INNER AUTOMORPHISMS OF GROUPOIDS

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Abstract. Bergman has given the following abstract characterisation of the inner automorphisms of a group G: they are exactly those automorphisms of G which can be extended functorially along any homomorphism $G \to H$ to an automorphism of H. This leads naturally to a definition of "inner automorphism" applicable to the objects of any category. Bergman and Hofstra–Parker–Scott have computed these inner automorphisms for various structures including k-algebras, monoids, lattices, unital rings, and quandles—showing that, in each case, they are given by an obvious notion of conjugation.

In this note, we compute the inner automorphisms of groupoids, showing that they are exactly the automorphisms induced by conjugation by a bisection. The twist is that this result is *false* in the category of groupoids and homomorphisms; to make it true, we must instead work with the less familiar category of groupoids and *comorphisms* in the sense of Higgins and Mackenzie. Besides our main result, we also discuss generalisations to topological and Lie groupoids, to categories and to partial automorphisms, and examine the link with the theory of inverse semigroups.

1. Background

In [2], Bergman gave an element-free characterisation of the inner automorphisms of a group G: they are exactly those automorphisms of G which can be extended in a functorial way along any group homomorphism $f\colon G \to H$ to an automorphism of H. More precisely, Bergman defines an *extended inner automorphism* β of G to be a family of group automorphisms $(\beta_f\colon H \to H)$, one for each homomorphism $f\colon G \to H$, with the property that the square

$$\begin{array}{ccc} H & \xrightarrow{\beta_f} & H \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ K & \xrightarrow{\beta_{gf}} & K \end{array} \tag{1.1}$$

commutes for all $f\colon G \to H$ and $g\colon H \to K$. It is easy to see that each $a \in G$ gives rise to an extended inner automorphism by taking

$$\beta_f(x) = f(a)xf(a)^{-1} \quad \text{for all} \quad f\colon G \to H\ . \tag{1.2}$$

Less obvious is the fact, proved in [2], that every extended inner automorphism of G is of this form for a *unique* $a \in G$; in particular, an automorphism of G is

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inner just when it is the component β_{1_G} of some extended inner automorphism. In light of this characterisation, it is natural for Bergman to define:

Definition 1 ([2]). An *extended inner automorphism* of an object G of a category $\mathcal{C}$ is a family of automorphisms $(\beta_f \colon H \to H)$, one for each map $f \colon G \to H$ in $\mathcal{C}$, which render commutative the square (1.1) for each $f \colon G \to H$ and $g \colon H \to K$.

Bergman goes on to characterise the extended inner automorphisms in the category of k-algebras over a field k, as well as the extended inner *endomorphisms* (dropping the requirement of invertibility of the maps β_f) for groups and k-algebras, along with further variants such as "inner derivations".

In [6], the calculation of extended inner automorphism groups is pursued further, with these being worked out in full for the categories of monoids, abelian groups, lattices, unital rings, racks, and quandles; in each case, the extended inner automorphisms capture a natural notion of *conjugation*. Besides these calculations, [6] also draws the connection between Bergman's extended inner automorphisms and the *isotropy group* [4] of a topos $\mathcal{E}$; this is the universal group object in $\mathcal{E}$ which acts naturally on every $X \in \mathcal{E}$. The link is that, in the case of a presheaf topos $\mathcal{E} = [\mathcal{C}, \mathcal{S}\mathrm{et}]$, the isotropy group of $\mathcal{E}$ is the functor $\mathcal{Z} \colon \mathcal{C} \to \mathcal{G}\mathrm{rp}$ sending each $X \in \mathcal{C}$ to its group of extended inner automorphisms.

In this note we show that, in the context of *groupoids*, extended inner automorphisms once again capture a natural notion of conjugation: not now by a single element, but rather by a suitable family of elements of a kind which is well-known from the study of Lie and topological groupoids.

Definition 2. A *bisection* α of a groupoid $\mathbb{G}$ is an $\mathbb{G}_0$-indexed family of morphisms $(\alpha_u \colon u \to \tilde{\alpha}(u))$ for which the function $\tilde{\alpha} \colon \mathbb{G}_0 \to \mathbb{G}_0$ is a bijection.

(Here, and subsequently, we write $\mathbb{G}_0$ and $\mathbb{G}_1$ for the sets of objects and morphisms of a groupoid.) Each bisection induces a conjugation automorphism $c_\alpha \colon \mathbb{G} \to \mathbb{G}$ with action on objects $\tilde{\alpha}$ and action on morphisms

$$x \colon u \to v \qquad \mapsto \qquad \alpha_v \circ x \circ \alpha_u^{-1} \colon \tilde{\alpha}(u) \to \tilde{\alpha}(v)\ ; \tag{1.3}$$

however, the characterisation of these as the extended inner automorphisms of groupoids is slightly delicate. It is *not* true that $c_\alpha \colon \mathbb{G} \to \mathbb{G}$ is the $1_\mathbb{G}$-component of an extended inner automorphism of $\mathbb{G}$ in the category $\mathcal{G}\mathrm{rpd}$ of small groupoids and homomorphisms. There is an intuitively clear explanation for this fact: given a homomorphism $f \colon \mathbb{G} \to \mathbb{H}$, we should like to define $\beta_f = c_{f\alpha}$, as in the case of groups, but there is no obvious way of defining the pushforward $f\alpha$ of the bisection α along f. While this does not rule out the possibility that there is some *non*-obvious way of defining the pushfoward, we find that, in fact:

Proposition 3. *There are no non-trivial extended inner automorphisms in* $\mathcal{G}\mathrm{rpd}$.

Proof. Let β be an extended inner automorphism of the small groupoid $\mathbb{G}$. Consider the coproduct $\mathbb{G}+1$ of $\mathbb{G}$ with the terminal groupoid, and $\iota \colon \mathbb{G} \to \mathbb{G}+1$ the coproduct injection. By (1.1), we have $\beta_\iota \circ \iota = \iota \circ \beta_{1_\mathbb{G}}$, and so the automorphism $\beta_\iota \colon \mathbb{G}+1 \to \mathbb{G}+1$ must map the full subcategory $\mathbb{G}$ of $\mathbb{G}+1$ into itself; thus, to be bijective on objects, it must map the remaining object $\star$ of $\mathbb{G}+1$ to itself.

Now for any homomorphism $f\colon \mathbb{G} \to \mathbb{H}$ and any $x \in \mathbb{H}$, there is a (unique) homomorphism $\langle f, x\rangle \colon \mathbb{G} + 1 \to \mathcal{H}$ such that $\langle f, x\rangle \circ \iota = f$ and $\langle f, x\rangle(\star) = x$. The first condition implies using (1.1) that $\beta_f \circ \langle f, x\rangle = \langle f, x\rangle \circ \beta_\iota$, whence

$$\beta_f(x) = \beta_f(\langle f, x\rangle(\star)) = \langle f, x\rangle(\beta_\iota(\star)) = \langle f, x\rangle(\star) = x \tag{1.4}$$

so that each β_f is the identity on objects.

Consider now the groupoid $\mathbb{G}+\mathcal{J}$ and coproduct injection $\jmath\colon \mathbb{G} \to \mathbb{G}+\mathcal{J}$, where here $\mathcal{J}$ is the groupoid with two objects and a single isomorphism between them. Writing $\alpha\colon \upsilon \to \nu$ for the image of this isomorphism in $\mathbb{G} + \mathcal{J}$, we conclude from the fact that $\beta_\jmath\colon \mathbb{G} + \mathcal{J} \to \mathbb{G} + \mathcal{J}$ is the identity on objects that $\beta_\jmath(\alpha) = \alpha$.

Now for any homomorphism $f\colon \mathbb{G} \to \mathbb{H}$ and any arrow $a\colon u \to v \in \mathbb{H}$, there is a (unique) homomorphism $\langle f, a\rangle \colon \mathbb{G}+\mathcal{J} \to \mathcal{H}$ with $\langle f, a\rangle \circ \iota = f$ and $\langle f, x\rangle(\alpha) = a$. Repeating the calculation (1.4), *mutatis mutandis*, we conclude that $\beta_f(a) = a$ so that each β_f is also the identity on morphisms. □

Despite this negative result, we *can* exhibit the conjugation automorphisms $c_\alpha\colon \mathbb{G} \to \mathbb{G}$ as the values of extended inner automorphisms: to do so, we need to alter the kind of morphism that we consider between groupoids. Rather than homomorphisms of groupoids, we must consider the *comorphisms* of Higgins and Mackenzie [5]. A comorphism $f\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ between groupoids comprises a function $f\colon \mathbb{H}_0 \to \mathbb{G}_0$ together with the assignation to each $a\colon fu \to v$ in $\mathbb{G}$ of a map $f(a)_u\colon u \to \tilde{f}(a, u)$ in $\mathbb{H}$, subject to suitable axioms. As explained in [1], bisections *do* transport along comorphisms; this rectifies the problem we observed earlier, allowing us to prove that:

Theorem. *The extended inner automorphisms of an object $\mathbb{G}$ of the category of small groupoids and comorphisms are in bijection with the bisections of $\mathbb{G}$. The extended inner automorphism corresponding to a bisection α is given by the family of congjuation automorphisms $(c_{f\alpha} \mid f\colon \mathbb{G} \rightsquigarrow \mathbb{H})$.*

This is our main result, and will be proven in Section 3 below. Preceding this is Section 2, which sets up the necessary background on [5]'s notion of comorphism, the relation to the usual groupoid homomorphisms, and the link with bisections. Finally, after proving our main result, we describe in Section 4 various natural generalisations—to topological and Lie groupoids, and to categories—and discuss an alternative perspective involving inverse semigroups.

2. Comorphisms and bisections

The definition of comorphism we give here is not the original one of [5], but a reformulation due to [1].

Definition 4. A *comorphism* (sometimes also called *cofunctor*) $f\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ between small groupoids comprises a function $f\colon \mathbb{H}_0 \to \mathbb{G}_0$ together with functions

$$\begin{array}{ccc} \sum_{v\in\mathbb{G}} \mathbb{G}(fu, v) & \to & \sum_{v'\in\mathbb{H}} \mathbb{H}(u, v') \\ fu \xrightarrow{a} v & \mapsto & u \xrightarrow{f(a)_u} \tilde{f}(a, u) \end{array}$$

for each $u \in \mathbb{H}_0$, subject to the following axioms:

(i) $f(\tilde{f}(a,u)) = v$ for all $a\colon fu \to v$ in $\mathbb{G}$;
(ii) $f(1_{fu})_u = 1_u$ for all $u \in \mathbb{H}_0$;
(iii) $f(b)_{\tilde{f}(a,u)} \circ f(a)_u = f(ba)_u$ for all $a\colon fu \to v$ and $b\colon v \to w$ in $\mathbb{G}$.

Comorphisms may be composed in the evident manner, and in this way we obtain a category $\mathcal{G}\mathrm{rpd}_{\mathrm{co}}$ of small groupoids and comorphisms.

There are two ways in which an ordinary homomorphism of groupoids can give rise to a comorphism. On the one hand, any bijective-on-objects homomorphism $f\colon \mathbb{G} \to \mathbb{H}$ induces a comorphism $f_*\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ which on objects acts as the inverse to $f\colon \mathbb{G}_0 \to \mathbb{H}_0$, and on maps is given by the assignation

$$f_*(u) \xrightarrow{a} v \quad \mapsto \quad u \xrightarrow{f(a)} f(v) \ .$$

On the other hand, we can obtain a comorphism from any *discrete opfibration*. Recall that a homomorphism of groupoids $f\colon \mathbb{G} \to \mathbb{H}$ is a discrete opfibration if, for each $u \in \mathbb{H}_0$ and map $a\colon fu \to v$ in $\mathbb{G}$, there is a unique map $f(a)_u\colon u \to \tilde{f}(a,u)$ in $\mathbb{H}$ whose domain is u and whose image under f is a. In this situation, the action on objects and unique liftings of arrows for $f\colon \mathbb{G} \to \mathbb{H}$ provide the data of a comorphism $f^*\colon \mathbb{H} \rightsquigarrow \mathbb{G}$.

In fact, as explained in [1, Section 4.4], all comorphisms are generated from those in the image of $(-)_*$ and $(-)^*$:

Proposition 5. *Any comorphism $f\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ can be decomposed as*

$$f = \mathbb{G} \overset{(f_1)^*}{\rightsquigarrow} \mathbb{K} \overset{(f_2)_*}{\rightsquigarrow} \mathbb{H} \ . \tag{2.1}$$

Proof. Let $\mathbb{K}$ be the groupoid whose objects are those of $\mathbb{H}$ and whose maps $u \twoheadrightarrow v$ are maps $a\colon fu \to fv$ of $\mathbb{G}$ such that $\tilde{f}(a,u) = v$. Composition is inherited from $\mathbb{G}$, and is well-defined by axioms (ii) and (iii) for a comorphism. There is an identity-on-objects homomorphism $f_2\colon \mathbb{K} \to \mathbb{H}$ given on maps by $f_2(a\colon u \twoheadrightarrow v) = f(a)_u\colon u \to v$; and so we can form $(f_2)_*\colon \mathbb{K} \rightsquigarrow \mathbb{H}$. There is also a homomorphism $f_1\colon \mathbb{K} \to \mathbb{G}$ with the same action as f on objects, and action on morphisms $f_1(a\colon u \twoheadrightarrow v) = a\colon fu \to fv$. Note that, for any map $a\colon fu \to v$ in $\mathbb{G}$, the unique map of $\mathbb{K}$ with domain u whose f_1-image is a is $a\colon u \twoheadrightarrow \tilde{f}(a,u)$. So f_1 is a discrete opfibration, and we can form $(f_1)^*\colon \mathbb{G} \rightsquigarrow \mathbb{K}$. It is now direct from the definitions that $f = (f_2)_*(f_1)^*$ as in (2.1). ☐

In fact, we can equally define comorphisms $\mathbb{G} \rightsquigarrow \mathbb{H}$ as (equivalence classes of) spans $\mathbb{G} \leftarrow \mathbb{K} \to \mathbb{H}$ with left leg a discrete opfibration and right leg bijective-on-objects; this is essentially the original definition of [5]. The following "Beck–Chevalley lemma" shows that composition of comorphisms corresponds to the composition of the representing spans by pullback.

Lemma 6. *Given a commuting square of homomorphisms*

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{f} & \mathbb{H} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ \mathbb{K} & \xrightarrow{g} & \mathbb{L} \end{array}$$

where both f and g are bijective on objects and both h and k are discrete opfibrations[1]*, we have $f_*h^* = k^*g_* \colon \mathbb{K} \rightsquigarrow \mathbb{H}$.*

Proof. On objects, the two composites act by $x \mapsto g^{-1}k(x)$ and $x \mapsto hf^{-1}(x)$; these coincide since $gh = kf$ on objects. For a map $a \colon g^{-1}k(x) = hf^{-1}(x) \to y$ in $\mathbb{K}$, its image $(k^*g_*)(a)_x$ is the unique map of $\mathbb{H}$ with domain x and k-image $g(a)$. On the other hand, $(f_*h^*)(a)_x = f(a')$, where a' is the unique map of $\mathbb{G}$ with domain $f^{-1}(x)$ and h-image a. It follows that $f(a')$ has domain x and k-image $kf(a'') = gh(a'') = g(a)$, and so we have $(f_*h^*)_x(a) = (k^*g_*)_x(a)$ as desired. □

Since we will be interested in automorphisms in the category $\mathcal{G}\mathrm{rpd}_{\mathrm{co}}$, the following lemma will be useful; its straightforward proof is left to the reader.

Lemma 7. *If $f \colon \mathbb{G} \rightsquigarrow \mathbb{H}$ is an invertible map in $\mathcal{G}\mathrm{rpd}_{\mathrm{co}}$, then $f = g_*$ for a unique invertible functor $g \colon \mathbb{G} \to \mathbb{H}$; moreover, we have $f^{-1} = g^*$.*

Finally in this section, we discuss the relationship between comorphisms and bisections. This is most clearly expressed in the terms of the fully faithful functor

$$\Sigma \colon \mathcal{G}\mathrm{rp} \to \mathcal{G}\mathrm{rpd}_{\mathrm{co}}$$

from the category of groups which on objects takes G to the corresponding one-object groupoid ΣG. We will show that the taking of bisections provides a right adjoint to this functor.

First observe that the set $\mathrm{Bis}(\mathbb{G})$ of bisections of a groupoid $\mathbb{G}$ is indeed a group under the operation on bisections $\beta, \alpha \mapsto \beta \cdot \alpha$ given by

$$(\beta \cdot \alpha)_u = u \xrightarrow{\alpha_u} \tilde{\alpha}(u) \xrightarrow{\beta_{\tilde{\alpha}(u)}} \tilde{\beta}(\tilde{\alpha}(u)) \ .$$

The identity element is the bisection $\mathbf{1}$ with $(\mathbf{1})_u = 1_u$. The inverse of the bisection α is the bisection α^{-1} determined by $(\alpha^{-1})_{\tilde{\alpha}(u)} = (\alpha_u)^{-1}$.

Proposition 8. *The full embedding $\Sigma \colon \mathcal{G}\mathrm{rp} \to \mathcal{G}\mathrm{rpd}_{\mathrm{co}}$ has a right adjoint whose value at a groupoid $\mathbb{G}$ is given by the group of bisections $\mathrm{Bis}(\mathbb{G})$.*

Proof. Let H be a group and $\mathbb{G}$ a groupoid, and consider what it is to give a comorphism $f \colon \Sigma H \to \mathbb{G}$. On objects, f must send each object $u \in \mathbb{G}$ to the unique object $*$ of ΣH. On morphisms, we must give an assignation

$$(f(u) \xrightarrow{a} *) \quad \mapsto \quad (u \xrightarrow{f(a)_u} \tilde{f}(a,u)) \ ,$$

where $a \in H$ and $u \in \mathbb{G}_0$, subject to the axioms (i)–(iii) of Definition 4. Axiom (i) is trivial as ΣH has only one object. Axioms (ii) and (iii) state that

$$f(1_H)_u = 1_u \qquad \text{and} \qquad f(ba)_u = f(b)_{\tilde{f}(a,u)} \circ f(a)_u \tag{2.2}$$

for all $a, b \in H$ and all $u \in \mathbb{H}_0$. Taking codomains, we have

$$\tilde{f}(1_H, u) = u \qquad \text{and} \qquad \tilde{f}(ba, u) = \tilde{f}(b, \tilde{f}(a,u))$$

so that the assignation $(a,u) \mapsto \tilde{f}(a,u)$ is an H-action on $\mathbb{G}_0$. In particular, for each $a \in H$ the function $u \mapsto \tilde{f}(a,u)$ is invertible, so that the collection of maps $(f(a)_u \colon u \to \tilde{f}(a,u))$ constitutes a bisection $\bar{f}(a) \in \mathrm{Bis}(\mathbb{G})$. In these terms, the

[1] Such a square is necessarily a pullback.

conditions (2.2) state precisely that the mapping $\bar{f}\colon H \to \mathrm{Bis}(\mathbb{G})$ so obtained is a group homomorphism. In this way, we have produced bijections

$$(\overline{})\colon \mathcal{G}\mathrm{rpd}_{\mathrm{co}}(\Sigma H, \mathbb{G}) \to \mathcal{G}\mathrm{pd}(H, \mathrm{Bis}(\mathbb{G})) \ . \tag{2.3}$$

which are easily seen to be natural in H. This shows, as claimed, that Σ has a right adjoint whose value at the groupoid $\mathbb{G}$ is given by $\mathrm{Bis}(\mathbb{G})$. ☐

A consequence of the adjointness exhibited above is that the assignation $\mathbb{G} \mapsto \mathrm{Bis}(\mathbb{G})$ extends uniquely to a functor $\mathrm{Bis}\colon \mathcal{G}\mathrm{rpd}_{\mathrm{co}} \to \mathcal{G}\mathrm{rp}$ making the bijections (2.3) natural in $\mathbb{G}$ as well as H. In other words, there is a canonical way of transporting bisections along comorphisms. To read off an explicit formula for this transport, note that, since bisections of $\mathbb{G}$ correspond bijectively to group homomorphisms $\mathbb{Z} \to \mathrm{Bis}(\mathbb{G})$, they also correspond bijectively to comorphisms $\Sigma\mathbb{Z} \rightsquigarrow \mathbb{G}$. In these terms, the transport of a such a bisection along a comorphism $\mathbb{G} \rightsquigarrow \mathbb{H}$ is given simply by postcomposition. Spelling this out, we obtain:

Definition 9. Given a a comorphism $f\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ and a bisection α of $\mathbb{G}$, the *pushforward bisection* $f\alpha$ of $\mathbb{H}$ is defined by $(f\alpha)_u = f(\alpha_{fu})_u\colon u \to \tilde{f}(\alpha_{fu}, u)$.

In particular, if $f\colon \mathbb{G} \to \mathbb{H}$ is a bijective-on-objects functor, then pushing forward the bisection α of $\mathbb{G}$ along f_* yields the bisection $f_*\alpha$ of $\mathbb{H}$ whose components are determined by

$$(f_*\alpha)_{fu} = f(\alpha_u) \ . \tag{2.4}$$

On the other hand, if $f\colon \mathbb{H} \to \mathbb{G}$ is a discrete opfibration, then we can push forward α along f^* to obtain the bisection $f^*\alpha$ of $\mathbb{H}$ uniquely determined by

$$f((f^*\alpha)_u) = \alpha_{fu} \ . \tag{2.5}$$

3. Inner automorphisms of groupoids

We now have all the necessary background to prove our main result. We begin with the easier direction.

Proposition 10. *Each bisection α of the groupoid $\mathbb{G}$ gives an extended inner automorphism of $\mathbb{G}$ in $\mathcal{G}\mathrm{rpd}_{\mathrm{co}}$ whose component at $f\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ is the conjugation isomorphism $(c_{f\alpha})_*\colon \mathbb{H} \rightsquigarrow \mathbb{H}$.*

Proof. We must check that, for each bisection α of a groupoid $\mathbb{G}$, and each $f\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ and $g\colon \mathbb{H} \rightsquigarrow \mathbb{K}$, the square of comorphisms left below commutes:

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{(c_{f\alpha})_*} & \mathbb{H} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{K} & \xrightarrow{(c_{gf\alpha})_*} & \mathbb{K} \end{array} \qquad \begin{array}{ccc} \mathbb{G} & \xrightarrow{(c_{\alpha})_*} & \mathbb{G} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{K} & \xrightarrow{(c_{g\alpha})_*} & \mathbb{K} \end{array} \ . \tag{3.1}$$

Since $f\alpha$ is a bisection of $\mathbb{H}$, we can without loss of generality assume that $\mathbb{H} = \mathbb{G}$ and $f = 1$, and so reduce to checking commutativity as right above. By Proposition 5 we can in turn reduce to the cases where $g = f_*$ or where $g = f^*$.

If $g = f_*$ for a bijective-on-objects f, then we need only check commutativity to the left in:

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{c_\alpha} & \mathbb{G} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{K} & \xrightarrow{c_{f_*(\alpha)}} & \mathbb{K} \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{K} & \xrightarrow{c_{f^*(\alpha)}} & \mathbb{K} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{G} & \xrightarrow{c_\alpha} & \mathbb{G} \end{array}\ ;$$

and this holds at a map $a\colon u \to v$ of $\mathbb{G}$ since, by functoriality of f and (2.4),

$$f(\alpha_v \circ a \circ \alpha_u^{-1}) = f(\alpha_v) \circ fa \circ f(\alpha_u)^{-1} = (f_*\alpha)_v \circ fa \circ (f_*\alpha)_u\ .$$

On the other hand, if $g = f^*$ for a discrete opfibration f, then on replacing the horizontal maps $(c_\alpha)_*$ and $(c_{f^*\alpha})_*$ in (3.1) by their inverses $(c_\alpha)^*$ and $(c_{f^*\alpha})^*$, we may reduce to checking commutativity of the square right above. This equality is verified at $a\colon u \to v$ in $\mathbb{K}$ since, by functoriality of f and (2.5),

$$f\big((f^*\alpha)_v \circ a \circ (f^*\alpha)_u^{-1}\big) = f\big((f^*\alpha)_v\big) \circ fa \circ f\big((f^*\alpha)_u^{-1}\big) = \alpha_{fv} \circ fa \circ \alpha_{fu}^{-1}\ . \qquad \square$$

It remains to show that:

Proposition 11. *Each extended inner automorphism of $\mathbb{G}$ in $\mathcal{G}\mathrm{rpd}_{\mathrm{co}}$ is induced in the manner of Proposition 10 from a* unique *bisection α of $\mathbb{G}$.*

Proof. Suppose we are given an extended inner automorphism β of $\mathbb{G}$ with components $(\beta_f)_*\colon \mathbb{H} \rightsquigarrow \mathbb{H}$. To prove the result, we must exhibit a unique bisection α of $\mathbb{G}$ such that $\beta_f = c_{f\alpha}$ for each f.

We first construct α. For each $u \in \mathbb{G}$, consider the coslice groupoid $u/\mathbb{G}$, whose objects are arrows $a\colon u \to v$ of $\mathbb{G}$ with domain u, and whose morphisms are commuting triangles under u. The obvious codomain projection $\pi_u\colon u/\mathbb{G} \to \mathbb{G}$ is a discrete opfibration, and so among the data of β is an automorphism $\beta_{\pi_u^*}\colon u/\mathbb{G} \to u/\mathbb{G}$. Let $\alpha_u\colon u \to \tilde{\alpha}(u)$ be the image of $1_u \in u/\mathbb{G}$ under $\beta_{\pi_u^*}$.

Now as β is an extended inner automorphism, the square of comorphisms left below commutes. Replacing $(\beta_{1_\mathbb{G}})_*$ and $(\beta_{\pi_u^*})_*$ by their inverses $(\beta_{1_\mathbb{G}})^*$ and $(\beta_{\pi_u^*})^*$, this is to say that the square of homomorphisms to the right below commutes. (Henceforth we will make such reductions to homomorphisms without comment.)

(3.2) $$\begin{array}{ccc} u/\mathbb{G} & \overset{(\beta_{\pi_u^*})_*}{\rightsquigarrow} & u/\mathbb{G} \\ {\scriptstyle (\pi_u)^*}\wr & & \wr{\scriptstyle (\pi_u)^*} \\ \mathbb{G} & \underset{(\beta_{1_\mathbb{G}})_*}{\rightsquigarrow} & \mathbb{G} \end{array} \qquad\qquad \begin{array}{ccc} u/\mathbb{G} & \xrightarrow{\beta_{\pi_u^*}} & u/\mathbb{G} \\ {\scriptstyle \pi_u}\downarrow & & \downarrow{\scriptstyle \pi_u} \\ \mathbb{G} & \xrightarrow{\beta_{1_\mathbb{G}}} & \mathbb{G} \end{array}$$

Tracing 1_u around this square yields $\beta_{1_\mathbb{G}}(u) = \tilde{\alpha}(u)$. Thus, since $\beta_{1_\mathbb{G}}$ is invertible, so is the function $u \mapsto \tilde{\alpha}(u)$; whence $(\alpha_u)_{u \in \mathbb{G}}$ is a bisection of $\mathbb{G}$.

We now show that $\beta_{1_\mathbb{G}} = c_\alpha\colon \mathbb{G} \to \mathbb{G}$. Consider a map $a\colon u \to v$ of $\mathbb{G}$. This induces by precomposition a functor $(-) \circ a\colon v/\mathbb{G} \to u/\mathbb{G}$, which fits into a commuting triangle of discrete opfibrations as left below. Since β is an extended inner automorphism, this implies the commutativity of the square of homomorphisms to the right.

$$
\begin{array}{ccc}
v/\mathbb{G} & \xrightarrow{(-)\circ a} & u/\mathbb{G} \\
 & {\scriptstyle \pi_v} \searrow \quad \swarrow {\scriptstyle \pi_u} & \\
 & \mathbb{G} &
\end{array}
\qquad\qquad
\begin{array}{ccc}
v/\mathbb{G} & \xrightarrow{\beta_{\pi_v^*}} & v/\mathbb{G} \\
{\scriptstyle (-)\circ a}\downarrow & & \downarrow{\scriptstyle (-)\circ a} \\
u/\mathbb{G} & \xrightarrow{\beta_{\pi_u^*}} & u/\mathbb{G}
\end{array}
$$

Tracing 1_v around this square, we find that $\beta_{\pi_u^*}$ sends the object $a \in u/\mathbb{G}$ to $\alpha_v \circ a \in u/\mathbb{G}$. Thus $\beta_{\pi_u^*}$ sends the *map* $a\colon 1_u \to a$ of $u/\mathbb{G}$ to a map $f\colon \alpha_u \to \alpha_v \circ a$ of $u/\mathbb{G}$. Note that $\pi_u(f)\colon \tilde{\alpha}(u) \to \tilde{\alpha}(v)$ satisfies $\pi_u(f) \circ \alpha_u = \alpha_v \circ a$, and so necessarily $\pi_u(f) = \alpha_v \circ a \circ \alpha_u^{-1}$. Thus, tracing $a\colon 1_u \to a$ around the right square of (3.2), we see that $\beta_{1_\mathbb{G}}(a\colon u \to v) = \alpha_v \circ a \circ \alpha_u^{-1}$, and so $\beta_{1_\mathbb{G}} = c_\alpha$ as claimed.

It remains to show that $\beta_f = c_{f\alpha}$ for *all* comorphisms $f\colon \mathbb{G} \rightsquigarrow \mathbb{H}$. For this, it suffices to show that the bisection associated to the extended inner automorphism $\beta_{(-)f}$ of $\mathbb{H}$ is $f\alpha$, since then $\beta_f = \beta_{(1_\mathbb{H})f} = c_{f\alpha}$ as desired. That is, we must prove:

$$
\beta_{\pi_u^* f}(1_u) = (f\alpha)_u \qquad \text{for all } f\colon \mathbb{G} \rightsquigarrow \mathbb{H} \text{ and } u \in \mathbb{H}_0 \ . \tag{3.3}
$$

Step 1. Suppose first that $f = g^*$ for some discrete opfibration $g\colon \mathbb{H} \to \mathbb{G}$. We then have a commuting square of discrete opfibrations as to the left in:

$$
\begin{array}{ccc}
u/\mathbb{H} & \xrightarrow{u/g} & gu/\mathbb{G} \\
{\scriptstyle \pi_u}\downarrow & & \downarrow{\scriptstyle \pi_{gu}} \\
\mathbb{H} & \xrightarrow{g} & \mathbb{G}
\end{array}
\qquad\qquad
\begin{array}{ccc}
u/\mathbb{H} & \xrightarrow{\beta_{\pi_u^* g^*}} & u/\mathbb{H} \\
{\scriptstyle u/g}\downarrow & & \downarrow{\scriptstyle u/g} \\
gu/\mathbb{G} & \xrightarrow{\beta_{\pi_{gu}^*}} & gu/\mathbb{G}
\end{array} \ ,
$$

and so, since β is an extended inner automorphism, a commuting square of homomorphisms as to the right. Tracing 1_u around this square yields $g(\beta_{\pi_u^* g^*}(1_u)) = \beta_{\pi_{gu}^*}(1_{gu}) = \alpha_{gu}$, and so by (2.5) that $\beta_{\pi_u^* g^*}(1_u) = (g^*\alpha)_u$ as required for (3.3).

Step 2. Suppose next that $f = h_*$ for some bijective-on-objects functor $h\colon \mathbb{G} \to \mathbb{H}$. We form the outer square, the pullback and the induced comparison map as to the left in:

$$
\begin{array}{ccccc}
u/\mathbb{G} & & \xrightarrow{u/h} & & hu/\mathbb{H} \\
 & {\scriptstyle r}\searrow & & \nearrow{\scriptstyle q} & \\
{\scriptstyle \pi_u}\downarrow & & \mathbb{P} \ \lrcorner & & \downarrow{\scriptstyle \pi_{hu}} \\
 & \swarrow{\scriptstyle p} & & & \\
\mathbb{G} & & \xrightarrow{h} & & \mathbb{H}
\end{array}
\qquad
\begin{array}{ccc}
u/\mathbb{G} & \xrightarrow{\beta_{\pi_{u*}}} & u/\mathbb{G} \\
{\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r} \\
\mathbb{P} & \xrightarrow{\beta_{p^*}} & \mathbb{P} \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\
hu/\mathbb{G} & \xrightarrow{\beta_{q_* p^*}} & hu/\mathbb{G}
\end{array}
\qquad
\begin{array}{ccc}
u/\mathbb{G} & \xrightarrow{\beta_{\pi_{u*}}} & u/\mathbb{G} \\
{\scriptstyle u/h}\downarrow & & \downarrow{\scriptstyle u/h} \\
hu/\mathbb{G} & \xrightarrow{\beta_{\pi_{hu}^* h_*}} & hu/\mathbb{G}
\end{array}
$$

In this square, π_u and π_{hu} are discrete opfibrations, and so is p, since it is a pullback of π_{hu}. It follows that the comparison map r is also a discrete opfibration. Since $pr = \pi_u$ and β is an extended inner automorphism, we have that the top square centre above commutes. On the other hand, q is bijective-on-objects as a pullback of f, and so, since β is an extended inner automorphism, we have that the bottom square centre above commutes.

By applying Lemma 6 to the pullback square left above, we have $q_*p^* = \pi^*_{hu}h_*$, and so the composite of the two centre squares is equally the square right above. Tracing the object 1_u around both sides yields $\beta_{\pi^*_{hu}h_*}(1_{hu}) = h(\alpha_u)$, and so by (2.4) we conclude that $\beta_{\pi^*_{hu}h_*}(1_u) = (h_*\alpha)_u$ as required for (3.3).

Step 3. We now prove for a general $f\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ that the bisection associated to the inner automorphism $\beta_{(-)f}$ of $\mathbb{H}$ is $f\alpha$. We first apply Proposition 5 to decompose f as $h_*g^*\colon \mathbb{G} \rightsquigarrow \mathbb{K} \rightsquigarrow \mathbb{H}$. By Step 1 applied to β and g, the bisection associated to the inner automorphism $\beta_{(-)g^*}$ of $\mathbb{K}$ is $g^*\alpha$. Now by Step 2 applied to $\beta_{(-)g^*}$ and h, the bisection associated to the inner automorphism $\beta_{(-)h_*g^*} = \beta_{(-)f}$ of $\mathbb{K}$ is $h_*g^*\alpha = f\alpha$, as required. □

We have thus proved the theorem stated in the introduction. In fact, we can do slightly better. The extended inner automorphisms of any object in any category form a group under the operation of composition. We noted above that the bisections of a groupoid also form a group. It is easily seen that these two group structures are related by the equation $c_\beta \circ c_\alpha = c_{\beta\cdot\alpha}$, and so we have:

Theorem 12. *The group of extended inner automorphisms of $\mathbb{G} \in \mathcal{G}\mathrm{rpd}_{\mathrm{co}}$ is isomorphic to the group* $\mathrm{Bis}(\mathbb{G})$*. The extended inner automorphism corresponding to $\alpha \in \mathrm{Bis}(\mathbb{G})$ is given by the family of automorphisms $((c_{f\alpha})_*\colon \mathbb{H} \rightsquigarrow \mathbb{H})$ as f ranges over comorphisms $\mathbb{G} \rightsquigarrow \mathbb{H}$.*

4. Generalisations and further perspectives

4.1. **Topological and Lie groupoids.** As mentioned in the introduction, bisections show up frequently in the study of Lie and topological groupoids. It is therefore natural to ask if our results generalise to those settings. The answer is yes. As the adaptations in the two cases are so similar, we concentrate on the topological one.

First we must adapt the basic notions. For a topological groupoid $\mathbb{G}$, we restrict attention to *continuous* bisections α: those for which the assignation $u \mapsto \alpha_u$ is continuous as a map $\mathbb{G}_0 \to \mathbb{G}_1$. This implies, easily, that the associated conjugation homomorphism $c_\alpha\colon \mathbb{G} \to \mathbb{G}$ is a continuous map of topological groupoids. We should like to identify these c_α's as the extended inner automorphisms of $\mathbb{G}$ in a suitable category.

The morphisms of this category will be comorphisms $f\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ between topological groupoids which are *continuous*, in the sense of rendering continuous the following maps of spaces:

$$\begin{aligned} \mathbb{H}_0 &\to \mathbb{G}_0 & \qquad \mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{H}_0 &\to \mathbb{H}_1 \\ u &\mapsto fu & (a,u) &\mapsto f(a)_u \ ; \end{aligned}$$

here, the fibre product $\mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{H}_0$ is taken along the source map $s\colon \mathbb{G}_1 \to \mathbb{G}_0$ and the action on objects $f\colon \mathbb{H}_0 \to \mathbb{G}_0$. Much like before, we obtain continuous comorphisms $\mathbb{G} \rightsquigarrow \mathbb{H}$ from continuous functors $\mathbb{G} \to \mathbb{H}$ which are homeomorphic-on-objects; and from functors $\mathbb{H} \to \mathbb{G}$ which are *continuous discrete opfibrations*, meaning that the operation of forming the unique lifting $f(a)_u\colon u \to \tilde{f}(a,u)$ of a map $a\colon fu \to v$ is a continuous map $\mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{H}_0 \to \mathbb{H}_1$. As in Proposition 5, every continuous comorphism arises by composing ones of these two special kinds.

In this situation, we also have an analogue of Proposition 8: the functor $\Sigma\colon \mathcal{G}\mathrm{rp} \to \mathcal{T}\mathrm{op}\mathcal{G}\mathrm{rpd}_{\mathrm{co}}$ embedding each (discrete!) group G as a one-object discrete topological groupoid has a right adjoint, sending a topological groupoid $\mathbb{G}$ to its discrete group of continuous bisections $\mathrm{Bis}(\mathbb{G})$[2]. In particular, continuous bisections of a topological groupoid can be transported along continuous comorphisms, with the same formulae as before. Using this Proposition 10 carries over, *mutatis mutandis*, showing that every continuous bisection of $\mathbb{G} \in \mathcal{T}\mathrm{op}\mathcal{G}\mathrm{rpd}_{\mathrm{co}}$ induces an extended inner automorphism.

All that remains is to adapt the proof of Proposition 11, showing that every extended inner automorphism β of $\mathbb{G}$ arises in this manner. All of the constructions in this proof continue to work in the topological context, and so we can conclude immediately that β must be of the form $\beta_f = c_{f\alpha}$ for a unique, but not necessarily continuous, bisection α of $\mathbb{G}$. To prove continuity, we consider the *décalage* [7] of $\mathbb{G}$. This is the topological groupoid $\mathrm{Dec}(\mathbb{G})$ whose underlying topological graph is given by

$$\mathbb{G}_1 \, {}_t\!\times_s \mathbb{G}_1 \overset{\pi_1}{\underset{\mu}{\rightrightarrows}} \mathbb{G}_1 \quad ,$$

where μ is the composition map of $\mathbb{G}$. The composition and units of $\mathrm{Dec}(\mathbb{G})$ itself are determined by requiring that its underlying discrete groupoid be the disjoint union of the coslice categories $u/\mathbb{G}$. There is a continuous discrete opfibration $\pi\colon \mathrm{Dec}(\mathbb{G}) \to \mathbb{G}$ which projects onto the codomain; and for each $u \in \mathbb{G}_0$ this fits into a commuting triangle as to the left in

$$\begin{array}{ccc} u/\mathbb{G} & \xrightarrow{\iota} & \mathrm{Dec}(\mathbb{G}) \\ & {\scriptstyle \pi_u}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & \mathbb{G} & \end{array} \qquad\qquad \begin{array}{ccc} u/\mathbb{G} & \xrightarrow{\beta_{\pi_u^*}} & u/\mathbb{G} \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\ \mathrm{Dec}(\mathbb{G}) & \xrightarrow{\beta_{\pi^*}} & \mathrm{Dec}(\mathbb{G}) \end{array} \; .$$

It follows that each square as right above is commutative in $\mathcal{T}\mathrm{op}\mathcal{G}\mathrm{rpd}$; so in particular, $\beta_{\pi^*}(1_u) = \beta_{\pi_u^*}(1_u) = \alpha_u$ for each $u \in \mathbb{G}_0$. This shows that composing the continuous identities map $1_{(-)}\colon \mathbb{G}_0 \to \mathbb{G}_1$ with the continuous map $\mathbb{G}_1 \to \mathbb{G}_1$ giving the action on objects of β_{π^*} yields the assignation $u \mapsto \alpha_u$—which is thus continuous, as desired. We thus obtain:

Theorem 13. *The group of extended inner automorphisms of* $\mathbb{G} \in \mathcal{T}\mathrm{opGrpd}_{\mathrm{co}}$ *is isomorphic to the group of continuous bisections* $\mathrm{Bis}(\mathbb{G})$*, under the same correspondence as in Theorem 12.*

4.2. **Internal groupoids.** Topological and Lie groupoids are particular examples of *internal groupoids* in a category $\mathcal{C}$. It is therefore natural to ask if our results generalise further to groupoids internal to any category $\mathcal{C}$. The answer is no.

To see this, consider the category $\mathcal{S}\mathrm{et}^{\mathbb{Z}_2}$ whose objects are sets X endowed with an involution $\tau\colon X \to X$, and whose maps are equivariant functions (i.e., ones commuting with the involutions). A groupoid internal to $\mathcal{S}\mathrm{et}^{\mathbb{Z}_2}$ is an

[2]Note that any attempt to construct a right adjoint to the embedding $\mathcal{T}\mathrm{op}\mathcal{G}\mathrm{rp} \to \mathcal{T}\mathrm{op}\mathcal{G}\mathrm{rpd}_{\mathrm{co}}$ would fall foul of the failure of topological spaces to be cartesian closed.

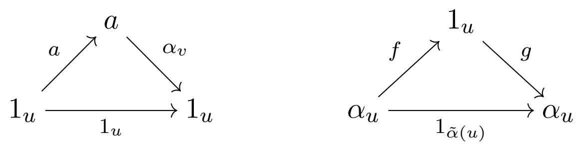
ordinary groupoid $\mathbb{G}$ with a (strict) involution $\tau\colon \mathbb{G} \to \mathbb{G}$; internal functors and comorphisms are just ordinary functors and comorphisms which commute with the involutions.

Now if $(\mathbb{G}, \tau)$ is an involutive groupoid, then the functor τ is easily seen to be equivariant $(\mathbb{G}, \tau) \to (\mathbb{G}, \tau)$; it follows that $(\mathbb{G}, \tau)$ has an extended inner automorphism β whose component at any $(\mathbb{G}, \tau) \rightsquigarrow (\mathbb{H}, \sigma)$ is $\sigma_*\colon (\mathbb{H}, \sigma) \rightsquigarrow (\mathbb{H}, \sigma)$. However, this β need not arise from any bisection α of $\mathbb{G}$. For example, if $\mathbb{G}$ is the discrete groupoid on two objects, and τ is the swap map, then $(\mathbb{G}, \tau)$ has no non-identity bisections, and yet the β defined above is not the identity.

The reason that things work differently in this case is really that objects in the indexing category $\mathbb{Z}_2$ can have their own non-trivial extended inner automorphisms. A more general formulation of our results would have to take this into account—but, lacking as we do any compelling reasons for developing such a generalisation, we have not pursued this further.

4.3. **Categories.** Another obvious direction of generalisation involves replacing groupoids everywhere by categories. There is not so much to say here; everything works without fuss. Comorphisms are defined exactly as before, and factorise in exactly the same way. For bisections, we must add the requirement that each map $\alpha_u\colon u \to \tilde{\alpha}(u)$ is invertible, and can then induce conjugation automorphisms in exactly the same way. Once again, bisections transport along comorphisms, with this now being evidenced by an an adjunction $\Sigma\colon \mathcal{G}\mathrm{rp} \leftrightarrows \mathcal{C}\mathrm{at}_{\mathrm{co}}\colon \mathrm{Bis}$.

Proposition 10 continues to work; and the only adaptation required in Proposition 11 is at the very start. Given an extended inner automorphism β of a category $\mathbb{C}$, with components $(\beta_f)_*$, we have as before the collection of maps $\alpha_u = \beta_{\pi_u^*}(1_u)\colon u \to \tilde{\alpha}(u)$. The argument showing the assignation $u \mapsto \tilde{\alpha}(u)$ is invertible still holds; but we must now also show that each α_u is invertible. For this, we first show as before that the automorphism $\beta_{\pi_u^*}\colon u/\mathbb{C} \to u/\mathbb{C}$ is given on objects by $\beta_{\pi_u^*}(a\colon u \to v) = \alpha_v \circ a$. Being an automorphism, there is in particular some such a for which $\alpha_v \circ a = 1_u$. So we have a commuting triangle as to the left below in $u/\mathbb{C}$. Applying $\beta_{\pi_u^*}$ yields a commuting triangle as to the right.



Since $g\colon 1_u \to \alpha_u$ we have $g = g \circ 1_u = \alpha_u$; since $f\colon \alpha_u \to 1_u$ we have $f \circ \alpha_u = 1_u$; and since the triangle commutes we have $\alpha_u \circ f = 1_u$. So f is an inverse for α_u. The remainder of the argument now follows exactly as before, and so we have:

Theorem 14. *The group of extended inner automorphisms of* $\mathbb{C} \in \mathcal{C}\mathrm{at}_{\mathrm{co}}$ *is isomorphic to the group of bisections* $\mathrm{Bis}(\mathbb{C})$*, under the same correspondence as in Theorem 12.*

4.4. **Inverse semigroups.** Our main results seem to diverge from the pattern for the computation of extended inner automorphism groups in [2, 6]. In this prior work, the categories under consideration have as objects, the models of an equational algebraic theory $\mathbb{T}$, and as morphisms, the obvious structure-preserving maps. This allows the extended inner automorphisms of a $\mathbb{T}$-model X to be characterised via universal algebra: they correspond to those invertible unary operations of the diagram theory $\mathbb{T}_X$[3] which *commute* with each $\mathbb{T}$-operation.

By contrast, our main result concerns the category of groupoids and comorphisms; and while the objects of this category are algebraic in nature—they are the models of an *essentially-algebraic* theory in the sense of [3]—the morphisms are *not* the obvious structure-preserving ones (which led only to our negative Proposition 3). This means that our argument for computing the extended inner automorphisms is necessarily different in nature.

In fact, there is a way of reconciling our results with those of [2, 6]: we adopt a different perspective on groupoid structure in which the comorphisms *are* the natural structure-preserving maps. More precisely, we take as the basic data of a groupoid not its objects and morphisms, but its *partial* bisections:

Definition 15. A *partial bisection* α of a groupoid $\mathbb{G}$ comprises subsets $s(\alpha)$ and $t(\alpha)$ of $\mathbb{G}_0$ together with an $s(\alpha)$-indexed family of morphisms $(\alpha_u \colon u \to \tilde{\alpha}(u))$ with the property that the assignation $u \mapsto \tilde{\alpha}(u)$ is a bijection $s(\alpha) \to t(\alpha)$.

The set $\mathrm{PBis}(\mathbb{G})$ of partial bisections of a groupoid $\mathbb{G}$ can be endowed with the structure of a *pseudogroup*—a special kind of inverse semigroup—and this structure allows $\mathrm{PBis}(\mathbb{G})$ to represent $\mathbb{G}$ faithfully. This fits into the pattern of a well-known correspondence between étale topological groupoids and pseudogroups, detailed, for example, in [9, 8]. A fact about this correspondence which does not appear to have been noted previously is that it equates the natural structure-preserving maps of pseudogroups with the *comorphisms* between the corresponding groupoids[4]. Thus, we may consider our result about groupoids and comorphisms instead as a result about pseudogroups and their structure-preserving morphisms, so fitting it in to the general pattern established in [2, 6].

To make the preceding claims more precise, we now define the category $\mathcal{P}\mathrm{s}\mathcal{G}\mathrm{rp}$ of pseudogroups, and sketch a proof that the assignation $\mathbb{G} \mapsto \mathrm{PBis}(\mathbb{G})$ yields a full embedding of $\mathcal{G}\mathrm{rpd}_{\mathrm{co}}$ into $\mathcal{P}\mathrm{s}\mathcal{G}\mathrm{rp}$.

Definition 16. An *inverse monoid* is a unital semigroup M such that, for every $m \in M$, there is a unique $m^* \in M$ with $mm^*m = m$ and $m^*mm^* = m^*$. The *natural partial order* $\leqslant$ and the *compatibility relation* $\sim$ on M are given by

$$m \leqslant n \quad \text{iff} \quad mn^*n = n$$
$$m \sim n \quad \text{iff} \quad mn^* \text{ and } n^*m \text{ are idempotent.}$$

A (abstract) *pseudogroup* is an inverse monoid M such that any family $S \subseteq M$ of pairwise-compatible elements admits a join $\bigvee S$ (with respect to $\leqslant$) which is preserved by each function $m \cdot (-) \colon M \to M$ and $(-) \cdot m \colon M \to M$. Pseudogroups form a category $\mathcal{P}\mathrm{s}\mathcal{G}\mathrm{rp}$ wherein maps are monoid homomorphisms that preserve joins of compatible families.

[3] i.e., the theory obtained by extending $\mathbb{T}$ with new constants for each element of X, and new equations describing the value of each $\mathbb{T}$-operation on those constants.

[4] Indeed, in [9], the correspondence is not defined on morphisms; while in [8], the morphisms considered are on one side, the *discrete opfibrations* of groupoids, and on the other, a slightly delicate class of morphisms between pseudogroups.

Example 17. For any groupoid $\mathbb{G}$, the set of partial bisections $\mathrm{PBis}(\mathbb{G})$ is a pseudogroup under the binary operation $\beta, \alpha \mapsto \beta \cdot \alpha$, where $\beta \cdot \alpha$ has

$$s(\beta \cdot \alpha) = s(\alpha) \cap \tilde{\alpha}^{-1}(s(\beta)) \qquad t(\beta \cdot \alpha) = \tilde{\beta}(t(\alpha)) \cap t(\beta)$$

and components $(\beta \cdot \alpha)_u = \beta_{\tilde{\alpha}(u)} \circ \alpha_u \colon u \to \tilde{\beta}(\tilde{\alpha}(u))$. The unit for this operation is the identity bisection $\mathbf{1}$, and the partial inverse α^* of α has $s(\alpha^*) = t(\alpha)$, $t(\alpha^*) = s(\alpha)$ and components determined by $(\alpha^*)_{\tilde{\alpha}(u)} = (\alpha_u)^{-1}$.

Two partial bisections α, β are compatible if $\alpha_u = \beta_u$ for all $u \in s(\alpha) \cap s(\beta)$, while $\alpha \leqslant \beta$ if $\alpha \sim \beta$ and $s(\alpha) \subseteq s(\beta)$. The join α of a pairwise-compatible family of partial bisections $(\alpha^i : i \in S)$ has $s(\alpha) = \bigcup_i s(\alpha^i)$, $t(\alpha) = \bigcup_i t(\alpha^i)$ and components $\alpha_u = \alpha^i_u$, for any $i \in S$ with $u \in s(\alpha^i)$.

Proposition 18. *The assignation* $\mathbb{G} \mapsto \mathrm{PBis}(\mathbb{G})$ *is the action on objects of a full embedding of categories* $\mathcal{G}\mathrm{rpd}_{\mathrm{co}} \to \mathcal{P}\mathrm{s}\mathcal{G}\mathrm{rp}$.

Proof. Let $f \colon \mathbb{G} \rightsquigarrow \mathbb{H}$ be a comorphism of groupoids, and $\alpha \in \mathrm{PBis}(\mathbb{G})$. Generalising Definition 9, we can define a pushforward partial bisection $f\alpha \in \mathrm{PBis}(\mathbb{H})$ by taking $s(f\alpha)$ and $t(f\alpha)$ to be the inverse images of $s(\alpha)$ and $t(\alpha)$ under the function $f \colon \mathbb{H}_0 \to \mathbb{G}_0$, and with components given like before by

$$(f\alpha)_u = f(\alpha_{fu})_u \colon u \to \tilde{f}(\alpha_{fu}, u) \ .$$

Straightforward checking shows that the assignation $\alpha \mapsto f\alpha$ is a pseudogroup morphism $\mathrm{PBis}(f) \colon \mathrm{PBis}(\mathbb{G}) \to \mathrm{PBis}(\mathbb{H})$ and that the assignation $f \mapsto \mathrm{PBis}(f)$ is functorial; so we have a functor $\mathrm{PBis} \colon \mathcal{G}\mathrm{rpd}_{\mathrm{co}} \to \mathcal{P}\mathrm{s}\mathcal{G}\mathrm{rp}$.

To see this functor is faithful, note that we can recover the action on objects of $f \colon \mathbb{G} \rightsquigarrow \mathbb{H}$ from $\varphi := \mathrm{PBis}(f) \colon \mathrm{PBis}(\mathbb{G}) \to \mathrm{PBis}(\mathbb{H})$ by the formula

$$f(v) = u \qquad \text{iff} \qquad v \in s(\varphi([1_u])) \ ; \tag{4.1}$$

here, if $a \colon u \to v$ is any map of $\mathbb{G}$ then we write $[a]$ for the partial bisection whose sole component is the map a. In a similar way, we can recover the action of the comorphism f on maps by the formula

$$f(a)_u = b \qquad \text{iff} \qquad \varphi([a])_u = b \ . \tag{4.2}$$

It remains only to show that PBis is full. So let $\varphi \colon \mathrm{PBis}(\mathbb{G}) \to \mathrm{PBis}(\mathbb{H})$ be any pseudogroup morphism. In $\mathrm{PBis}(\mathbb{G})$ we have that

$$\mathbf{1} = \textstyle\bigvee_{u \in \mathbb{G}_0} [1_u] \qquad \text{and} \qquad [1_u] \cdot [1_v] = \bot \text{ for } u \neq v \ ;$$

since φ is a pseudogroup morphism, it follows that in $\mathrm{PBis}(\mathbb{H})$ we have

$$\mathbf{1} = \textstyle\bigvee_{u \in \mathbb{G}_0} \varphi([1_u]) \qquad \text{and} \qquad \varphi([1_u]) \cdot \varphi([1_v]) = \bot \text{ for } u \neq v \ ,$$

so that the sets $s(\varphi([1_u]))$ are a partition of $\mathbb{H}_0$. We thus have a well-defined function $f \colon \mathbb{H}_0 \to \mathbb{G}_0$ determined by (4.1); whereupon we obtain the assignations on morphisms required for a comorphism $f \colon \mathbb{G} \rightsquigarrow \mathbb{H}$ by the formula (4.2). The comorphism axioms now follow easily from the homomorphism axioms for φ together with the observation that $[a] \cdot [b] = [a \circ b]$ in $\mathrm{PBis}(\mathbb{G})$ whenever a and b are composable maps. Finally, to see that $\mathrm{PBis}(f) = \varphi$, we observe that $\mathrm{PBis}(f)([a]) = \varphi([a])$ by construction; now since for any $\alpha \in \mathrm{PBis}(\mathbb{G})$, we have $\alpha = \bigvee_{u \in s(\alpha)} [\alpha_u]$, and since both $\mathrm{PBis}(f)$ and φ preserve joins, it follows that $\mathrm{PBis}(f)(\alpha) = \varphi(\alpha)$ for all $\alpha \in \mathrm{PBis}(\mathbb{G})$, as desired. □

It is not too hard to characterise the essential image of the embedding functor $\mathcal{G}\mathrm{rpd}_{\mathrm{co}} \to \mathcal{P}\mathrm{s}\mathcal{G}\mathrm{rp}$; it comprises the *complete atomic* pseudogroups—those whose partially ordered set of idempotents forms a complete atomic Boolean algebra (i.e., a power-set lattice). Thus, our main result, concerning the "non-algebraic" category of groupoids and comorphisms, can be recast as one about the "algebraic" category of complete atomic pseudogroups and pseudogroup homomorphisms; and following [8], we may recast the generalisation of our main result to étale topological groupoids in terms of more general pseudogroups.

There are a couple of points worth noting here. Firstly, when translated into the language of pseudogroups, our main result states that every extended inner automorphism of a complete atomic pseudogroup M is induced by conjugation (in the usual sense) by an invertible element of the monoid M—indeed, such invertible elements correspond to *total* bisections of the corresponding groupoid. So in this sense, our result fits into the pattern established in [2, 6].

On the other hand, if we translate the proof of our main Theorem 12 into the language of complete atomic pseudogroups, then it is still not a proof in the same mould as [2, 6]. If it were, then the first step in determining the components of an extended inner automorphism β of M would be to adjoin freely a new element x to M and consider the component $\beta_\iota \colon M[x] \to M[x]$ at the resulting inclusion map $\iota \colon M \to M[x]$. This is quite different from what is done in Proposition 11: in the language of pseudogroups, the first step there is to consider an atomic idempotent $u \in M$, and now consider the component $\beta_\jmath$ corresponding to the homomorphism

$$\begin{aligned} \jmath \colon M &\to \mathrm{PBij}(M_u) \\ x &\mapsto x \cdot (-) \end{aligned}$$

into the pseudogroup of all partial bijections of $M_u = \{m \in M : m^* m = u\}$. It may be interesting to compare these approaches more thoroughly, but would take us too far afield here.

4.5. **Extended partial inner automorphisms.** In a pseudogroup M, *any* element $a \in M$ induces a conjugation map $c_a(x) = axa^*$. However, $c_a \colon M \to M$ is not typically an automorphism of M, nor even a well-defined homomorphism, since it does not preserve the monoid unit 1 unless $a \in M$ is genuinely invertible.

Nonetheless, we would like to think of c_a as a *partial* automorphism of M, hoping for a result to the effect that every extended inner partial automorphism of a pseudogroup is induced by conjugation by an element. In the world of groupoids this would translate into the statement that every extended partial automorphism in $\mathcal{G}\mathrm{rpd}_{\mathrm{co}}$ comes from conjugating by a partial bisection.

Making this precise is delicate because, just as conjugation on a pseudogroup does not give a pseudogroup homomorphism, so conjugation on a groupoid by a partial bisection does not give a comorphism. Thus, much as in Section 6 of [2], we must proceed in an essentially *ad hoc* manner.

Definition 19. A *partial automorphism* $\varphi \colon \mathbb{G} \rightarrow\mkern-14mu\mapstochar\mkern14mu \mathbb{G}$ of a groupoid $\mathbb{G}$ is given by full subcategories $s(\varphi), t(\varphi) \subseteq \mathbb{G}$ together with an isomorphism of groupoids

$\varphi\colon s(\varphi) \to t(\varphi)$. Given a comorphism $f\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ and partial automorphisms $\varphi\colon \mathbb{G} \nrightarrow \mathbb{G}$ and $\psi\colon \mathbb{H} \nrightarrow \mathbb{H}$, we say that

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{\ \varphi\ } & \mathbb{G} \\ {\scriptstyle f}\Big\downarrow\wr & & \wr\Big\downarrow{\scriptstyle f} \\ \mathbb{H} & \xrightarrow[\ \psi\ ]{} & \mathbb{H} \end{array} \tag{4.3}$$

is a *commuting square* if:

(i) On objects, we have $u \in s(\psi)$ if and only if $f(u) \in s(\varphi)$; and for those u where this does hold, we have that $\varphi(f(u)) = f(\psi(u))$ in $t(\varphi)$.

(ii) For all $a\colon fu \to v$ in $s(\varphi)$, we have $\psi(f(a)_u) = f(\varphi(a))_{\psi(u)}$ in $t(\psi)$.

Now by an *extended inner partial automorphism* of $\mathbb{G}$, we mean a family of partial automorphisms $(\beta_f\colon \mathbb{H} \nrightarrow \mathbb{H})$, one for each comorphism $f\colon \mathbb{G} \rightsquigarrow \mathbb{H}$, such that for all comorphisms $f\colon \mathbb{G} \rightsquigarrow \mathbb{H}$ and $g\colon \mathbb{H} \rightsquigarrow \mathbb{K}$ we have a commuting square:

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\ \beta_f\ } & \mathbb{H} \\ {\scriptstyle g}\Big\downarrow\wr & & \wr\Big\downarrow{\scriptstyle g} \\ \mathbb{K} & \xrightarrow[\ \beta_{gf}\ ]{} & \mathbb{K}\ . \end{array}$$

Using the fact that commuting squares of the form (4.3) stack vertically and horizontally to give commuting squares, we can now follow through the same argument as before, *mutatis mutandis*, to show that:

Theorem 20. *The monoid of extended partial inner automorphisms of a groupoid* $\mathbb{G}$ *is isomorphic to the monoid of partial bisections* $\mathrm{PBis}(\mathbb{G})$*. The extended inner automorphism corresponding to* $\alpha \in \mathrm{PBis}(\mathbb{G})$ *is given by the family of partial automorphisms* $\big(c_{f\alpha}\colon \mathbb{H} \rightsquigarrow \mathbb{H}\big)$ *as* f *ranges over comorphisms* $\mathbb{G} \rightsquigarrow \mathbb{H}$*.*

The details are sufficiently similar that we leave them to the interested reader to reconstruct.

References

[1] Aguiar, M. *Internal categories and quantum groups.* PhD thesis, Cornell University, 1997.

[2] Bergman, G. M. An inner automorphism is only an inner automorphism, but an inner endomorphism can be something strange. *Publicacions Matemàtiques 56*, 1 (2012), 91–126.

[3] Freyd, P. J. Aspects of topoi. *Bulletin of the Australian Mathematical Society 7* (1972), 1–76.

[4] Funk, J., Hofstra, P., and Steinberg, B. Isotropy and crossed toposes. *Theory and Applications of Categories 26* (2012), 660–709.

[5] Higgins, P. J., and Mackenzie, K. C. H. Duality for base-changing morphisms of vector bundles, modules, Lie algebroids and Poisson structures. *Mathematical Proceedings of the Cambridge Philosophical Society 114*, 3 (1993), 471–488.

[6] Hofstra, P., Parker, J., and Scott, P. Isotropy of algebraic theories. *Electronic Notes in Theoretical Computer Science 341* (2018), 201–217.

[7] Illusie, L. *Complexe cotangent et déformations. II*, vol. 283 of *Lecture Notes in Mathematics.* Springer, 1972.

[8] Lawson, M. V., and Lenz, D. H. Pseudogroups and their étale groupoids. *Advances in Mathematics 244* (2013), 117–170.

[9] Resende, P. Lectures on étale groupoids, inverse semigroups and quantales. Lecture Notes for the GAMAP IP Meeting, Antwerp, 4–18 September, 2006. Available at https://www.math.tecnico.ulisboa.pt/~pmr.

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