s_f = s_{f1_i} = s_f * s_{1_i} = s_f * \eta_i = \mu_{ijj}(1, \eta_i\sigma_{ij})s_f$$

where the last equality is (5.5). Finally, for associativity, given $f \in \mathcal{A}(i, j)$, $g \in \mathcal{A}(j, k)$ and $h \in \mathcal{A}(k, \ell)$ we have that

$$s_{hgf} = s_h * (s_g * s_f) = \mu_{ik\ell} \circ (\mu_{ijk} \times 1) \circ \mu_{ijk}^* \tau_{ik}^*(s_h) \circ \tau_{ij}^*(s_g) \circ s_f$$

and $s_{hgf} = (s_h * s_g) * s_f = \mu_{ij\ell} \circ (1 \times \mu_{jkl}) \circ \mu_{ijk}^* \tau_{ik}^*(s_h) \circ \tau_{ij}^*(s_g) \circ s_f$,

where the last step in each line comes from (5.6), respectively (5.7). The composition axiom follows since, by Lemma 3.18, the family of partial sections $\{\mu_{ijk}^* \tau_{ik}^*(s_h) \circ \tau_{ij}^*(s_g) \circ s_f\}$ constitutes a basis for the local homeomorphism

$$\Psi P_{k\ell} \times_{P_k} \Psi P_{jk} \times_{P_j} \Psi P_{ij} \xrightarrow{\pi_{23}} \Psi P_{jk} \times_{P_j} \Psi P_{ij} \xrightarrow{\pi_2} \Psi P_{ij} \xrightarrow{\sigma_{ij}} P_i.$$

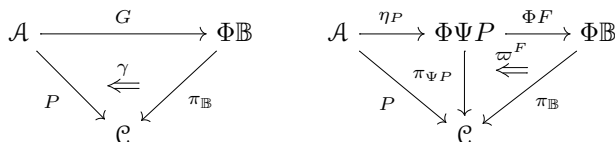
$\eta_P : \mathcal{A} \rightarrow \Phi(\Psi P)$ is a restriction functor, and (5.14) commutes. Unitality of η_P is the fact that $s_{1_i} = \eta_i$; binary functoriality is precisely the fact (5.13) that $s_g * s_f = s_{gf}$. Preservation of restriction follows as $s_{\bar{f}} = s_{1_i\bar{f}} = s_{1\bar{f}} = \eta_{1_i}\bar{f} = \tilde{s}_{\bar{f}}$. Finally, (5.14) clearly commutes on objects and commutes on maps by (5.12). \square

We are now in a position to prove:

Theorem 5.14. For each $P : \mathcal{A} \rightarrow \mathcal{C}$ in rCat/\mathcal{C} , the map (5.14) exhibits ΨP as the value at P of a left adjoint to $\Phi : \text{pCat}_{\mathcal{C}}(\mathcal{C}) \rightarrow \text{rCat}/\mathcal{C}$, yielding an adjunction

$$\text{pCat}_{\mathcal{C}}(\mathcal{C}) \xleftarrow[\Phi]{\Psi} \text{rCat}/\mathcal{C}. \tag{5.16}$$

Proof. Let \mathbb{B} be a J -partite internal category in \mathcal{C} . We must show that every lax-commuting triangle as on the left in



factors as to the right for a unique partite cofunctor $F : \Psi P \rightsquigarrow \mathbb{B}$. We first show that the desired factorisation forces the definition of F .

F must act on component-sets by $i \mapsto Gi$. This is as $\Phi F \circ \eta_P = G$ on objects.

F must have action on objects given by $F_i = \gamma_i : B_{Gi} \rightarrow P_i$. This follows since $\gamma = \varpi^F \circ \eta_P$ and η_P is the identity on objects, so that $\gamma_i = \varpi_i^F = F_i$.

F's action on arrows $F_{ij} : \Psi P_{ij} \times_{P_i} B_{G_i} \rightarrow B_{G_i, G_j}$ **is uniquely determined by**

$$Gf = B_{G_i} \xrightarrow{\gamma_i^*(s_f)} \Psi P_{ij} \times_{P_i} B_{G_i} \xrightarrow{F_{ij}} B_{G_i, G_j} \quad \text{for all } f \in \mathcal{A}(i, j). \tag{5.17}$$

Indeed, the displayed equalities are forced since $\Phi F \circ \eta_P = G$ on morphisms; but by Lemma 3.18, $\{\gamma_i^*(s_f) : f \in \mathcal{A}(i, j)\}$ is a basis of partial sections for the local homeomorphism $\pi_2 : \Psi P_{ij} \times_{P_i} B_{G_i} \rightarrow B_{G_i}$, so that there is at most one map F_{ij} verifying these equalities. So the data of F are uniquely determined; we now check that these data underlie a well-defined cofunctor.

The F_{ij} 's are well-defined. The local atlas ψ for $\pi_2 : \Psi P_{ij} \times_{P_i} B_{G_i} \rightarrow B_{G_i}$ associated to the basis of local sections $\{\gamma_i^*(s_f)\}$ is, by Lemma 3.18, given by

$$\psi_{f,g} = \overline{[f =_P g] \gamma_i} \quad \text{for } f, g \in \mathcal{A}(i, j).$$

We must verify that the family of maps $(Gf : f \in \mathcal{A}(i, j))$ satisfies the conditions of Lemma 3.4 with respect to this local atlas. First, by naturality and totality of γ , and totality of τ_{ij} , we have for any $f \in \mathcal{A}(i, j)$ that

$$\overline{Pf \circ \gamma_i} = \overline{\gamma_j \circ \pi_{\mathbb{B}}(G(f))} = \overline{\gamma_j \circ \tau_{ij} \circ Gf} = \overline{Gf}$$

It follows by definition of $[f =_P g]$ and distributivity of joins that

$$\psi_{fg} = \overline{[f =_P g] \gamma_i} = \bigvee \{ \overline{Pe \circ \gamma_i} : e \leq \bar{f}\bar{g}, fe = ge \} = \bigvee \{ \overline{Ge} : e \leq \bar{f}\bar{g}, fe = ge \}$$

whence $\psi_{ff} = \overline{Gf}$ and

$$\begin{aligned} Gg \circ \psi_{fg} &= \bigvee \{ Gg \circ \overline{Ge} : e \leq \bar{f}\bar{g}, fe = ge \} = \bigvee \{ Gg * Ge : e \leq \bar{f}\bar{g}, fe = ge \} \\ &= \bigvee \{ G(ge) : e \leq \bar{f}\bar{g}, fe = ge \} = \bigvee \{ G(fe) : e \leq \bar{f}\bar{g}, fe = ge \} \leq Gf \end{aligned}$$

where on the first line, we have $Gg \circ \overline{Ge} = Gg * \widetilde{Ge} = Gg * Ge$ (composition in \mathcal{C}) just as in the proof of Proposition 5.8.

The maps F_{ij} are compatible with source and target. For the source axiom $\sigma_{G_i, G_j} F_{ij} = \pi_2$, we calculate that

$$\sigma_{G_i, G_j} F_{ij} \gamma_i^*(s_f) = \sigma_{G_i, G_j} Gf = \overline{Gf} = \psi_{ff} = \pi_2 \gamma_i^*(s_f)$$

and apply Lemma 3.4. For the target axiom $\tau_{ij} \pi_1 = \gamma_j \tau_{G_i, G_j} F_{ij}$, we have

$$\tau_{ij} \pi_1 \gamma_i^*(s_f) = \tau_{ij} s_f \gamma_i = Pf \gamma_i = \gamma_j \tau_{G_i, G_j} Gf = \gamma_j \tau_{G_i, G_j} \widetilde{G}_{ij} \gamma_i^*(s_f)$$

whence the result, again by Lemma 3.4.

F satisfies the unit and multiplication axioms. For the unit axiom, we have

$$\eta_{G_i} = G(1_i) = F_{ii}\gamma_i^*(s_{1_i}) = F_{ii}(\eta_i G_i, 1)$$

as required. Finally, for the composition axiom, let $f \in \mathcal{A}(i, j)$ and $g \in \mathcal{A}(j, k)$. By exactly the same calculations as in (5.10) and (5.11), we have that

$$\begin{aligned} Gg * Gf &= \mu_{G_i, G_j, G_k} \circ (F_{jk} \times 1) \circ (1 \times F_{ij}) \circ \pi_1^* \tau_{ij}^*(s_g) \circ \gamma_i^*(s_f) \\ G(gf) &= F_{ik} \circ (1 \times \mu_{jkl}) \circ \pi_1^* \tau_{ij}^*(s_g) \circ \gamma_i^*(s_f), \end{aligned}$$

and since $Gg * Gf = G(gf)$, the composites to the right are equal. But by Lemma 3.18, the partial sections $\{\pi_1^* \tau_{ij}^*(s_g) \gamma_i^*(s_f)\}$ are a basis for the local homeomorphism $\pi_3: \Psi P_{jk} \times P_j \Psi P_{ij} \times P_i B_{P_i} \rightarrow B_{P_i}$, we conclude by Lemma 3.4 that $\mu_{G_i, G_j, G_k} \circ (F_{jk} \times 1) \circ (1 \times F_{ij}) = F_{ik} \circ (1 \times \mu_{jkl})$ as required. \square

5.4. The fixpoints

We have thus succeeded in constructing the adjunction (5.16); we now identify its left and right fixpoints.

Proposition 5.15. *The unit (5.14) at $P: \mathcal{A} \rightarrow \mathcal{C}$ of the adjunction $\Psi \dashv \Phi$ is invertible if and only if P lies in $\text{jrCat}/\!/_{h}\mathcal{C}$.*

Proof. If η_P is invertible then \mathcal{A} is in $\text{jrCat}/\!/_{h}\mathcal{C}$ since the isomorphic $\Phi\Psi P$ is so. Suppose conversely that P is in $\text{jrCat}/\!/_{h}\mathcal{C}$; we must show that η_P is invertible. Since it is already the identity on objects, we need only prove full fidelity. Looking at the action on maps of η_P , this means showing that each local section of $\sigma_{ij}: \Psi P_{ij} \rightarrow P_i$ is of the form s_f for a unique $f \in \mathcal{A}(i, j)$.

As in Example 4.4, since P is hyperconnected, the hom-set $\mathcal{A}(i, j)$ is a sheaf on $P_i \in \mathcal{C}$, and by Remark 5.10 the equality $\llbracket f = g \rrbracket$ of this sheaf is the same as the P -valued equality $\llbracket f =_P g \rrbracket$. Thus, the local homeomorphism $\sigma_{ij}: \Psi P_{ij} \rightarrow P_i$ is exactly $\Delta(\mathcal{A}(i, j)) \rightarrow P_i$. So by Lemma 4.5, each partial section of σ_{ij} is of the form s_f for a unique $f \in \mathcal{A}(i, j)$ as desired. \square

Proposition 5.16. *The counit $\varepsilon_{\mathbb{A}}: \Psi\Phi\mathbb{A} \rightarrow \mathbb{A}$ of the adjunction $\Psi \dashv \Phi$ is invertible at a partite internal category \mathbb{A} if and only if \mathbb{A} is source-étale.*

Proof. The partite category $\Psi\Phi\mathbb{A}$ has the same indexing set I as \mathbb{A} and the same objects of objects $(A_i : i \in I)$. Since $\pi_{\mathbb{A}}: \Phi\mathbb{A} \rightarrow \mathcal{C}$ is hyperconnected, the discussion of the preceding proposition entails that the source projections $\Psi\Phi\mathbb{A}_{ij} \rightarrow A_i$ of $\Psi\Phi\mathbb{A}$ are obtained as the glueing of the sheaf $\Phi\mathbb{A}(i, j)$ on $A_i \in \mathcal{C}$. By inspection, this is precisely the sheaf $\Gamma(\sigma_{ij})$ of sections of $\sigma_{ij}: A_{ij} \rightarrow A_i$, so that $\Psi\Phi\mathbb{A}_{ij} \rightarrow A_i$ is exactly $p_{\Gamma(\sigma_{ij})}: \Delta\Gamma(\sigma_{ij}) \rightarrow A_i$.

In these terms, the counit $\varepsilon_{\mathbb{A}} : \Psi\Phi\mathbb{A} \rightarrow \mathbb{A}$ is the partite cofunctor which is the identity on components, the identity on objects, and has action on arrows $\Psi\Phi\mathbb{A}_{ij} \rightarrow A_{ij}$ given by the counit maps

$$\begin{array}{ccc} \Delta\Gamma(\sigma_{ij}) & \xrightarrow{\varepsilon_{\sigma_{ij}}} & A_{ij} \\ & \searrow p_{\Gamma(\sigma_{ij})} & \swarrow \sigma_{ij} \\ & & A_i \end{array}$$

of the adjunction $\Delta \dashv \Gamma$ of (4.2). Clearly $\varepsilon_{\mathbb{A}}$ is invertible if and only if each $\varepsilon_{\sigma_{ij}}$ as displayed is so; which, by Proposition 4.14, happens just when each σ_{ij} is a local homeomorphism, as desired. \square

From this our main theorem immediately follows:

Theorem 5.17. *Restricting (5.16) to its fixpoints yields an equivalence*

$$\text{peCat}_c(\mathcal{C}) \xrightleftharpoons[\Phi]{\Psi} \text{jrCat} // {}_h\mathcal{C} \tag{5.18}$$

between the category of source-étale partite internal categories in \mathcal{C} , and the category of join restriction categories hyperconnected over \mathcal{C} .

Since each ΨP is source-étale, hence a left fixpoint, we also obtain:

Corollary 5.18. *The adjunction (5.16) is Galois. In particular, $\text{jrCat} // {}_h\mathcal{C}$ is reflective in $\text{rCat} // \mathcal{C}$ via $\Phi\Psi$, while $\text{peCat}_c(\mathcal{C})$ is coreflective in $\text{pCat}_c(\mathcal{C})$ via $\Psi\Phi$.*

By analogy with Section 4.3, the reflector $\Phi\Psi : \text{rCat} // \mathcal{C} \rightarrow \text{jrCat} // {}_h\mathcal{C}$ can be seen as “sheafification”, and much as there, we can describe it explicitly.

Proposition 5.19. *For any $P : \mathcal{A} \rightarrow \mathcal{C}$ in $\text{rCat} // \mathcal{C}$, its reflection $\pi_{\Psi P} : \Phi\Psi P \rightarrow \mathcal{C}$ into $\text{jrCat} // {}_h\mathcal{C}$ can be described as follows.*

- **Objects** of $\Phi\Psi P$ are those of \mathcal{A} ;
- **Morphisms** $\theta \in \Phi\Psi P(i, j)$ are families $(\theta_f \in \mathcal{O}(Pi) : f \in \mathcal{A}(i, j))$ such that

$$\theta_g \theta_f \leq \llbracket f =_P g \rrbracket \quad \text{and} \quad \theta_g \llbracket f =_P g \rrbracket \leq \theta_f \quad \text{for all } f, g \in \mathcal{A}(i, j) ; \tag{5.19}$$

- The **identity map** $1_i \in \Phi\Psi P(i, i)$ is given by the family $(1_i)_f = \llbracket 1_i =_P f \rrbracket$;
- **Composition** of $\theta \in \Phi\Psi P(i, j)$ and $\psi \in \Phi\Psi P(j, k)$ is given by the family

$$(\psi \circ \theta)_h = \bigvee_{\substack{f \in \mathcal{A}(i, j) \\ g \in \mathcal{A}(j, k)}} \llbracket gf =_P h \rrbracket \overline{\psi_g P f} \theta_f ;$$

- **Restriction** of $\theta \in \Phi\Psi P(i, j)$ is the family $\bar{\theta}_f = \llbracket 1_i =_P f \rrbracket \bigvee_g \theta_g$;
- The **projection** functor $\pi_{\Psi P} : \Phi\Psi P \rightarrow \mathcal{C}$ acts as P does on objects, and on morphisms by sending $\theta \in \Phi\Psi P(i, j)$ to $\bigvee_{f \in \mathcal{A}(i, j)} Pf \circ \theta_f$;
- The **unit** functor $\eta_P : \mathcal{A} \rightarrow \Phi\Psi P$ is the identity on objects, and on morphisms sends $g \in \mathcal{A}(i, j)$ to $(\llbracket f =_P g \rrbracket : f \in \mathcal{A}(i, j))$.

Proof. Objects of $\Phi\Psi P$ are clearly those of \mathcal{A} , while morphisms $i \rightarrow j$ are partial sections of $\sigma_{ij} : \Psi P_{ij} \rightarrow Pi$. This map has basis of local sections $(s_f : f \in \mathcal{A}(i, j))$ and associated local atlas $(\llbracket f =_P g \rrbracket : f, g \in \mathcal{A}(i, j))$; so by Lemma 3.16, its partial sections s correspond to families (θ_f) as in (5.19) via the assignments

$$s \mapsto (p_f s : f \in \mathcal{A}(i, j)) \quad \text{and} \quad (\theta_f : f \in \mathcal{A}(i, j)) \mapsto \bigvee_f s_f \theta_f .$$

Under this bijection, the identity section $s_{1_i} \in \Phi\Psi P(i, i)$ corresponds to the family with components $p_f s_{1_i} = \llbracket 1_i =_P f \rrbracket$, as desired.

For composition, if $\theta \in \Phi\Psi P(i, j)$ and $\psi \in \Phi\Psi P(j, k)$, then their composite is the family with component at $h \in \mathcal{A}(i, k)$ given by

$$p_h \left(\left(\bigvee_g s_g \psi_g \right) * \left(\bigvee_f s_f \theta_f \right) \right) = \bigvee_{f, g} p_h \left((s_g \psi_g) * (s_f \theta_f) \right) .$$

Now clearly $(s_g \psi_g) * (s_f \theta_f) \leq s_g * s_f = s_{gf}$, and its restriction satisfies

$$\pi_{\Psi P} \left(\overline{(s_g \psi_g) * (s_f \theta_f)} \right) = \overline{\pi_{\Psi P}(s_g \psi_g) \pi_{\Psi P}(s_f \theta_f)} = \overline{P_g \psi_g P_f \theta_f} = \overline{\psi_g P_f \theta_f}$$

so that we have $(s_g \psi_g) * (s_f \theta_f) = s_{gf} \overline{\psi_g P_f \theta_f}$ and thus $(\psi \circ \theta)_h$ given by

$$\bigvee_{f, g} p_h \left((s_g \psi_g) * (s_f \theta_f) \right) = \bigvee_{f, g} p_h s_{gf} \overline{\psi_g P_f \theta_f} = \bigvee_{f, g} \llbracket gf =_P h \rrbracket \overline{\psi_g P_f \theta_f}$$

as required. Further, the restriction of $\theta \in \Phi\Psi P(i, j)$ is the family

$$\bar{\theta}_f = p_f \eta_i \overline{\bigvee_g s_g \theta_g} = p_f s_{1_i} \bigvee_g \theta_g = \llbracket 1_i =_P f \rrbracket \bigvee_g \theta_g .$$

This proves that $\Phi\Psi P$ is as described. Next consider $\pi_{\Psi P} : \Phi\Psi P \rightarrow \mathcal{C}$. This acts as P does on objects, while on maps it sends $\theta \in \Phi\Psi P(i, j)$, corresponding to the partial section $\bigvee_f s_f \theta_f$, to $\tau_{ij} \bigvee_f s_f \theta_f = \bigvee_f Pf \circ \theta_f$ as desired.

Finally, consider $\eta_P : \mathcal{A} \rightarrow \Phi\Psi P$; this is the identity on objects, and sends $g \in \mathcal{A}(i, j)$ to the partial section s_g , corresponding to the family of idempotents $(p_f s_g : f \in \mathcal{A}(i, j)) = (\llbracket f =_P g \rrbracket : f \in \mathcal{A}(i, j))$ as claimed. \square

6. The groupoid case

As explained in the introduction, our main result generalises along four different axes the correspondence between étale topological groupoids and pseudogroups. We now begin

to examine the effect of rolling back these generalisations. Once again, \mathcal{C} will be any join restriction category with local glueings.

It is trivial to see that (5.18) restricts back to an equivalence between source-étale (1-partite) internal categories in \mathcal{C} and join restriction monoids hyperconnected over \mathcal{C} . More interesting are the adaptations required for the groupoidal case; the goal of this section is to describe these.

6.1. Groupoids and étale join restriction categories

To one side of our restricted adjunction and equivalence will be the partite internal groupoids:

Definition 6.1. An *I-partite internal groupoid* in the join restriction category \mathcal{C} is an *I-partite internal category* \mathbb{A} endowed with total maps $\iota_{ij} : A_{ij} \rightarrow A_{ji}$ which are involutions in the sense that $\iota_{ij}\iota_{ji} = 1$, are compatible with source and target in the sense that $\sigma_{ji}\iota_{ij} = \tau_{ij}$ (and so also $\tau_{ji}\iota_{ij} = \sigma_{ij}$), and which render commutative each square to the left in:

$$\begin{array}{ccc}
 A_{ij} & \xrightarrow{(\iota_{ij}, 1)} & A_{ji} \times_{A_j} A_{ij} \\
 \sigma_{ij} \downarrow & & \downarrow \mu_{iji} \\
 A_i & \xrightarrow{\eta_i} & A_{ii}
 \end{array}
 \qquad
 \begin{array}{ccc}
 A_{ij} & \xrightarrow{(1, \iota_{ij})} & A_{ij} \times_{A_i} A_{ji} \\
 \tau_{ij} \downarrow & & \downarrow \mu_{jij} \\
 A_j & \xrightarrow{\eta_j} & A_{jj}
 \end{array}$$

(and so also each square to the right). The usual argument adapts to show that the inverse maps ι_{ij} are unique, if they exist; and this justifies us in writing $\text{pSpd}_c(\mathcal{C})$ and $\text{peSpd}_c(\mathcal{C})$ for the full subcategories of $\text{pCat}_c(\mathcal{C})$ and $\text{peCat}_c(\mathcal{C})$ on the partite internal groupoids.

Remark 6.2. Since a source-étale partite internal groupoid satisfies $\sigma_{ij}\iota_{ij} = \tau_{ij}$, its target maps, as well as its source maps, are local homeomorphisms; and in fact, the groupoid axioms together with Lemma 4.2 ensure that *all* of the groupoid structure maps are local homeomorphisms. Thus, we may say “étale” rather than “source-étale” when dealing with partite internal groupoids.

The entities to the other side of the equivalence can be described in two different ways: either as the join inverse categories hyperconnected over \mathcal{C} , or as the étale join restriction categories hyperconnected over \mathcal{C} (recall from Definition 2.23 that a join restriction category is *étale* if every map therein is a join of partial isomorphisms). While the two formulations are equivalent by Corollary 2.24, each has its own advantages, and so we will discuss both. We first consider the one involving étale join restriction categories; this has the advantage of being a genuine special case of (5.18).

Theorem 6.3. *The equivalence (5.18) restricts back to an equivalence*

$$\text{peGpd}_c(\mathcal{C}) \begin{array}{c} \xleftarrow{\Psi} \\ \sim \\ \xrightarrow{\Phi} \end{array} \text{ejCat} // {}_h\mathcal{C} \tag{6.1}$$

between the category of étale partite internal groupoids in \mathcal{C} and the category of étale join restriction categories hyperconnected over \mathcal{C} .

Before proving this, we need a couple of preliminary lemmas. The first is a purely technical result about equality of partial isomorphisms; the second is rightly a part of the theorem, but we separate it out for later re-use.

Lemma 6.4. *If $f, g: A \rightarrow B$ are partial isomorphisms in a join restriction category then $f \llbracket f = g \rrbracket = \llbracket f^* = g^* \rrbracket f$.*

Proof. Recall from Lemma 2.8 that $f \llbracket f = g \rrbracket$ is the meet $f \wedge g$ in $\mathcal{C}(A, B)$, so it suffices to prove $\llbracket f^* = g^* \rrbracket f$ is too. Clearly it is below f , while from $g^* \llbracket f^* = g^* \rrbracket = f^* \llbracket f^* = g^* \rrbracket$ we conclude by taking partial inverses that $\llbracket f^* = g^* \rrbracket f = \llbracket f^* = g^* \rrbracket g \leq g$.

Suppose now that $h \in \mathcal{C}(A, B)$ is below f and g . Such an h is itself a partial isomorphism with $h^* = \overline{h}f^* = \overline{h}g^*$. It follows that $h^* \leq f^* \wedge g^*$, i.e., $\overline{h^*} \leq \llbracket f^* = g^* \rrbracket$, and so that $h = h\overline{f} = hf^*f = h\overline{h}f^*f = hh^*f = \overline{h^*}f \leq \llbracket f^* = g^* \rrbracket f$ as desired. \square

Lemma 6.5. *If \mathbb{A} is a partite internal groupoid in \mathcal{C} , then $\pi_{\mathbb{A}}: \Phi\mathbb{A} \rightarrow \mathcal{C}$ reflects partial isomorphisms and étale maps.*

Proof. Let $f \in \Phi\mathbb{A}(i, j)$ be such that $\pi_{\mathbb{A}}(f) = \tau_{ij}f: A_i \rightarrow A_j$ is a partial isomorphism in \mathcal{C} . Consider the composite

$$g := A_j \xrightarrow{(\tau_{ij}f)^*} X_i \xrightarrow{f} A_{ij} \xrightarrow{\iota_{ij}} A_{ji}$$

in \mathcal{C} . Note that $\sigma_{ji}g = \sigma_{ji}\iota_{ij}f(\tau_{ij}f)^* = \tau_{ij}f(\tau_{ij}f)^* = \overline{(\tau_{ij}f)^*} \leq 1$ so that g is a partial section of σ_{ji} with $\overline{g} = \overline{(\tau_{ij}f)^*}$. We claim that $g \in \Phi\mathbb{A}(j, i)$ is the desired partial inverse of $f \in \Phi\mathbb{A}(i, j)$. First we calculate that

$$\begin{aligned} g\tau_{ij}f &= \iota_{ij}f(\tau_{ij}f)^*\tau_{ij}f = \iota_{ij}f\overline{(\tau_{ij}f)^*} = \iota_{ij}f\overline{f} = \iota_{ij}f \\ \text{and } f\tau_{ji}g &= f\tau_{ji}\iota_{ij}f(\tau_{ij}f)^* = f\sigma_{ij}f(\tau_{ij}f)^* = f\overline{f}(\tau_{ij}f)^* = f(\tau_{ij}f)^* \end{aligned}$$

with respective restrictions

$$\overline{g\tau_{ij}f} = \overline{\iota_{ij}f} = \overline{f} \quad \text{and} \quad \overline{f\tau_{ji}g} = \overline{f(\tau_{ij}f)^*} = \overline{\tau_{ij}f(\tau_{ij}f)^*} = \overline{(\tau_{ij}f)^*} = \overline{g}$$

from which we obtain the desired equalities

$$\begin{aligned}
 f * g &= \mu_{jij} \tau_{ji}^*(f)g = \mu_{jij}(f \tau_{ji} g, g \overline{f \tau_{ji} g}) = \mu_{jij}(f(\tau_{ij} f)^*, g) \\
 &= \mu_{jij}(1, \iota_{ij})f(\tau_{ij} f)^* = \eta_j \tau_{ij} f(\tau_{ij} f)^* = \eta_j \bar{g} = \tilde{g} \\
 \text{and } g * f &= \mu_{iji} \tau_{ij}^*(g)f = \mu_{iji}(g \tau_{ij} f, f \overline{g \tau_{ij} f}) = \mu_{iji}(\iota_{ij} f, f) \\
 &= \mu_{iji}(\iota_{ij}, 1)f = \eta_i \sigma_{ij} f = \eta_i \bar{f} = \tilde{f}.
 \end{aligned}$$

So $\pi_{\mathbb{A}}$ reflects partial isomorphisms as claimed. Since it is also hyperconnected, it follows that it reflects étale maps. Indeed, if $\pi_{\mathbb{A}}(f : i \rightarrow j)$ is a join of partial isomorphisms $\bigvee_k p_k$, then taking $e_k \in \mathcal{O}(i)$ to be unique such that $\pi_{\mathbb{A}}(e_k) = \overline{p_k}$, we have $\pi_{\mathbb{A}}(f e_k) = p_k$ for each i . Since $\pi_{\mathbb{A}}$ reflects partial isomorphisms, each $f e_k$ is a partial isomorphism, and so $f = \bigvee_k f e_k$ is étale. \square

We are now ready to give:

Proof of Theorem 6.3. Suppose first that \mathbb{A} is an étale partite internal groupoid in \mathcal{C} ; we will show that every map in $\Phi \mathbb{A}$ is étale. Since each ι_{ij} is invertible and each σ_{ij} is a local homeomorphism, each $\tau_{ij} = \sigma_{ji} \iota_{ij}$ is a local homeomorphism. Now any $s \in \Phi \mathbb{A}(i, j)$ is a partial section of σ_{ij} , and hence a partial isomorphism, whence for each $s \in \Phi \mathbb{A}(i, j)$, the map $\pi_{\mathbb{A}}(s) = \tau_{ij} s$ is étale. Since by Lemma 6.5, $\pi_{\mathbb{A}}$ reflects étale maps, s is itself étale, as desired.

Suppose conversely that $P : \mathcal{A} \rightarrow \mathcal{C}$ is an étale join restriction category hyperconnected over \mathcal{C} ; we show that ΨP is a partite internal groupoid. As in Proposition 5.15, each partial section of the source map $\sigma_{ij} : \Psi P_{ij} \rightarrow Pi$ is of the form s_f for some $f \in \mathcal{A}(i, j)$; but since \mathcal{A} is étale, we can write each such s_f as $\bigvee_i s_{f_i}$ where each f_i is a partial isomorphism. Thus by Lemma 3.15 the family of sections $\{s_f : f \in \mathcal{A}(i, j) \text{ a partial isomorphism}\}$ is a basis for σ_{ij} . We can thus determine ι_{ij} by asking it to be the unique total map $\Psi P_{ij} \rightarrow \Psi P_{ji}$ such that

$$\iota_{ij} \circ s_f = s_{f^*} \circ f \quad \text{for all } f \in \mathcal{A}(i, j) \text{ a partial isomorphism.}$$

For this to be well-defined, we must check compatibility of the family of maps $s_{f^*} f : Pi \rightarrow \Psi P_{ji}$ with the associated local atlas for σ_{ij} ; by Proposition 5.15 again, this atlas is given by $\varphi_{fg} = \llbracket f = g \rrbracket$, and so we have

$$\begin{aligned}
 s_{g^*} g \llbracket f = g \rrbracket &= s_{g^*} f \llbracket f = g \rrbracket = s_{g^*} \llbracket f^* = g^* \rrbracket f = s_{g^*} \llbracket f^* = g^* \rrbracket f \leq s_{f^*} f \\
 \text{and } \overline{s_{f^*} f} &= \overline{s_{f^*} f} = \overline{f^* f} = \bar{f} = \llbracket f = f \rrbracket
 \end{aligned}$$

as required. So ι_{ij} is well-defined; it remains to check the groupoid axioms.

First, $\iota_{ji} \iota_{ij} s_f = \iota_{ji} s_{f^*} f = s_f f^* f = s_f \bar{f} = s_f$ for each partial isomorphism f , so that $\iota_{ji} \iota_{ij} = 1$ by Lemma 3.15. Similarly, we have that $\sigma_{ij} \iota_{ij} s_f = \sigma_{ij} s_{f^*} f = \overline{f^* f} = f f^* f = f = \tau_{ij} s_f$, so that $\sigma_{ij} \iota_{ij} = \tau_{ij}$. Finally, we have

$$\begin{aligned} \mu_{iji}(l_{ij}, 1)s_f &= \mu_{iji}(s_f * f, s_f \overline{f}) = \mu_{iji}(s_f * \tau_{ij} s_f, s_f \overline{s_f * \tau_{ij} s_f}) = \mu_{iji}(s_f * \tau_{ij}, \overline{s_f * \tau_{ij}}) s_f \\ &= \mu_{iji} \tau_{ij}^*(s_f^*) s_f = s_f^* * s_f = s_f * f = s_f \overline{f} = \eta_i \overline{f} = \eta_i \sigma_{ij} s_f \end{aligned}$$

for each partial isomorphism f , so that $\mu_{iji}(l_{ij}, 1) = \eta_i \sigma_{ij}$, as required. \square

6.2. Groupoids and join inverse categories

We now turn to the formulation of the equivalence (6.1) involving inverse categories. This account has two advantages: it matches up more closely with the classical étale groupoid–pseudogroup correspondence, and it allows us to give a groupoid version not only of the equivalence (5.18), but also of the adjunction (5.16). The disadvantage is that we have to alter the adjunction slightly.

Definition 6.6. Let \mathbb{A} be a partite internal groupoid in \mathcal{C} . A *partial bisection* of the source map $\sigma_{ij} : A_{ij} \rightarrow A_i$ is a partial section $s : A_i \rightarrow A_{ij}$ for which the composite $\tau_{ij}s$ is a partial isomorphism. We write $\Phi_g \mathbb{A}$ for the subcategory of $\Phi \mathbb{A}$ with the same objects, but only those maps $s \in \Phi \mathbb{A}(i, j)$ which are partial bisections, and reuse the notation $\pi_{\mathbb{A}} : \Phi_g \mathbb{A} \rightarrow \mathcal{C}$ for the functor sending s to $\tau_{ij}s$.

Proposition 6.7. *The assignment $\mathbb{A} \mapsto (\pi_{\mathbb{A}} : \Phi_g \mathbb{A} \rightarrow \mathcal{C})$ is the action on objects of a well-defined functor $\Phi_g : \text{pSpd}_{\mathcal{C}}(\mathcal{C}) \rightarrow \text{iCat} // \mathcal{C}$ taking values in $\text{jiCat} // {}_h \mathcal{C}$.*

Proof. Note that $\Phi_g \mathbb{A}$ comprises just those maps of $\Phi \mathbb{A}$ which $\pi_{\mathbb{A}} : \Phi \mathbb{A} \rightarrow \mathcal{C}$ sends to partial isomorphisms. Since by Lemma 6.5, $\pi_{\mathbb{A}}$ reflects partial isomorphisms, we conclude that $\Phi_g \mathbb{A} = \text{PIso}(\Phi \mathbb{A})$. We can thus obtain Φ_g as

$$\Phi_g = \text{pSpd}_{\mathcal{C}}(\mathcal{C}) \xrightarrow{\Phi} \text{rCat} // \mathcal{C} \xrightarrow{\text{PIso} // \mathcal{C}} \text{iCat} // \mathcal{C} \tag{6.2}$$

where the second component is the functor which acts on objects by restricting back $P : \mathcal{A} \rightarrow \mathcal{C}$ along the inclusion $\text{PIso}(\mathcal{A}) \subseteq \mathcal{A}$. By Proposition 2.19 and the fact that $\text{PIso}(\mathcal{A}) \subseteq \mathcal{A}$ is a hyperconnected inclusion, $\text{PIso} // \mathcal{C}$ maps $\text{jrCat} // {}_h \mathcal{C}$ into $\text{jiCat} // {}_h \mathcal{C}$; it follows that the image of Φ_g must land in $\text{jiCat} // {}_h \mathcal{C}$. \square

We now use this result to exhibit an adjunction whose fixpoints will provide the desired equivalence $\text{peSpd}_{\mathcal{C}}(\mathcal{C}) \simeq \text{jiCat} // {}_h \mathcal{C}$.

Theorem 6.8. *The functor $\Psi : \text{rCat} // \mathcal{C} \rightarrow \text{pCat}_{\mathcal{C}}(\mathcal{C})$ maps $\text{iCat} // \mathcal{C}$ into $\text{pSpd}_{\mathcal{C}}(\mathcal{C})$, and the restricted functor induced in this way provides a left adjoint to Φ_g in*

$$\text{pSpd}_{\mathcal{C}}(\mathcal{C}) \begin{array}{c} \xleftarrow{\Psi} \\ \perp \\ \xrightarrow{\Phi_g} \end{array} \text{iCat} // \mathcal{C} . \tag{6.3}$$

Proof. Let $P: \mathcal{A} \rightarrow \mathcal{C}$ be an object of iCat/\mathcal{C} and consider $\Psi P \in \text{rCat}(\mathcal{C})$. Each $\sigma_{ij}: \Psi P_{ij} \rightarrow Pi$ has a basis $\{s_f : f \in \mathcal{A}(i, j) \text{ a partial isomorphism}\}$ since every map in \mathcal{A} is a partial isomorphism. So the same proof as in Theorem 6.3 shows that ΨP is a partite internal groupoid. For the final claim, consider the adjunction

$$\text{pCat}_c(\mathcal{C}) \begin{matrix} \xleftarrow{\Psi} \\ \perp \\ \xrightarrow{\Phi} \end{matrix} \text{rCat}/\mathcal{C} \begin{matrix} \xleftarrow{\text{include}} \\ \perp \\ \xrightarrow{\text{PIso}/\mathcal{C}} \end{matrix} \text{iCat}/\mathcal{C}.$$

We just showed that the left adjoint lands in $\text{pSpd}_c(\mathcal{C})$, and by (6.2), the right adjoint agrees with Φ_g on $\text{pSpd}_c(\mathcal{C})$. This yields the desired adjunction (6.3). \square

And we now deduce:

Theorem 6.9. *Restricting (6.3) to its fixpoints yields an equivalence*

$$\text{peSpd}_c(\mathcal{C}) \begin{matrix} \xleftarrow{\Psi} \\ \sim \\ \xrightarrow{\Phi_g} \end{matrix} \text{jiCat}/_h\mathcal{C} \tag{6.4}$$

between the category of étale partite internal groupoids in \mathcal{C} , and the category of join inverse categories hyperconnected over \mathcal{C} .

Proof. The left and right fixpoints are contained in the respective essential images of Ψ and Φ_g , which are in turn contained in peSpd_c and $\text{jiCat}/_h\mathcal{C}$ respectively. So we have a restricted adjunction as in (6.4), and it suffices to show this is an equivalence. But the right adjoint Φ_g is equally the composite of the equivalence $\Phi: \text{peSpd}_c(\mathcal{C}) \rightarrow \text{ejCat}/_h\mathcal{C}$ of Theorem 6.3 with the equivalence $\text{PIso}/_h\mathcal{C}: \text{ejCat}/_h\mathcal{C} \rightarrow \text{jiCat}/_h\mathcal{C}$ obtained from Corollary 2.24. \square

Corollary 6.10. *The adjunction (6.3) is Galois. In particular, $\text{jiCat}/_h\mathcal{C}$ is reflective in iCat/\mathcal{C} via $\Phi_g\Psi$, while $\text{peSpd}_c(\mathcal{C})$ is coreflective in $\text{pSpd}_c(\mathcal{C})$ via $\Psi\Phi_g$.*

Like in Section 5.4, we can describe “sheaffication” $\Phi_g\Psi: \text{iCat}/\mathcal{C} \rightarrow \text{jiCat}/_h\mathcal{C}$ explicitly. As in Proposition 6.7, we have $\Phi_g\Psi P = \text{PIso}(\Phi\Psi P)$, and so our description of the sheaffication will follow from:

Lemma 6.11. *Let $P: \mathcal{A} \rightarrow \mathcal{C}$ in iCat/\mathcal{C} . A map $\theta \in \Phi\Psi P(i, j)$, given by a family $(\theta_f \in \mathcal{O}(Pi) : f \in \mathcal{A}(i, j))$ satisfying (5.19), is a partial isomorphism just when*

$$(Pf \circ \theta_f \circ Pf^*)(Pg \circ \theta_g \circ Pg^*) \leq \llbracket f^* =_P g^* \rrbracket \quad \text{for all } f, g \in \mathcal{A}(i, j). \tag{6.5}$$

Proof. Since ΨP is an internal groupoid by Theorem 6.8, $\pi_{\Psi P}: \Phi\Psi P \rightarrow \mathcal{C}$ reflects partial isomorphisms by Lemma 6.5. So $\theta \in \Phi\Psi P(i, j)$ is a partial isomorphism just when

$\pi_{\Psi P}(\theta) = \bigvee_f Pf \circ \theta_f$ is so. In turn, this will be so just when the compatible family of partial isomorphisms $(Pf \circ \theta_f : f \in \mathcal{A}(i, j))$ is bicompatible. This is true just when, for all $f, g \in \mathcal{A}(i, j)$, we have $\theta_f \circ Pf^* \circ \overline{\theta_g \circ Pg^*} \leq Pg^*$, or, equivalently, when

$$\theta_f \circ Pf^* \circ Pg \circ \theta_g \circ Pg^* \leq Pg^* .$$

But $\theta_f \circ Pf^* = Pf^* \circ \overline{\theta_f \circ Pf^*} = Pf^* \circ Pf \circ \theta_f \circ Pf^*$, so this happens just when

$$Pf^*(Pf \circ \theta_f \circ Pf^*)(Pg \circ \theta_g \circ Pg^*) \leq Pg^* ,$$

i.e., precisely when the condition (6.5) holds, as claimed. \square

It follows that the reflector $\Phi_g \Psi : \text{iCat}/\mathcal{C} \rightarrow \text{jiCat}/\!/_h \mathcal{C}$ has exactly the same description as the reflector $\Phi \Psi : \text{rCat}/\mathcal{C} \rightarrow \text{jrCat}/\!/_h \mathcal{C}$ in Proposition 5.19, except that maps of $\Phi_g \Psi P$ are families satisfying not only (5.19), but also (6.5).

6.3. The one-object case

By combining the groupoidal and the one-object specialisations of our main result, we obtain an equivalence between étale internal groupoids in \mathcal{C} , and join restriction monoids hyperconnected over \mathcal{C} . Since this is the form of our theorem which most resembles the classical situation, we spell it out in detail. In the following definition, as in the introduction, we write $\mathcal{J}(X)$ for the join inverse monoid of partial automorphisms of some $X \in \mathcal{C}$.

Definition 6.12. The category $\mathcal{PsGrp}(\mathcal{C})$ has:

- As objects, *complete \mathcal{C} -pseudogroups* (S, X, θ) , comprising a join inverse monoid S , an object $X \in \mathcal{C}$ and a monoid homomorphism $\theta : S \rightarrow \mathcal{J}(X)$ which restricts to an isomorphism between the idempotents of S and the idempotents of $\mathcal{J}(X)$.
- As maps $(S, X, \theta) \rightarrow (T, Y, \gamma)$, pairs (f, g) of a monoid homomorphism $f : S \rightarrow T$ and a total map $g : Y \rightarrow X$ in \mathcal{C} such that, for all $s \in S$ with $f(s) = t$, we have $\theta(s)g = g\gamma(t)$.

Theorem 6.13. *The equivalence (6.4) restricts to an equivalence*

$$\text{eSpd}_{\mathcal{C}}(\mathcal{C}) \xrightleftharpoons[\Phi_g]{\Psi} \mathcal{PsGrp}(\mathcal{C}) \tag{6.6}$$

between the category of étale internal groupoids and internal cofunctors in \mathcal{C} , and the category of complete \mathcal{C} -pseudogroups and their homomorphisms.

When $\mathcal{C} = \mathcal{L}oc_p$, this result recovers exactly the correspondence between localic étale groupoids and their comorphisms, and *complete abstract pseudogroups*, established by Resende in [29,30]: see Section 8.2 below. In the closely related case where $\mathcal{C} = \mathcal{T}op_p$, our result goes beyond [25] due to the extended functoriality of our correspondence. We now explain how the narrower functoriality may be re-found.

7. Localic and hyperconnected maps

7.1. Covering functors and localic morphisms

In [25], the morphisms considered between étale topological groupoids are not cofunctors, but the so-called *covering functors*. We now describe the analogue of this notion in our context.

Definition 7.1. Let \mathbb{B} be an I -partite internal category and \mathbb{A} a J -partite internal category in \mathcal{C} . A *partite internal functor* $G: \mathbb{B} \rightarrow \mathbb{A}$ comprises an assignation on components $i \mapsto Gi$, assignations on objects $G_i: B_i \rightarrow A_{Gi}$, and assignations on morphisms $G_{ij}: B_{ij} \rightarrow A_{G_i, G_j}$, subject to the following partite analogues of the usual functor axioms (for all $i, j, k \in I$):

$$\begin{aligned} G_i \sigma_{ij} &= \sigma_{G_i, G_j} G_{ij} & G_{ii} \eta_i &= \eta_{G_i} G_i \\ G_j \tau_{ij} &= \tau_{G_i, G_j} G_{ij} & G_{ik} \mu_{ijk} &= \mu_{G_i, G_j, G_k} (G_{jk} \times_{G_j} G_{ij}) . \end{aligned}$$

A partite internal functor is a *covering functor* if the assignation on components $i \mapsto Gi$ is invertible, and each square as below is a pullback in \mathcal{C} . We write $\text{pCat}_{\text{cov}}(\mathcal{C})$ for the category of partite internal categories and partite covering functors in \mathcal{C} .

$$\begin{array}{ccc} B_{ij} & \xrightarrow{G_{ij}} & A_{G_i, G_j} \\ \sigma_{ij} \downarrow & & \downarrow \sigma_{G_i, G_j} \\ B_i & \xrightarrow{G_i} & A_{G_i} . \end{array} \tag{7.1}$$

This is an appropriate generalisation of the notion in [25]: indeed, when $\mathcal{C} = \mathcal{T}op_p$, the covering functors between 1-partite internal groupoids are precisely the covering functors of [25], p. 126. We now show that partite covering functors can be identified with a special class of partite internal cofunctors.

Definition 7.2. Let \mathbb{A} and \mathbb{B} be internal partite categories in \mathcal{C} , and $F: \mathbb{A} \rightsquigarrow \mathbb{B}$ an internal cofunctor. We say that F is:

- *Bijjective on components* if the function $i \mapsto Fi$ is invertible;
- *Bijjective on objects* if each map $F_i: B_{Fi} \rightarrow A_i$ is invertible;

- *Bijjective on arrows* if each map $F_{ij} : A_{ij} \times_{A_i} B_{F_i} \rightarrow B_{F_i, F_j}$ is invertible.

Proposition 7.3. *The category $\text{pCat}_{\text{cov}}(\mathcal{C})$ is contravariantly isomorphic to the subcategory of $\text{pCat}_c(\mathcal{C})$ comprising all objects together with the maps which are bijective on components and arrows.*

Proof. Given a partite cofunctor $F : \mathbb{A} \rightsquigarrow \mathbb{B}$ which is bijective on components and arrows, the corresponding covering functor $G : \mathbb{B} \rightarrow \mathbb{A}$ has action on components and objects determined by the formulae $G(Fi) = i$ and $G_{F_i} = F_i : B_{F_i} \rightarrow A_i$, while its action on arrows is determined by

$$G_{F_i, F_j} = B_{F_i, F_j} \xrightarrow{F_{ij}^{-1}} A_{ij} \times_{A_i} B_{F_i} \xrightarrow{\pi_1} A_{ij}.$$

Conversely, if $G : \mathbb{B} \rightarrow \mathbb{A}$ is a partite covering functor, the corresponding partite cofunctor $F : \mathbb{A} \rightsquigarrow \mathbb{B}$ has action on components and objects determined by $F(Gi) = i$ and $F_{G_i} = G_i : B_i \rightarrow A_{G_i}$, and action on arrows determined by

$$F_{G_i, G_j} = (\pi_1, \pi_2) : A_{G_i, G_j} \times_{A_{G_i}} B_i \xrightarrow{\cong} B_{ij}$$

the unique isomorphism induced by (5.19)’s being a pullback. It is easy to check that $F \mapsto G$ and $G \mapsto F$ are well-defined, mutually inverse assignments. \square

Now, under our main equivalence $\text{peCat}_c(\mathcal{C}) \simeq \text{jrCat} // {}_h\mathcal{C}$, the subcategory of $\text{peCat}_c(\mathcal{C})$ on the bijective-on-components-and-arrows cofunctors corresponds to some subcategory of $\text{jrCat} // {}_h\mathcal{C}$ —one which, by the preceding result, is contravariantly equivalent to $\text{pCat}_{\text{cov}}(\mathcal{C})$. We now identify the maps in this subcategory.

Definition 7.4. A join restriction functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between join restriction categories is said to be *localic* if it is bijective on objects, and moreover:

- (i) For all $f, g \in \mathcal{A}(i, j)$, we have $Ff \wedge Fg = F(f \wedge g)$;
- (ii) For all $g \in \mathcal{B}(Fi, Fj)$, we have $g = \bigvee_{f \in \mathcal{A}(i, j)} g \wedge Ff$.

Here, \wedge denotes meet with respect to the natural partial order on each hom; these meets always exist by Lemma 2.8. Note that these morphisms are the natural generalisation of the *hypercallitic* morphisms of [25, p. 128].

We now show that, under the equivalence $\text{peCat}_c(\mathcal{C}) \simeq \text{jrCat} // {}_h\mathcal{C}$, the bijective-on-components-and-arrows partite cofunctors correspond to the maps

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\
 & \searrow & \swarrow \\
 & P & Q \\
 & \searrow & \swarrow \\
 & & \mathcal{C}
 \end{array}
 \quad (7.2)$$

of $\text{jrCat}/\!/_{h}\mathcal{C}$ for which F is localic. In fact, it will be convenient for later use to prove something slightly more general.

Proposition 7.5. *Consider a map (7.2) of $\text{jrCat}/\!/_{h}\mathcal{C}$. We have that:*

- $\Psi(F, \alpha)$ is bijective on components precisely when F is bijective on objects;
- $\Psi(F, \alpha)$ is bijective on objects precisely when α is invertible;
- $\Psi(F, \alpha)$ is bijective on arrows precisely when:
 - (i) For all $f, g \in \mathcal{A}(i, j)$, we have $Q(Ff \wedge Fg) = QF(f \wedge g)$;
 - (ii) For all $g \in \mathcal{B}(Fi, Fj)$, we have $Qg = \bigvee_{f \in \mathcal{A}(i, j)} Q(g \wedge Ff)$.

Proof. From the proof of Theorem 5.14, we see that $\Psi(F, \alpha)$ is the partite cofunctor which acts:

- On component-sets via the action of F on objects;
- On objects via the maps $F_i = \alpha_i: QFi \rightarrow Pi$;
- On arrows via the unique maps $F_{ij}: \Psi P_{ij} \times_{Pi} QFi \rightarrow \Psi Q_{Fi, Fj}$ for which

$$s_{Ff} = QFi \xrightarrow{\alpha_i^*(s_f)} \Psi P_{ij} \times_{Pi} QFi \xrightarrow{F_{ij}} \Psi Q_{Fi, Fj} \text{ for all } f \in \mathcal{A}(i, j). \tag{7.3}$$

The first two claims of the proposition are therefore immediate. For the third, recall that by Lemma 3.18, the family of sections $\{\alpha_i^*(s_f)\}$ constitute a basis for the local homeomorphism $\pi_2: \Psi P_{ij} \times_{Pi} QFi \rightarrow QFi$ with induced local atlas

$$\overline{[f =_P g]} \alpha_i = \overline{P[f = g]} \alpha_i = \overline{\alpha_i QF[f = g]} = QF[f = g],$$

here using Remark 5.10, naturality of α , and totality of α . Thus, by Lemma 3.4, F_{ij} is invertible precisely when the family of sections (7.3) are a basis for $\sigma_{Fi, Fj}: \Psi Q_{Fi, Fj} \rightarrow QFi$, and the induced local atlas is $QF[f = g]$.

Now, by construction, the family (7.3) induces the local atlas $[[Ff =_Q Fg]] = Q[[Ff = Fg]]$; so the latter requirement says that $Q[[Ff = Fg]] = QF[f = g]$, or equivalently, by Lemma 2.8(iv), that $Q(Ff \wedge Fg) = QF(f \wedge g)$. On the other hand, given that the sections $(s_g : g \in \mathcal{B}(Fi, Fj))$ are a basis for $\sigma_{Fi, Fj}$, to ask that $(s_{Ff} : f \in \mathcal{A}(i, j))$ is also a basis is equally, by Lemma 3.15, to ask that

$$s_g = \bigvee_{f \in \mathcal{A}(i, j)} s_{Ff} [[s_g = s_{Ff}]]: QFi \rightarrow \Psi Q_{Fi, Fj} \text{ for all } g \in \mathcal{B}(Fi, Fj).$$

Now, by Lemma 2.8(iv) we always have the inequality \geq above, and so have equality just when both sides have the same restriction. Since the sections $(s_g : g \in \mathcal{B}(Fi, Fj))$ generate the local atlas $Q[f = g]$, this is equally to ask that $Q\bar{g} = \bigvee_{f \in \mathcal{A}(i, j)} Q[g = Ff]$ which, by Lemma 2.8(iv), is equivalent to asking that $Qg = \bigvee_{f \in \mathcal{A}(i, j)} Q(g \wedge Ff)$. \square

Corollary 7.6. *Consider a morphism (7.2) of rCat/\mathcal{C} . If F is localic then $\Psi(F, \alpha)$ is bijective on components and arrows, and the converse is true whenever Q is hyperconnected. In particular, under the equivalence $\text{peCat}_c(\mathcal{C}) \simeq \text{jrCat}/_h\mathcal{C}$, the cofunctors which are bijective on components and arrows correspond to the maps (7.2) of $\text{jrCat}/_h\mathcal{C}$ in which F is localic.*

Proof. Clearly, if F is localic, then it satisfies conditions (i) and (ii) of Proposition 7.5, so that $\Psi(F, \alpha)$ is bijective on components and arrows. Suppose now that Q is hyperconnected, and that $\Psi(F, \alpha)$ is bijective on components and arrows. Thus $Q(Ff \wedge Fg) = QF(f \wedge g)$, so that $Q[[Ff = Fg]] = QF[[f = g]]$; but now $[[Ff = Fg]] = F[[f = g]]$ by hyperconnectedness of Q , and so $Ff \wedge Fg = Ff \wedge g$. So F satisfies condition (i) to be localic, and a similar argument verifies (ii). \square

Putting together Proposition 7.3 and Corollary 7.6, we obtain the desired:

Theorem 7.7. *The main equivalence (5.18) restricts back to an equivalence*

$$\text{peCat}_{\text{cov}}(\mathcal{C})^{\text{op}} \begin{array}{c} \xleftarrow{\Psi} \\ \xrightarrow[\Phi]{\sim} \end{array} \text{jrCat}_\ell/_h\mathcal{C}$$

between the opposite of the category of partite internal categories and internal covering functors, and the subcategory of $\text{jrCat}/_h\mathcal{C}$ comprising all the objects and the localic morphisms between them.

This result transcribes perfectly to the groupoidal situation. Any join inverse category has joins of restriction idempotents, and so admits equalities $[[f = g]]$ and meets $f \wedge g$, so that Definition 7.4 makes perfect sense for join-preserving functors between join inverse categories. Tracing through the rest of the arguments, *mutatis mutandis*, now gives:

Theorem 7.8. *The equivalence (6.4) restricts back to an equivalence*

$$\text{peSpd}_{\text{cov}}(\mathcal{C})^{\text{op}} \begin{array}{c} \xleftarrow{\Psi} \\ \xrightarrow[\Phi_g]{\sim} \end{array} \text{jiCat}_\ell/_h\mathcal{C}$$

between the opposite of the category of partite internal étale groupoids and internal covering functors, and the subcategory of $\text{jiCat}/_h\mathcal{C}$ comprising all the objects and the localic morphisms between them.

Finally, in the one-object case, we have:

Theorem 7.9. *The equivalence (6.6) restricts back to an equivalence*

$$\mathcal{Gpd}_{\text{cov}}(\mathcal{C})^{\text{op}} \begin{array}{c} \xleftarrow{\Psi} \\ \xrightarrow[\Phi_g]{\sim} \\ \end{array} \mathcal{PsGrp}(\mathcal{C})_\ell$$

between the opposite of the category of internal groupoids in \mathcal{C} and covering functors, and the category of complete \mathcal{C} -pseudogroups and localic homomorphisms.

7.2. *The (localic, hyperconnected) factorisation system on jrCat*

As an application of the concepts developed in the previous section, we show that there is a *factorisation system* on the category of join restriction categories jrCat whose two classes are the localic and the hyperconnected morphisms. We first recall:

Definition 7.10. A *factorisation system* $(\mathcal{E}, \mathcal{M})$ on a category \mathcal{D} comprises two classes \mathcal{E} and \mathcal{M} of maps, each of which is closed under composition with isomorphisms, and which satisfy the following two axioms.

- (i) *Factorisation:* each $f: A \rightarrow B$ can be written as $f = me: A \rightarrow C \rightarrow B$ for some $e \in \mathcal{E}$ and $m \in \mathcal{M}$;
- (ii) *Orthogonality:* each $e \in \mathcal{E}$ is orthogonal to each $m \in \mathcal{M}$; this is to say that, for any commuting square as in the solid part of

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ e \downarrow & \nearrow j & \downarrow m \\ B & \xrightarrow{g} & D \end{array} \tag{7.4}$$

there is a unique map $j: B \rightarrow C$ as displayed making both triangles commute.

The two classes of a factorisation system $(\mathcal{E}, \mathcal{M})$ determine each other: a map lies in \mathcal{E} *precisely* when it is orthogonal to each map in \mathcal{M} , and vice versa. Thus, given any class of maps \mathcal{M} in a category \mathcal{D} , there is at most one factorisation system on \mathcal{D} whose right class is given by \mathcal{M} . In [6, Section 2], the present authors showed that when \mathcal{M} is the class of hyperconnected maps in rCat, such a factorisation system exists. We will now show that the same is true when \mathcal{M} is the class of hyperconnected maps in jrCat.

Theorem 7.11. *(Localic, hyperconnected) is a factorisation system on jrCat.*

Proof. First, given a commuting square in jrCat , as to the left in:

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\
 L \downarrow & \nearrow J & \downarrow H \\
 \mathcal{B} & \xrightarrow{G} & \mathcal{D}
 \end{array}
 \qquad
 \begin{array}{ccc}
 (\mathcal{A}, GL) & \xrightarrow{(F,1)} & (\mathcal{C}, H) \\
 (L,1) \downarrow & \nearrow (J,1) & \\
 (\mathcal{B}, G) & &
 \end{array}$$

with L localic and H hyperconnected, we will exhibit a unique J as shown.

Suppose first that \mathcal{D} admits local glueings. We can encode the data of the given commuting square as a span in rCat/\mathcal{D} , as right above, and a map $J: \mathcal{B} \rightarrow \mathcal{C}$ is a filler for the square as to the left just when it fits into a commuting triangle as to the right. But as L is bijective on objects, any (J, α) fitting into such a commuting triangle must have $\alpha = 1$. So we are left with proving that there is a unique factorisation of $(F, 1)$ through $(L, 1)$. Since (\mathcal{C}, H) lies in the reflective subcategory jrCat/\mathcal{D} of rCat/\mathcal{D} , it suffices to show that $(L, 1)$ is inverted by the reflector $\Phi\Psi$ into this subcategory. In fact, by Proposition 7.5, $\Psi(L, 1)$ is already invertible, since L is localic and 1 is invertible.

If \mathcal{D} does not admit local glueings, then we embed it into its local glueing completion via $\iota: \mathcal{D} \rightarrow \text{Gl}(\mathcal{D})$ as in Proposition 3.12, and apply the preceding argument to obtain a unique diagonal filler in

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\
 L \downarrow & \nearrow J & \downarrow \iota H \\
 \mathcal{B} & \xrightarrow{\iota G} & \text{Gl}(\mathcal{D}) .
 \end{array}$$

Now as $\iota: \mathcal{D} \rightarrow \text{Gl}(\mathcal{D})$ is monic, this J is also a unique diagonal filler for the original square. This proves the orthogonality of localic and hyperconnected maps; it remains to show factorisation.

Consider a map $F: \mathcal{C} \rightarrow \mathcal{D}$ in jrCat ; we again start by assuming that \mathcal{D} admits local glueings. In this case, we can form the reflection

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{\eta_F} & \Phi\Psi F \\
 F \searrow & = & \swarrow \pi_{\Psi F} \\
 & \mathcal{D} &
 \end{array}$$

of $F: \mathcal{C} \rightarrow \mathcal{D}$ in rCat/\mathcal{C} into jrCat/\mathcal{D} . By construction, $\pi_{\Psi F}$ is hyperconnected; further, since $\Psi \dashv \Phi$ is a Galois adjunction, the unit $\eta_{\mathcal{C}}: (\mathcal{C}, F) \rightarrow (\Phi\Psi F, \pi_{\Psi F})$ is inverted by Ψ , whence, by Corollary 7.6, η_F is localic. So we have the desired (localic, hyperconnected) factorisation of F .

If \mathcal{D} does not admit local glueings, we can like before consider the embedding $\iota: \mathcal{D} \rightarrow \text{Gl}(\mathcal{D})$ into the local glueing completion. By the preceding argument we have a (localic, hyperconnected) factorisation of ιF , as in the solid part of:

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
 L \downarrow & \nearrow K & \downarrow \iota \\
 \mathcal{E} & \xrightarrow{H} & \text{Gl}(\mathcal{D}) .
 \end{array}$$

Since L is localic, and ι is (fully faithful and hence) hyperconnected, there is a unique diagonal filler as displayed. Since ι and $\iota K = H$ is hyperconnected, so is K , and so KL is the required (localic, hyperconnected) factorisation of F . \square

Note that, in the case where \mathcal{D} has local glueings, Proposition 5.19 provides an explicit description of the (localic, hyperconnected) factorisation of $F: \mathcal{C} \rightarrow \mathcal{D}$ in jrCat , and it is easy to see that this same description is also valid for a general \mathcal{D} . The following result gives a corresponding explicit description of the orthogonal liftings of localic maps against hyperconnected ones.

Proposition 7.12. *Given a commuting square in jrCat*

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\
 L \downarrow & \nearrow J & \downarrow H \\
 \mathcal{B} & \xrightarrow{G} & \mathcal{D}
 \end{array}$$

with L localic and H hyperconnected, the unique diagonal filler $J: \mathcal{B} \rightarrow \mathcal{C}$ is determined as follows.

- On **objects** by requiring that $J(Li) = Fi$;
- On **morphisms** by requiring that $J(g: Li \rightarrow Lj) = \bigvee_{f \in \mathcal{A}(i,j)} Ff \circ \varphi_{fg}$, where $\varphi_{fg} \in \mathcal{O}(Fi)$ is unique such that $H\varphi_{fg} = G[g = Lf]$.

Proof. The given conditions completely specify J 's action since L is bijective on objects. The condition on objects is clearly necessary. As for the condition on morphisms, given $g: Li \rightarrow Lj$ in \mathcal{B} , we have since L is localic that

$$g = \bigvee_{f \in \mathcal{A}(i,j)} g \wedge Lf = \bigvee_{f \in \mathcal{A}(x,y)} Lf \llbracket g = Lf \rrbracket$$

and so $Jg = \bigvee_{f \in \mathcal{A}(x,y)} JLf \circ J \llbracket g = Lf \rrbracket = \bigvee_{f \in \mathcal{A}(x,y)} Ff \circ J \llbracket g = Lf \rrbracket .$

Writing $\varphi_{fg} := J[g = Lf]$, we have that $H\varphi_{fg} = HJ[g = Lf] = G[g = Lf]$. Since H is hyperconnected, this property uniquely determines $\varphi_{fg} \in \mathcal{O}(Fi)$. \square

7.3. The (localic, hyperconnected) factorisation system on $\text{jrCat}/_h\mathcal{C}$

We now show that the factorisation system of the previous section lifts to a factorisation system on $\text{jrCat}/_h\mathcal{C}$ whenever \mathcal{C} is a join restriction category with local glueings. The classes of maps are given by:

Definition 7.13. A map $(F, \alpha): (\mathcal{A}, P) \rightarrow (\mathcal{B}, Q)$ of $\text{jrCat}/_h\mathcal{C}$ is said to be *hyperconnected* if α is invertible, and *localic* if F is localic in jrCat .

Note that, if (F, α) is hyperconnected, then $QF \cong P$ is hyperconnected; since Q is hyperconnected, it follows that F is also hyperconnected. However F could be hyperconnected without α being invertible, so that hyperconnectedness of (F, α) is *strictly* stronger than hyperconnectedness of F .

Proposition 7.14. Let \mathcal{C} be a join restriction category with local glueings. (Localic, hyperconnected) is a factorisation system on $\text{jrCat}/_h\mathcal{C}$.

Proof. Given a map $(F, \alpha): (\mathcal{A}, P) \rightarrow (\mathcal{B}, Q)$ of $\text{jrCat}/_h\mathcal{C}$, we may form a (localic, hyperconnected) factorisation $F = HL: \mathcal{A} \rightarrow \mathcal{D} \rightarrow \mathcal{B}$ of $F: \mathcal{A} \rightarrow \mathcal{B}$ in jrCat ; the desired factorisation in $\text{jrCat}/_h\mathcal{C}$ is now given by

$$(\mathcal{A}, P) \xrightarrow{(L, \alpha)} (\mathcal{D}, QH) \xrightarrow{(H, 1)} (\mathcal{D}, Q).$$

For orthogonality, consider a square in $\text{jrCat}/_h\mathcal{C}$ as to the left in

$$\begin{array}{ccc}
 (\mathcal{A}, P) & \xrightarrow{(F, \alpha)} & (\mathcal{D}, R) \\
 (L, \gamma) \downarrow & \nearrow (J, \theta) & \downarrow (H, \delta) \\
 (\mathcal{B}, Q) & \xrightarrow{(G, \beta)} & (\mathcal{E}, S)
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{A} & \xrightarrow{F} & \mathcal{D} \\
 L \downarrow & \nearrow J & \downarrow H \\
 \mathcal{B} & \xrightarrow{G} & \mathcal{E}
 \end{array}$$

where L is localic and δ is invertible. As argued above, it follows that H is hyperconnected, and so using orthogonality in jrCat , we obtain a unique diagonal filler as to the right above. We wish to lift to this a filler as to the left. Since δ is invertible, the *unique* possible θ satisfying $\theta \circ \delta J = \beta$ is $\theta = \beta \circ \delta^{-1}J$, and this also satisfies $\gamma \circ \theta L = \gamma \circ \beta L \circ \delta^{-1}JL = \alpha \circ \delta F \circ \delta^{-1}F = \alpha$. So (J, θ) is the desired unique filler for the square to the left. \square

By transporting this factorisation system across the equivalence $\text{jrCat}/_h\mathcal{C} \simeq \text{peCat}_c(\mathcal{C})$, we obtain a factorisation system on $\text{peCat}_c(\mathcal{C})$. In fact, this is easy to describe.

Proposition 7.15. *Under the equivalence $\text{jrCat}/_h\mathcal{C} \simeq \text{peCat}_c(\mathcal{C})$, the (localic, hyper-connected) factorisation system on $\text{jrCat}/_h\mathcal{C}$ corresponds to the factorisation system (bijective on components and arrows, bijective on objects) on $\text{peCat}_c(\mathcal{C})$.*

Proof. This is immediate from Proposition 7.5 and Corollary 7.6. \square

We leave it as an instructive exercise to the reader to give an explicit description of factorisations and orthogonal liftings for this factorisation system on $\text{peCat}_c(\mathcal{C})$.

8. Applications

In this section, we instantiate our main result and its variants at particular choices of a join restriction category \mathcal{C} . This will allow us to recapture and extend existing correspondences in the literature, and also to construct various *completions*.

8.1. Haefliger groupoids

In [9], Ehresmann describes for a pair of spaces E and E' , the space $\Pi(E, E')$ of germs of partial homeomorphisms $E \rightarrow E'$, and explains that, when $E = E'$, we have a groupoid $\Pi(E)$. Later, Haefliger [16] focused on the *smooth* variant of the germ groupoid—that is, the groupoid of all local diffeomorphisms of a smooth manifold M —and this has subsequently come to be known as the *Haefliger groupoid*. In our language, the Haefliger groupoid is the internal étale groupoid in Smooth_p associated to the complete Smooth_p -pseudogroup $\mathcal{J}(M) \rightarrow \mathcal{J}(M)$. We can adapt this construction to the more general setting of our main theorem as follows.

Definition 8.1. Let \mathcal{C} be a join restriction category with local glueings. The *Haefliger category* of \mathcal{C} is the partite source-étale internal category $\mathcal{H}(\mathcal{C})$ in \mathcal{C} associated via Ψ to the join restriction functor $1_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$.

The components of $\mathcal{H}(\mathcal{C})$ are indexed by objects of \mathcal{C} ; the object of objects associated to $A \in \text{ob}\mathcal{C}$ is A itself; and the object of morphisms $\mathcal{H}(\mathcal{C})_{AB}$ is the appropriate analogue of Ehresmann's $\Pi(A, B)$.

Example 8.2.

- When $\mathcal{C} = \text{Set}_p$, elements of $\mathcal{H}(\mathcal{C})_{AB}$ with source $a \in A$ and target $b \in B$ are germs of partial functions $A \rightarrow B$ sending a to b . There is exactly *one* such germ—represented by the partial function with graph $\{(a, b)\}$ —so that $\mathcal{H}(\mathcal{C})_{AB} = A \times B$. (Alternatively,

since every map in Set_p is étale, we can derive this conclusion from Proposition 8.4 below).

- When $\mathcal{C} = \text{Top}_p$, elements of $\mathcal{H}(\mathcal{C})_{AB}$ with source a and target b are germs of partial continuous maps $A \rightarrow B$ sending a to b .
- When $\mathcal{C} = \text{Smooth}_{\mathbb{R}}$, $\mathcal{H}(\mathcal{C})_{AB}$ is the space of germs of partial smooth maps. In particular, $\mathcal{H}(\mathcal{C})_{M\mathbb{R}}$ incarnates the ring of smooth functions on M ; more specifically, elements of $\mathcal{H}(\mathcal{C})_{M\mathbb{R}}$ with source and target projections $m \in M$ and $r \in \mathbb{R}$ are the germs at m of elements $f \in C_{\infty}(U)$ where $U \subseteq M$ is open, $m \in U$ and $f(m) = r$.
- In a similar spirit, when $\mathcal{C} = \text{Sch}_p$ and $\mathbf{A}^1 = (\text{Spec } \mathbb{Z}[x], \mathcal{O}_{\text{Spec } \mathbb{Z}[x]})$ we have that $\mathcal{H}(\mathcal{C})_{X,\mathbf{A}^1}$ is $(\mathcal{O}_X, \mathcal{O}_X^* \mathcal{O}_X)$, i.e., the structure sheaf of X seen as a local isomorphism over X .
- Let $\mathcal{C} = \text{Bun}(\text{Set}_p)$, the join restriction category of discrete bundles. The source-target span of its Haefliger category at objects $\xi: X' \rightarrow X$ and $\gamma: Y' \rightarrow Y$ is a diagram of sets and (total) functions of the form

$$\begin{array}{ccccc}
 X' & \xleftarrow{\pi_2} & \mathcal{H}(\mathcal{C})_{\xi\gamma} \times_X X' & \xrightarrow{\tau'_{\xi\gamma}} & Y' \\
 \xi \downarrow & & \downarrow \pi_1 & & \gamma \downarrow \\
 X & \xleftarrow{\sigma_{\xi\gamma}} & \mathcal{H}(\mathcal{C})_{\xi\gamma} & \xrightarrow{\tau_{\xi\gamma}} & Y,
 \end{array} \tag{8.1}$$

and we can calculate that elements of $\mathcal{H}(\mathcal{C})_{\xi\gamma}$ with source $x \in X$ and target $y \in Y$ are functions $f: \xi^{-1}(x) \rightarrow \gamma^{-1}(y)$ between the fibres. The map $\tau'_{\xi\gamma}$ sends such an element f and an element $x' \in \xi^{-1}(x)$ to $f(x') \in \gamma^{-1}(y)$. In particular, for a single object $\xi: X \rightarrow X'$, the internal category $\mathcal{H}(\mathcal{C})_{\xi\xi} \rightrightarrows X$ in Set is the so-called *internal full subcategory* associated to the map ξ , and $\tau'_{\xi\xi}$ exhibits ξ as an internal presheaf over this internal full subcategory.

- Let $\mathcal{C} = \text{Bun}(\text{Top}_p)$, the join restriction category of topological bundles. Again, the Haefliger category involves diagram of the form (8.1), where this time elements of $\mathcal{H}(\mathcal{C})_{\xi\gamma}$ over $x \in X$ and $y \in Y$ are germs of partial continuous maps sending x to y in the base, and lifting to a partial continuous map on the fibres. The maps $\tau'_{\xi\gamma}$ encode an action by $\mathcal{H}(\mathcal{C})$ on the family of all bundles $\xi: X' \rightarrow X$. This generalises [15]’s construction of the “groupoid of germs of local automorphisms of a fibred space $p: E \rightarrow B$ ” (§I.4) and the action of this groupoid on p (§I.5).

Remark 8.3. The Haefliger category of \mathcal{C} is “generic” among partite internal categories built out of \mathcal{C} , in the following sense. Suppose that $F: \mathcal{C} \rightarrow \mathcal{D}$ is a hyperconnected join restriction functor. On the one hand, we can view F as an object of $\text{jrCat} //_{\hbar} \mathcal{D}$, and construct a source-étale partite internal category $\Phi(F)$ in \mathcal{D} . On the other hand, we can form the source-étale partite internal category $\mathcal{H}(\mathcal{C})$ in \mathcal{C} . Now F preserves total maps, local homeomorphisms, and pullbacks along local homeomorphisms, and so also

preserves source-étale partite internal categories. It is now easy to see that applying F to the source-étale internal category $\mathcal{H}(\mathcal{C})$ in \mathcal{C} yields, to within isomorphism, $\Phi(F)$ in \mathcal{D} .

Of course, we can also associate a *Haefliger groupoid* $\Pi(\mathcal{C})$ to any join restriction category with local glueings, taking it to be the partite étale internal groupoid $\Pi(\mathcal{C})$ in \mathcal{C} associated via Ψ to the join inverse category $\text{PIso}(\mathcal{C})$ over \mathcal{C} . So, for example, the Haefliger groupoid $\Pi(\text{Top}_p)$ has spaces of morphisms $\Pi(\text{Top}_p)_{AB}$ given *exactly* by Ehresmann’s spaces $\Pi(A, B)$.

In fact, the Haefliger groupoid of a join restriction category \mathcal{C} is a special case of a Haefliger category. Indeed, by Theorem 6.3 we can describe $\Pi(\mathcal{C})$ as the partite étale groupoid obtained by applying Ψ to the étale join restriction category $\text{Et}(\mathcal{C}) \rightarrow \mathcal{C}$ over \mathcal{C} where, as in Corollary 2.24, we write $\text{Et}(\mathcal{C})$ for the category of étale maps in \mathcal{C} . Since $\text{Et}(\mathcal{C})$ is closed in \mathcal{C} under local glueings, this is equally the result of applying Ψ to $1: \text{Et}(\mathcal{C}) \rightarrow \text{Et}(\mathcal{C})$; thus we have $\Pi(\mathcal{C}) = \mathcal{H}(\text{Et}(\mathcal{C}))$. We can use this observation to obtain an alternative understanding of the “objects of arrows” $\Pi(\mathcal{C})_{AB}$ in the Haefliger groupoid.

Proposition 8.4. *Let \mathcal{C} be a join restriction category with glueings. The source-target span $\sigma_{AB}: A \leftarrow \Pi(\mathcal{C})_{AB} \rightarrow B: \tau_{AB}$ exhibits $\Pi(\mathcal{C})_{AB}$ as a product of A and B in the category $\text{Lh}(\mathcal{C})$ of local homeomorphisms between objects of \mathcal{C} .*

Proof. Let $u: A \leftarrow X \rightarrow B: v$ be a span in $\text{Lh}(\mathcal{C})$. We must show there is a *unique* local homeomorphism h rendering commutative the diagram:

$$\begin{array}{ccc}
 & X & \\
 u \swarrow & \downarrow h & \searrow v \\
 & \Pi(\mathcal{C})_{AB} & \\
 \swarrow \sigma_{AB} & & \searrow \tau_{AB} \\
 A & & B.
 \end{array} \tag{8.2}$$

Now, since \mathcal{C} admits local glueings, the category $\text{Lh}(\mathcal{C})$ has a pullback-stable initial object 0 , obtained by glueing the empty atlas on *any* object of \mathcal{C} . We can thus construct a two-component source-étale partite internal category \mathbb{X} in $\text{Et}(\mathcal{C})$, where $\mathbb{X}_0 = A, \mathbb{X}_1 = B, \mathbb{X}_{01} = X, \mathbb{X}_{10} = 0, \mathbb{X}_{00} = A, \mathbb{X}_{11} = B$, and with remaining data obtained in the obvious manner. This \mathbb{X} is such that maps h rendering (8.2) commutative correspond precisely to *identity-on-objects* partite cofunctors $\mathbb{X} \rightsquigarrow \Pi(\mathcal{C})$ in $\text{Et}(\mathcal{C})$. Since $\Pi(\mathcal{C}) = \Phi(1_{\text{Et}(\mathcal{C})})$, such maps correspond in turn to maps $(\Phi\mathbb{X}, \pi_{\mathbb{X}}) \rightarrow (\text{Et}(\mathcal{C}), 1_{\text{Et}(\mathcal{C})})$ of $\text{jrCat}/_h\text{Et}(\mathcal{C})$ whose 2-cell component is the *identity*. Clearly $(\pi_{\mathbb{X}}, 1)$ is the unique such map. \square

Examples 8.5.

- When $\mathcal{C} = \mathcal{J}op_p$, this recaptures a result of Selinger [31], proving the existence of binary products in the category of local homeomorphisms between topological spaces.
- When $\mathcal{C} = \mathcal{S}et_p$ every map is already étale, so that $\Pi(\mathcal{C}) = \mathcal{H}(\mathcal{C})$; likewise every total map is already a local homeomorphisms, so that $\Pi(\mathcal{C})_{AB} = \mathcal{H}(\mathcal{C})_{AB} = A \times B$, the cartesian product of sets.
- When $\mathcal{C} = \mathcal{B}un(\mathcal{S}et_p)$, the category $\mathcal{L}h(\mathcal{C})$ is the category $\mathcal{C}art$ whose objects are functions between sets, and whose morphisms are pullback squares. It follows that $\mathcal{C}art$ has binary products. Explicitly, if $\xi: X' \rightarrow X$ and $\gamma: Y' \rightarrow Y$ in $\mathcal{C}art$, then their product is $\zeta: Z' \rightarrow Z$ where

$$Z = \{(x, y, \theta) : x \in X, y \in Y, \theta: \xi^{-1}(x) \cong \gamma^{-1}(y)\}$$

and an element of $\zeta^{-1}(x, y, \theta)$ is a pair $(x' \in \xi^{-1}(x), y' \in \gamma^{-1}(y))$ with $\theta(x') = y'$.

8.2. *The Resende correspondence*

In [29,30], Resende establishes a correspondence between join inverse monoids—*abstract complete pseudogroups* in his terminology—and localic étale groupoids, by way of a third notion which he terms an *inverse quantal frame*. More precisely, he establishes in [29, Theorem 4.15] an equivalence between the categories of join inverse monoids and inverse quantal frames; and in [30, Theorem 5.15], an equivalence between the category of inverse quantal frames, and the category of localic étale groupoids and cofunctors.

In fact, Resende’s abstract complete pseudogroups are exactly our complete $\mathcal{L}oc_p$ -pseudogroups. Indeed, if $\theta: S \rightarrow \mathcal{J}(X)$ is a complete $\mathcal{L}oc_p$ -pseudogroup, then since θ restricts back to an isomorphism $E(S) \rightarrow E(\mathcal{J}(X)) \cong X$, the locale X can be identified with the locale of idempotents of S ; whereupon functoriality of θ forces $\theta(s): E(S) \rightarrow E(S)$ to be the partial locale map given by $e \mapsto s^*es$. Thus, our Theorem 6.13 specialised to the case $\mathcal{C} = \mathcal{L}oc_p$ re-finds Resende’s functorial equivalence between the category of join inverse monoids (= abstract complete pseudogroups) and the category of localic étale groupoids and cofunctors.

The identification of abstract complete pseudogroups and complete $\mathcal{L}oc_p$ -pseudogroups has an analogue for join restriction categories, described in [6]; using this, we may generalise Resende’s correspondence further. The key definition is:

Definition 8.6. (cf. [5, §4.1]) The *fundamental functor* $\mathcal{O}_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{L}oc_p$ of a join restriction category \mathcal{A} is given on objects by $X \mapsto \mathcal{O}(X)$, and on maps by sending $f: X \rightarrow Y$ to the partial locale map $\mathcal{O}(f): \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ with $\mathcal{O}(f)^*(e) = \overline{ef}$.

The crucial fact about the fundamental functor of \mathcal{A} is that it is the essentially-unique hyperconnected functor from \mathcal{A} to $\mathcal{L}oc_p$. The precise result is the following one, which combines Proposition 3.3 and Proposition 7.3 of [6].

Proposition 8.7. *The fundamental functor $\mathcal{O}_A: \mathcal{A} \rightarrow \mathcal{L}oc_p$ is a hyperconnected join restriction functor. For any $H: \mathcal{A} \rightarrow \mathcal{L}oc_p$, the following are equivalent:*

- (i) *H is hyperconnected;*
- (ii) *H is a terminal object in $\text{jrCat}(\mathcal{A}, \mathcal{L}oc_p)$;*
- (iii) *H is naturally isomorphic to the fundamental functor $\mathcal{O}_A: \mathcal{A} \rightarrow \mathcal{L}oc_p$.*

We thus obtain the following generalisation of the correspondence between abstract complete pseudogroups and complete $\mathcal{L}oc_p$ -pseudogroups.

Corollary 8.8. (*[6, Theorem 7.14]*) *There is an equivalence of categories*

$$\text{jrCat} // {}_h\mathcal{L}oc_p \xrightleftharpoons[U]{V} \text{jrCat}$$

where U forgets the projection down to $\mathcal{L}oc_p$, and where V sends \mathcal{A} to $(\mathcal{A}, \mathcal{O}_A)$ and sends $F: \mathcal{A} \rightarrow \mathcal{B}$ to $(F, \alpha): (\mathcal{A}, \mathcal{O}_A) \rightarrow (\mathcal{B}, \mathcal{O}_B)$, where α has components $\alpha_i: \mathcal{O}_B(Fi) \rightarrow \mathcal{O}_A(i)$ given by $\alpha_i^*(e) = Fe$ for all $e \in \mathcal{O}_A(i)$.

Taking this together with our main results, we thus obtain:

Theorem 8.9. *There are equivalences of categories*

$$\text{peCat}_c(\mathcal{L}oc_p) \xrightleftharpoons[U\Phi]{\Psi V} \text{jrCat} \quad \text{and} \quad \text{peSpd}_c(\mathcal{L}oc_p) \xrightleftharpoons[U\Phi_g]{\Psi V} \text{jiCat} \tag{8.3}$$

together with their obvious restrictions to the one-object case.

For the left-to-right direction of these equivalences, we cannot improve upon the descriptions given in Sections 5.2 and 6.2. For the right-to-left direction, we can use the characterisation of local glueings in $\mathcal{L}oc_p$ from Example 3.8 to say more. It suffices to do this in the most general case.

Proposition 8.10. *Let \mathcal{A} be a join restriction category with object-set I . The corresponding I -partite source-étale localic category has:*

- *Locale of objects $X_i = \mathcal{O}(i)$ for $i \in \mathcal{A}$;*
- *Locale of arrows X_{ij} for $i, j \in \mathcal{A}$ given by*

$$X_{ij} = \{ (\theta_f \in \mathcal{O}(i) : f \in \mathcal{A}(i, j)) \mid \theta_f e = \theta_{fe} \text{ for } f \in \mathcal{A}(i, j), e \in \mathcal{O}(i) \} . \tag{8.4}$$

- Source–target maps $\sigma_{ij}: X_i \leftarrow X_{ij} \rightarrow X_j: \tau_{ij}$ given by

$$\sigma_{ij}^*(d) = (\overline{fd} : f \in \mathcal{A}(i, j)) \quad \tau_{ij}^*(d) = (\overline{df} : f \in \mathcal{A}(i, j)) ;$$

- The partial section $s_f: X_i \rightarrow X_{ij}$ of σ_{ij} associated to the map $f \in \mathcal{A}(i, j)$ given by $s_f^*(\theta) = \theta_f$.
- Identities $\eta_i: X_i \rightarrow X_{ii}$ given by $\eta_i^*(\theta) = \theta_{1_i}$;
- Multiplication $\mu_{ijk}: X_{jk} \times_{X_j} X_{ij} \rightarrow X_{ik}$ given by

$$\mu_{ijk}^*(\theta) = (\theta_{gf} : f \in \mathcal{A}(i, j), g \in \mathcal{A}(j, k)) ,$$

under the identification of $X_{jk} \times_{X_j} X_{ij}$ with the locale of all families

$$\{ (\psi_{f,g} \in \mathcal{O}(i) : f \in \mathcal{A}(i, j), g \in \mathcal{A}(j, k)) \mid \psi_{f,g}e = \psi_{fe,g}, \psi_{f,g}\overline{df} = \psi_{f,gd} \} . \quad (8.5)$$

Proof. The identification of X_i is clear. By definition, X_{ij} is the glueing in $\mathcal{L}oc_p$ of the local atlas $(\llbracket f = g \rrbracket : f, g \in \mathcal{A}(i, j))$ on $\mathcal{O}(i)$, and so by Example 3.8 comprises the locale of all families

$$\{ (\theta_f \in \mathcal{O}(i) : f \in \mathcal{A}(i, j)) \mid \theta_f \leq \overline{f} \text{ and } \theta_g \llbracket f = g \rrbracket \leq \theta_f \} . \quad (8.6)$$

We claim these are the same families as in (8.4). First, given a family as in (8.4) we have $\theta_f = \theta_{f\overline{f}} = \theta_f\overline{f}$, so that $\theta_f \leq \overline{f}$, and also that $\theta_g \llbracket f = g \rrbracket = \theta_g \llbracket \underline{f} = \underline{g} \rrbracket = \theta_f \llbracket f = g \rrbracket = \theta_f \llbracket f = g \rrbracket \leq \theta_f$. Conversely, given a family as in (8.6), we have $\theta_{fe} = \theta_f\overline{f}e = \theta_f \llbracket f = fe \rrbracket \leq \theta_{fe}$ and $\theta_{fe} = \theta_{fe}\overline{f}ee = \theta_{fe} \llbracket fe = f \rrbracket e \leq \theta_{fe}$ so that $\theta_{fe} = \theta_{fe}$. So (8.4) is a correct description of X_{ij} .

The descriptions of σ_{ij} and the partial sections s_f now follow directly from Example 3.8; this gives also $\eta_i = s_{1_i}$. For the target map, it suffices by the description in Definition 5.12 to show that $\tau_{ij}s_f = \mathcal{O}(f): \mathcal{O}(i) \rightarrow \mathcal{O}(j)$ for all $f \in \mathcal{A}(i, j)$, which is so since $s_f^*\tau_{ij}^*(d) = \overline{df}$. Finally, for the multiplication, we observe that $X_{jk} \times_{X_j} X_{ij}$ is, as in the proof of Proposition 5.13, a glueing of the local atlas

$$(\llbracket f = h \rrbracket \llbracket \overline{g} = \overline{k} \rrbracket f : (f, g), (h, k) \in \mathcal{A}(i, j) \times \mathcal{A}(j, k))$$

with associated family of local sections $\tau_{ij}^*(s_g)s_f$. A short calculation similar in nature to that given above shows that (8.5) is a valid description of this local glueing, and that in these terms, the section $\tau_{ij}^*(s_g)s_f$ is given by $\theta \mapsto \theta_{(f,g)}$. Given this, the description of μ_{ijk} is validated by observing that it satisfies the condition $\mu_{ijk}\tau_{ij}^*(s_g)s_f = s_{gf}$ which uniquely characterises it in Definition 5.12. \square

8.3. The Lawson–Lenz correspondence

In [25], Lawson and Lenz describe a Galois adjunction between join inverse monoids and étale topological groupoids, inducing an equivalence between the categories of fix-

points. They call these fixpoints the *spatial* join inverse monoids and the *sober* étale groupoids; the nomenclature draws on the Galois adjunction between topological spaces and locales, whose fixpoints are the sober spaces and the spatial locales.

Definition 8.11. A topological space is *sober* if each irreducible closed set is the closure of a unique point. A *point* of a locale X is a function $p: X \rightarrow 2 = \{\perp \leq \top\}$ which preserves finite meets and all joins. A locale X is *spatial* if whenever $x \neq y \in X$ there is a point p with $p(x) \neq p(y)$.

We now explain how the Lawson–Lenz adjunction can be re-derived from our Theorem 7.9 by composing the equivalence given there with the space–locale adjunction. We begin by describing this latter adjunction in a manner which is amenable for our applications.

Lemma 8.12. *The fundamental functor $\mathcal{O}: \mathcal{J}\text{op}_p \rightarrow \mathcal{L}\text{oc}_p$ has a right adjoint $\text{pt}: \mathcal{L}\text{oc}_p \rightarrow \mathcal{J}\text{op}_p$ in jrCat . The underlying adjunction of categories $\mathcal{O} \dashv \text{pt}$ is Galois, and its fixpoints are the sober spaces, respectively, the spatial locales.*

Proof. Let X be a locale. We define $\text{pt}(X)$ to be the space of points of X endowed with the topology with open sets $[x] = \{p \in \text{pt}(X) : p(x) = \top\}$ for each $x \in X$. We define a total locale map $\varepsilon_X: \mathcal{O}(\text{pt}(X)) \rightarrow X$ by $\varepsilon_X^*(x) = [x]$. For any space Y and partial locale map $f: \mathcal{O}(Y) \rightarrow X$, we have a partial continuous map $g: Y \rightarrow \text{pt}(X)$ defined on $y \in f^*(\top) \subseteq Y$ by $g(y)(x) = \top$ just when $y \in f^*(x)$; this is the *unique* map with $\varepsilon_X \circ \mathcal{O}(g) = f$, and so $\mathcal{O} \dashv \text{pt}$ in Cat .

By construction, the counit ε is total; by inspection, the unit η is also total, and the functor $\text{pt}: \mathcal{L}\text{oc}_p \rightarrow \mathcal{J}\text{op}_p$ preserves restriction. It follows that $\mathcal{O} \dashv \text{pt}$ in rCat . Finally, by observing that

$$\begin{array}{ccc}
 \mathcal{L}\text{oc}_p & \xrightarrow{\text{pt}} & \mathcal{J}\text{op}_p \\
 & \swarrow \varepsilon & \searrow \mathcal{O} \\
 & \mathcal{L}\text{oc}_p &
 \end{array}$$

is a triangle in $\text{jrCat} // {}_h \mathcal{L}\text{oc}_p$ and applying Lemma 5.2, we see that pt is join-preserving, so that $\mathcal{O} \dashv \text{pt}$ in jrCat . To see that the underlying adjunction is Galois, and the fixpoints are as described, see, for example, [21, §II.1.7]. \square

Corollary 8.13. *There is a Galois adjunction*

$$\text{jrCat} // {}_h \mathcal{J}\text{op}_p \xleftarrow[W]{\perp} \text{jrCat} \xrightarrow[U]{} \text{jrCat} \tag{8.7}$$

where U forgets the projection down to $\mathcal{J}op_p$, and where W sends A to $(A, pt \circ \mathcal{O}_A)$ and sends $F: A \rightarrow B$ to $(F, pt \circ \alpha)$, where α is defined as in Corollary 8.8.

Proof. Compose the equivalence of Corollary 8.8 with the induced Galois adjunction $\mathcal{O} \circ (-): jrCat //_h \mathcal{J}op_p \rightleftarrows jrCat //_h \mathcal{L}oc_p: pt \circ (-)$. \square

Taking this together with the case $\mathcal{C} = \mathcal{J}op_p$ of our main result, we recover the Lawson–Lenz correspondence and its generalisations.

Theorem 8.14. *There are Galois adjunctions*

$$peCat_c(\mathcal{J}op_p) \begin{array}{c} \xleftarrow{\Psi W} \\ \perp \\ \xrightarrow{U\Phi} \end{array} jrCat \quad \text{and} \quad pe\mathcal{G}pd_c(\mathcal{J}op_p) \begin{array}{c} \xleftarrow{\Psi W} \\ \perp \\ \xrightarrow{U\Phi_g} \end{array} jiCat$$

together with their obvious restrictions to the one-object case. The fixpoints to each side are those $X \in peCat_c(\mathcal{J}op_p)$ with sober spaces of objects, and those $A \in jrCat$ with spatial locales of restriction idempotents.

As we have already discussed in Section 7, even the one-object inverse case of this theorem is more general than [25], due to the more generous notions of morphism to each side. To recapture the precise form of the Lawson–Lenz equivalence, we may employ Theorem 7.9 in place of Theorem 6.13.

Remark 8.15. In [25], the étale groupoid associated to a join inverse monoid S is described in terms of *completely prime filters* on S : subsets of S which are upwards closed and downwards directed, and which contain a join $\bigvee_{i \in I} s_i$ precisely when they contain at least one of the s_i 's. We can recover this description, and its generalisation to the other cases of our correspondence, by combining the explicit description of $pt: \mathcal{L}oc_p \rightarrow \mathcal{J}op_p$ from Lemma 8.12 and the description of local glueings in $\mathcal{J}op_p$ from Example 3.6.

8.4. The Ehresmann–Schein–Nambooripad correspondence

The correspondence we consider next was first made explicit by Lawson in [24, Chapter 4], bringing together contributions by the three named authors. Lawson’s version correlates inverse semigroups with a certain category of ordered groupoids; the version we state here, for inverse categories, is due to DeWolf and Pronk [8].

Definition 8.16. A partite internal groupoid in $\mathcal{P}os_p$ is *inductive* if each source (and hence each target) map is a discrete fibration, and each poset of objects has finite meets. We write $pJnd\mathcal{G}pd$ for the category of partite inductive groupoids, where maps are partite internal functors whose object part preserves finite meets.

Definition 8.17. If \mathcal{A} is an inverse category with object-set I , then the I -partite inductive groupoid $\mathcal{G}\mathcal{A}$ is defined as follows. The poset of objects $\mathcal{G}\mathcal{A}_i$ is $(\mathcal{O}(i), \leq)$ while the poset of arrows $\mathcal{G}\mathcal{A}_{ij}$ is $(\mathcal{A}(i, j), \leq)$. The source and target maps are given by $\sigma_{ij}(f) = f^*f$ and $\tau_{ij}(f) = ff^*$, while the identity and composition maps ι_i and μ_{ijk} are given by identities and composition in \mathcal{A} .

Theorem 8.18. ([8, Theorem 3.16]) *The assignment $\mathcal{A} \mapsto \mathcal{G}(\mathcal{A})$ is the action on objects of an equivalence of categories $\mathcal{G}: \text{iCat} \rightarrow \text{pJndGpd}$.*

Our goal is to explain how this equivalence can be obtained from our main correspondence. We should say up front that the proof we describe has many more moving parts than the approach of [24,8]; nonetheless, we consider it illuminating to see how this correspondence fits into our framework. In doing so, we make use of the following mild refinement of our main result.

Definition 8.19. Let \mathcal{C} be a join restriction category with local glueings, and let \mathcal{C}' be a subcategory of the category of total maps in \mathcal{C} . We write $\text{rCat}/\!/(\mathcal{C}', \mathcal{C})$ for the subcategory of $\text{rCat}/\!/ \mathcal{C}$ with:

- **Objects:** those $P: \mathcal{A} \rightarrow \mathcal{C}$ for which each object Pi lies in \mathcal{C}' ;
- **Morphisms:** those $(F, \alpha): (\mathcal{A}, P) \rightarrow (\mathcal{B}, Q)$ for which each component $\alpha_i: QFi \rightarrow Pi$ lies in \mathcal{C}' .

On the other hand, we write $\text{pCat}_c(\mathcal{C}', \mathcal{C})$ (resp., $\text{pCat}(\mathcal{C}', \mathcal{C})$) for the subcategory of $\text{pCat}_c(\mathcal{C})$ (resp., $\text{pCat}(\mathcal{C})$) with:

- **Objects:** those partite internal categories \mathbb{A} for which each A_i lies in \mathcal{C}' ;
- **Morphisms:** those partite internal cofunctors (resp., internal functors) whose object mappings all lie in \mathcal{C}' .

Theorem 8.20. *The adjunction (5.16) restricts back to a Galois adjunction as to the left below, and taking fixpoints yields an equivalence as to the right.*

$$\text{pCat}_c(\mathcal{C}', \mathcal{C}) \begin{array}{c} \xleftarrow{\Psi} \\ \perp \\ \xrightarrow{\Phi} \end{array} \text{rCat}/\!/(\mathcal{C}', \mathcal{C}) \qquad \text{peCat}_c(\mathcal{C}', \mathcal{C}) \begin{array}{c} \xleftarrow{\Psi} \\ \sim \\ \xrightarrow{\Phi} \end{array} \text{jrCat}/\!/h(\mathcal{C}', \mathcal{C}).$$

Proof. Direct from the constructions giving Theorem 5.14 and Theorem 5.17. \square

We begin by recalling from Example 3.1 that the local homeomorphisms in Pos_p are *exactly* the discrete fibrations. Thus, writing $\text{Msl} \subseteq \text{Pos}_p$ for the subcategory of meet-semilattices and finite-meet-preserving total maps, we have

$$\text{pJndGpd} = \text{peGpd}(\text{Msl}, \text{Pos}_p) . \tag{8.8}$$

We now transform the right-hand side of this equality by way of the fundamental functor $\mathcal{O}: \mathcal{P}os_p \rightarrow \mathcal{L}oc_p$. This takes a poset P to the locale $\mathcal{O}(P)$ of downsets in P , and takes a poset map $f: P \rightarrow Q$ to the locale map $\mathcal{O}(f): \mathcal{O}(P) \rightarrow \mathcal{O}(Q)$ given by $\mathcal{O}(f)^*(B \subseteq Q) = f^{-1}(B) \subseteq P$. It is easy to see that \mathcal{O} is faithful and is full on isomorphisms; it therefore establishes an equivalence between $\mathcal{P}os_p$ and its replete image in $\mathcal{L}oc_p$. Since, moreover, \mathcal{O} is hyperconnected, it induces as in Remark 4.17 an equivalence between local homeomorphisms over $P \in \mathcal{P}os_p$ and over $\mathcal{O}(P) \in \mathcal{L}oc_p$. It follows that a partite internal groupoid in $\mathcal{L}oc_p$ is in the replete image of \mathcal{O} just when each of its objects of objects is so. Thus, writing $\mathcal{O}(\mathcal{M}sl)$ for the replete image of $\mathcal{M}sl \subseteq \mathcal{P}os_p$ in $\mathcal{L}oc_p$, we conclude that the action of \mathcal{O} induces an equivalence of categories

$$\mathcal{O}: \text{pe}\mathcal{G}pd(\mathcal{M}sl, \mathcal{P}os_p) \rightarrow \text{pe}\mathcal{G}pd(\mathcal{O}(\mathcal{M}sl), \mathcal{L}oc_p) . \tag{8.9}$$

The next step is delicate: we transform the category of partite groupoids and *functors* to the right above into a category of partite groupoids and *cofunctors*. To do so, we must explicitly identify the subcategory $\mathcal{O}(\mathcal{M}sl) \subseteq \mathcal{L}oc_p$.

Definition 8.21. An element ℓ of a locale L is *supercompact* if $\ell \leq \bigvee D$ implies $\ell \leq d$ for some $d \in D$. We call L *supercoherent* if the supercompact elements form a meet-semilattice, and each $\ell \in L$ is a join of supercompact elements. We write $\text{sc}\mathcal{L}oc$ for the category whose objects are supercoherent locales, and whose maps $f: L \rightarrow M$ are total locale maps for which f^* preserves supercompactness.

The objects of $\mathcal{O}(\mathcal{M}sl)$ are exactly the supercoherent locales. On the other hand, a morphism $f: M \rightarrow L$ of $\mathcal{O}(\mathcal{M}sl)$ is a total locale map for which $f^*: L \rightarrow M$ has a finite-meet-preserving left adjoint $f_!: M \rightarrow L$. This $f_!$ is then the inverse image of a supercoherent locale map $f^\vee: M \rightarrow L$, and every supercoherent map arises in this way. We thus have an identity-on-objects isomorphism of categories $(-)^{\vee}: \mathcal{O}(\mathcal{M}sl)^{\text{op}} \rightarrow \text{sc}\mathcal{L}oc$. We claim that this induces an identity-on-objects isomorphism of categories

$$(-)^{\vee}: \text{pe}\mathcal{G}pd(\mathcal{O}(\mathcal{M}sl), \mathcal{L}oc_p) \rightarrow \text{pe}\mathcal{G}pd_c(\text{sc}\mathcal{L}oc, \mathcal{L}oc_p) . \tag{8.10}$$

The key observation is as follows. Given a total locale map $f: L \rightarrow M$, pullback along f gives a functor $\Delta_f: \mathcal{L}oc_p/\ell_h M \rightarrow \mathcal{L}oc_p/\ell_h L$. When f is a map of $\mathcal{O}(\mathcal{M}sl)$, we also have the adjoint map $f^\vee: M \rightarrow L$ and in this case, we have that $\Delta_{f^\vee} \dashv \Delta_f$. Indeed, on identifying $\mathcal{L}oc_p/\ell_h M$ with $\text{Sh}(M)$ via Theorem 4.15, this follows from the 2-functoriality [22, §C1.4] of the assignation $M \mapsto \text{Sh}(M)$.

In concrete terms, the adjointness $\Delta_{f^\vee} \dashv \Delta_f$ states that if $p: A \rightarrow L$ and $q: B \rightarrow M$ are local homeomorphisms, then there is a bijection between total locale maps g as to the left, and total locale maps \tilde{g} as to the right in:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xrightarrow{g} & B \\
 \downarrow p & & \downarrow q \\
 L & \xrightarrow{f} & M
 \end{array} & \rightsquigarrow & \begin{array}{ccc}
 A & \xleftarrow{(f^\vee)^*(A)} & B \\
 \downarrow p & & \downarrow q \\
 L & \xleftarrow{f^\vee} & M
 \end{array}
 \end{array}$$

using this, we can describe the isomorphism (8.10) as sending a partite internal functor $F: \mathbb{A} \rightarrow \mathbb{B}$ to the partite internal cofunctor $F^\vee: \mathbb{A} \rightsquigarrow \mathbb{B}$ with action on components, objects and arrows $F^\vee i = Fi$, $(F^\vee)_i = (F_i)^\vee$, and $(F^\vee)_{ij} = \widetilde{F}_{ij}$.

We now arrive at the final step, which is to exhibit the right-hand side of (8.10) as equivalent to the category of inverse categories. The starting point is the *join completion* of an inverse category.

Definition 8.22. Let \mathcal{J} be an inverse category. Its *join completion* $\mathfrak{j}(\mathcal{J})$ is the join inverse category with the same objects, and with maps $S: A \rightarrow B$ being downclosed bicompatible families $S \subseteq \mathcal{J}(A, B)$.

That $\mathfrak{j}(\mathcal{J})$ is a join inverse category can be verified by following exactly the same argument as in [24, Theorem 23]; while by following Theorem 24 of [24], we see that $\mathfrak{j}(\mathcal{J})$ is the free join restriction category on \mathcal{J} , in the sense of providing the value at \mathcal{J} of a left 2-adjoint $\mathfrak{j}: \mathfrak{iCat} \rightarrow \mathfrak{jiCat}$ to the obvious forgetful 2-functor.

We now wish to characterise the objects and morphisms in the image of \mathfrak{j} . We do so by following the approach of [25, Section 3].

Definition 8.23. A join inverse category \mathcal{C} is called *supercoherent* if $\mathcal{O}(A)$ is a supercoherent locale for all $A \in \mathcal{C}$. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between supercoherent join inverse categories is called *supercoherent* if each function $\mathcal{O}(A) \rightarrow \mathcal{O}(FA)$ is the inverse image of a supercoherent locale map. We write $\mathfrak{scjiCat}$ for the subcategory of \mathfrak{jiCat} determined by the supercoherent objects and morphisms.

Proposition 8.24. *The functor $\mathfrak{j}: \mathfrak{iCat} \rightarrow \mathfrak{jiCat}$ lands inside the subcategory $\mathfrak{scjiCat}$, and when restricted to this codomain yields an equivalence*

$$\mathfrak{scjiCat} \xrightleftharpoons[K_0]{\mathfrak{j}} \mathfrak{iCat} , \tag{8.11}$$

where K_0 sends \mathcal{C} to the subcategory $K_0(\mathcal{C})$ composed of all the objects, and all maps $s: A \rightarrow B$ for which $s^*s \in \mathcal{O}(A)$ is supercompact.

Proof. On substituting “coherent” for “supercoherent”, this is, *mutatis mutandis*, the argument of [25, Lemma 3.3, Lemma 3.4 & Proposition 3.5]. \square

Now, it is direct from the definition of supercoherence that the equivalence to the right of (8.3) restricts to an equivalence $\mathfrak{jiCat} //_h (\mathfrak{scLoc}, \mathcal{L}oc_p) \simeq \mathfrak{scjiCat}$; while from

Theorem 8.20 we have $\text{pe}\mathcal{G}\text{pd}(\text{sc}\mathcal{L}\text{oc}, \mathcal{L}\text{oc}_p) \simeq \text{jiCat}/\!/_h(\text{sc}\mathcal{L}\text{oc}, \mathcal{L}\text{oc}_p)$. Putting these together with (8.11), we obtain the desired equivalence

$$\text{pe}\mathcal{G}\text{pd}(\text{sc}\mathcal{L}\text{oc}, \mathcal{L}\text{oc}_p) \begin{array}{c} \xleftarrow{\Psi V_j} \\ \xrightarrow[\sim]{K_0 U \Phi_g} \end{array} \text{iCat}.$$

Combining this with the equality (8.8), the equivalence (8.9) and the isomorphism (8.10) reconstructs the Ehresmann–Schein–Nambooripad correspondence $\text{pJnd}\mathcal{G}\text{pd} \simeq \text{iCat}$. However, it is perhaps more illuminating to observe that we have a pseudo-commuting triangle of equivalences

$$\begin{array}{ccc} & \text{iCat} & \\ \mathcal{G} \swarrow & & \searrow \Psi V_j \\ \text{pJnd}\mathcal{G}\text{pd} & \xrightarrow{(-)^\vee \circ \mathcal{O}} & \text{pe}\mathcal{G}\text{pd}_c(\text{sc}\mathcal{L}\text{oc}, \mathcal{L}\text{oc}_p) \end{array} \tag{8.12}$$

On objects, this says that the (partite) inductive groupoid associated to an inverse category is really a presentation of the associated (partite) supercoherent localic groupoid. Equivalently, since supercoherent locales are spatial, it is a presentation of the associated supercoherent *topological* groupoid—which, from the definition of $\text{pt}: \mathcal{L}\text{oc}_p \rightarrow \mathcal{T}\text{op}_p$, we see to be Paterson’s universal groupoid of an inverse semigroup, described as in [26] as a groupoid of filters.

To conclude this section, let us comment briefly on the *non-groupoidal* analogue of the theory presented above. In [12], Gould and Hollings describe a version of the Ehresmann–Schein–Nambooripad theorem which, to the one side replaces inverse monoids by restriction monoids. To the other side, their result replaces inductive groupoids with what they call *inductive constellations*.

Although it appears that the details are even more delicate, it appears that, in exactly the same way that inductive groupoids present the étale localic groupoids associated to free join inverse monoids, inductive constellations present the source-étale localic categories associated to free join restriction categories. We will leave the details of this claim to future work.

8.5. Completion processes

Above we have discussed how our result reconstructs various equivalences from the literature. In this final section, we describe how we may exploit the larger adjunctions to construct various *completions*.

8.5.1. Full monoids

Let M be a (discrete) monoid which acts by continuous maps on a topological space X . We can view M as a one-object restriction category \mathcal{M} wherein every map is total,

and view the action of M on X as a restriction functor $\alpha: \mathcal{M} \rightarrow \mathcal{J}op_p$ sending the unique object of \mathcal{M} to the space X .

Applying the functor Ψ to $\alpha: \mathcal{M} \rightarrow \mathcal{J}op_p$ yields a source-étale topological category whose space of objects is X , and whose space of arrows is the glueing of the M -object local atlas φ with $\varphi_{mn} = 1_X$ if $m = n$ and $\varphi_{mn} = \perp$ otherwise. It is easy to see from Example 3.6 that this glueing is the product space $M \times X$, where M is endowed with the discrete topology. The source and target maps $\sigma, \tau: M \times X \rightarrow X$ are given by $\sigma(m, x) = x$ and $\tau(m, x) = m \cdot x$; the identity map is $\iota(x) = (e_M, x)$; while composition is given by $\mu((n, y), (m, x)) = (nm, x)$.

So $\Psi(\alpha)$ is the well-known *action category* of M acting on X . It follows that $\Phi\Psi(\alpha)$ is the join restriction monoid whose elements are pairs (U, s) , where $U \subseteq X$ and $s: U \rightarrow M$ is a continuous (i.e., locally constant) function. The identity is $(X, x \mapsto e)$, while the composite $(V, t)(U, s)$ is given by (W, u) , where

$$W = \{x \in X : x \in U \text{ and } s(x) \cdot x \in V\} \quad \text{and} \quad u(x) = t(s(x) \cdot x)s(x) .$$

In particular, the monoid of total elements in $\Phi\Psi(\alpha)$ comprises all continuous functions $s: X \rightarrow M$ under the multiplication $(t \circ s)(x) = t(s(x) \cdot x)s(x)$. When M acts faithfully on X , we can identify this monoid with a submonoid of $\mathcal{J}(X)$, comprising all endomorphisms of M which act locally like an element of M ; we might reasonably call this the *topological full monoid* of M in X .

If now M is a group, then \mathcal{M} is an inverse category, and so $\Psi(\alpha)$ is by Theorem 6.3 a groupoid. In this case, we have not only the étale join restriction monoid $\Phi\Psi(\alpha)$, but also the join inverse monoid $\Phi_g\Psi(\alpha)$; this has as elements those (U, s) for which the mapping $x \mapsto s(x) \cdot x$ is an open injection. Finally, the group of units of $\Phi_g\Psi(\alpha)$ may be identified with the group of those continuous functions $s: X \rightarrow M$ for which $x \mapsto s(x) \cdot x$ is a homeomorphism. Like before, when M acts faithfully on X , this yields the topological full group of M in X .

Of course, everything described above works equally well for actions on spaces by discrete categories or groupoids; we leave the adaptations to the reader.

8.5.2. Relative join completions

[25] introduces the notion of a *coverage* C on an inverse semigroup S , and describes in particular cases the *relative join completion* of S with respect to C ; this is the free join inverse semigroup admitting a map from S which sends each cover in C to a join.

The purpose of this section is to construct relative join completions in greater generality by exploiting our main result. We begin with the necessary definitions.

Definition 8.25. A *sieve* on a map $f: i \rightarrow j$ of a restriction category \mathcal{A} is a down-closed subset of $\downarrow g = \{g \in \mathcal{A}(i, j) : g \leq f\}$. A *coverage* on \mathcal{A} is given by specifying, for each $f \in \mathcal{A}(i, j)$, a collection $C(f)$ of sieves on f , called *covers*, satisfying the following axioms:

- (i) If $f \in \mathcal{A}(i, j)$ and $g \in \mathcal{A}(j, k)$, then $X \in C(g)$ implies $Xf \in C(gf)$.
- (ii) $X \in C(f)$ if and only if $\overline{X} \in C(\overline{f})$.

Our definition of coverage is modelled after the notion of coverage on a meet-semilattice in [21], and indeed reduces to it in the case where \mathcal{C} is the one-object join restriction category ΣM associated to a meet-semilattice M . We omit some of the additional clauses for a coverage listed in [25], which merely impose inessential additional “saturation” conditions on the class of covers. We will, however, note the following consequence of the axioms, which will be useful later.

Lemma 8.26. *Let C be a coverage on \mathcal{A} . If $f \in \mathcal{A}(i, j)$ and $g \in \mathcal{A}(j, k)$, then $X \in C(f)$ implies $gX \in C(gf)$.*

Proof. We first show that $\overline{gX} \in C(\overline{gf})$. First, since $\overline{gf} = f\overline{gf}$, we have by axiom (i) that $X\overline{gf} \in C(\overline{gf})$. But $X\overline{gf} = \{x\overline{gf} : x \in X\}$, and since $x \leq f$ we have $x\overline{gf} = f\overline{gf}x = \overline{gf}x = \overline{g}x$; whence $X\overline{gf} = \overline{g}X \in C(\overline{gf})$ as claimed.

We now show $gX \in C(gf)$. Since $\overline{gX} \in C(\overline{gf})$ we have by axiom (ii) that $\overline{\overline{gX}} \in C(\overline{\overline{gf}})$, whence $\overline{gX} \in C(\overline{gf})$, whence by axiom (ii) again, $gX \in C(gf)$. \square

Definition 8.27. Let \mathcal{A} be a restriction category with a coverage C . If \mathcal{B} is a join restriction category, then a restriction functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is called a *cover-to-join map* if, for each $X \in \mathcal{C}(f)$, we have $\bigvee_{x \in X} Fx = Ff$. By a *relative join completion* of \mathcal{A} with respect to C , we mean a join restriction category $\text{j}_C(\mathcal{A})$ endowed with a cover-to-join functor $\eta: \mathcal{A} \rightarrow \text{j}_C(\mathcal{A})$, such that any other cover-to-join map $F: \mathcal{A} \rightarrow \mathcal{B}$ factors through η via a unique join restriction functor $F': \text{j}_C(\mathcal{A}) \rightarrow \mathcal{B}$.

Our objective is to construct relative join completions by exploiting the adjunction of our main theorem together with the known construction of the relative join completion for meet-semilattices. We begin by discussing the latter. As noted above, a coverage in the sense of [21] on a meet-semilattice M is the same as a coverage in our sense on ΣM , so that by [21, §II.2.11] we have:

Proposition 8.28. *Let C be a coverage on a meet semilattice M . The relative join completion of ΣM with respect to C is $\Sigma(C\text{-Idl}(M))$, where $C\text{-Idl}(M)$ is the locale of C -closed ideals in M , whose elements are down-closed subsets $D \subseteq M$ with the property that $A \subseteq D$ and $A \in C(x)$ imply $x \in D$. The universal cover-to-join map $\eta: \Sigma M \rightarrow \Sigma(C\text{-Idl}(M))$ sends $m \in M$ to the C -closed ideal generated by m .*

We now exploit this result to construct a variant of the fundamental functor for a restriction category \mathcal{A} endowed with a coverage C .

- For each object $i \in \mathcal{A}$, the coverage C restricts to a coverage of the same name on the meet-semilattice $\mathcal{O}(i)$. We write $\mathcal{O}_C(i)$ for the locale $C\text{-Idl}(\mathcal{O}(i))$.

- For each map $f: i \rightarrow j$ in \mathcal{A} , we claim there is a unique partial locale map $\mathcal{O}_C(f): \mathcal{O}_C(i) \rightarrow \mathcal{O}_C(j)$ whose inverse image map renders commutative the square of binary-meet-preserving functions to the left in:

$$\begin{array}{ccc}
 \mathcal{O}(j) & \xrightarrow{\overline{(-)f}} & \mathcal{O}(i) \\
 \eta \downarrow & & \downarrow \eta \\
 \mathcal{O}_C(j) & \xrightarrow{\mathcal{O}_C(f)^*} & \mathcal{O}_C(i)
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{O}(j) & \xrightarrow{\overline{(-)f}} & \downarrow \overline{f} \\
 \eta \downarrow & & \downarrow \eta \\
 \mathcal{O}_C(j) & \dashrightarrow & \downarrow \eta(\overline{f}) .
 \end{array}
 \tag{8.13}$$

This is equally to say there is a unique *total* locale map whose inverse image renders commutative the right square of finite-meet-preserving functions. By Proposition 8.28, it suffices for this to show that the upper right composite is a cover-to-join map. But for all $e \in \mathcal{O}(j)$ and $X \in C(e)$, the coverage axioms imply that $\overline{Xf} \in C(\overline{ef})$; since η is a cover-to-join map, we conclude that $\eta(\overline{ef}) = \bigvee_{x \in X} \eta(\overline{xf})$ in $\mathcal{O}_C(i)$, and hence also in $\downarrow \eta(\overline{f})$, as required.

It follows from the unicity in (8.13) that the above assignments underlie a restriction functor $\mathcal{O}_C: \mathcal{A} \rightarrow \mathcal{L}oc_p$. We now show that \mathcal{O}_C has a universal characterisation similar in spirit to Proposition 8.7 above.

Proposition 8.29. *$\mathcal{O}_C: \mathcal{A} \rightarrow \mathcal{L}oc_p$ is a terminal object in the category whose objects are cover-to-join maps $\mathcal{A} \rightarrow \mathcal{L}oc_p$ and whose maps are total transformations. Furthermore, any $F: \mathcal{A} \rightarrow \mathcal{L}oc_p$ which does admit a total transformation $F \Rightarrow \mathcal{O}_C$ is necessarily a cover-to-join map.*

Proof. First we show that \mathcal{O}_C is indeed a cover-to-join map. Let $f \in \mathcal{A}(i, j)$ and $X \in C(f)$. We must show that $\mathcal{O}_C(f) = \bigvee_{x \in X} \mathcal{O}_C(x)$. By unicity in (8.13), it suffices for this to exhibit an equality of binary-meet-preserving maps

$$\eta \circ \overline{(-)f} = \bigvee_{x \in X} \eta \circ \overline{(-)x}: \mathcal{O}(j) \rightarrow \mathcal{O}_C(i) .$$

Evaluating at $e \in \mathcal{O}(j)$ and using the fact that η is a cover-to-join map, it suffices to show that $\overline{eX} \in C(\overline{ef})$ —which follows from $X \in C(f)$ using the axioms and Lemma 8.26.

We now show that \mathcal{O}_C is terminal among cover-to-join maps. Indeed, suppose that $F: \mathcal{A} \rightarrow \mathcal{L}oc_p$ is another such. If we were to have a total natural transformation $\alpha: F \rightarrow \mathcal{O}_C$, then for every $i \in \mathcal{A}$ and $e \in \mathcal{O}(e)$, we would have the naturality square of inverse image mappings as to the left in:

$$\begin{array}{ccc}
 Fi \xleftarrow{\alpha_i^*} \mathcal{O}_C(i) & & Fe^*(\top) = \alpha_i^*(\eta(e)) \xleftarrow{\alpha_i^*} \eta(e) \\
 \uparrow Fe^* & & \uparrow Fe^* \\
 Fi \xleftarrow{\alpha_i^*} \mathcal{O}_C(i) & & \top \xleftarrow{\alpha_i^*} \top \\
 & \uparrow \mathcal{O}_C(e)^* & \uparrow \mathcal{O}_C(e)^*
 \end{array}$$

and so evaluating at the top element of $\mathcal{O}_C(i)$, the equality to the right. So the join- and finite-meet-preserving map α_i^* must map $\eta(e)$ to $Fe^*(\top)$ for each $e \in \mathcal{O}(i)$, and by the universal property of $\eta: \mathcal{O}(i) \rightarrow \mathcal{O}_C(i)$, this completely determines α_i^* . So there is *at most* one total transformation $\alpha: F \rightarrow \mathcal{O}_C$. It remains to show that defining α_i^* in this way yields a well-defined total transformation.

For α_i^* to be well-defined, we must show that $e \mapsto Fe^*(\top)$ is a finite-meet-preserving cover-to-join map $\mathcal{O}(i) \rightarrow Fi$. Finite meet preservation is as in [6, Proposition 3.2]; on the other hand, if $e \in \mathcal{O}(i)$ and $X \in C(e)$, then, since F is a cover-to-join map, we must have $Fe = \bigvee_x Fx$, and so $Fe^*(\top) = \bigvee_x Fx^*(\top)$, as desired. Finally, for naturality of α , we again argue as in [6, Proposition 3.2].

It remains to prove the final claim of the proposition. We will prove more generally that, if $F \Rightarrow G: \mathcal{A} \rightarrow \mathcal{B}$ is a total transformation and G is a cover-to-join map, then F is too. So let $f \in \mathcal{A}(i, j)$ and $X \in C(f)$. We know that $\bigvee_{x \in X} Fx \leq Ff$ and so it suffices to show both sides have the same restriction, for which we calculate (using totality of α) that:

$$\overline{Ff} = \overline{\alpha_j Ff} = \overline{Gf \alpha_i} = \overline{\bigvee_x Gx \circ \alpha_i} = \overline{\bigvee_x \alpha_j Fx} = \overline{\bigvee_x Fx}. \quad \square$$

We can now use this to prove:

Theorem 8.30. *Let C be a coverage on the restriction category \mathcal{A} . The unit*

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{\eta} & \Phi\Psi(\mathcal{O}_C) \\
 & \searrow \mathcal{O}_C & \swarrow \pi_{\Psi \circ \mathcal{O}_C} = \mathcal{O}_{\Phi\Psi(\mathcal{O}_C)} \\
 & & \mathcal{Loc}_p
 \end{array}$$

at \mathcal{O}_C of our main adjunction (5.16) exhibits $\Phi\Psi(\mathcal{O}_C)$ as the relative join completion of \mathcal{A} with respect to \mathcal{C} .

Proof. We claim that, for any join restriction category \mathcal{B} , a restriction functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is a cover-to-join map if and only if it can be extended, necessarily uniquely, to a map

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\
 \searrow \mathcal{O}_C & \leftarrow & \swarrow \mathcal{O}_B \\
 & \mathcal{L}oc_p &
 \end{array}$$

of $\mathbf{rCat} // \mathcal{L}oc_p$. Indeed, from the fact that \mathcal{O}_B is hyperconnected it follows easily that F is a cover-to-join map if and only if $\mathcal{O} \circ F$ is so; and now by Proposition 8.29, $\mathcal{O} \circ F$ is a cover-to-join map if and only if it admits a, necessarily unique total transformation as displayed to \mathcal{O}_C .

The theorem now follows immediately from the above observation and the universal property of the adjunction $\Psi \dashv \Phi$. \square

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