

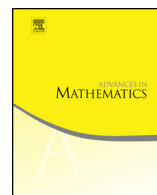


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The Isbell monad [☆]

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ABSTRACT

In 1966 [7], John Isbell introduced a construction on categories which he termed the “couple category” but which has since come to be known as the *Isbell envelope*. The Isbell envelope, which combines the ideas of contravariant and covariant presheaves, has found applications in category theory, logic, and differential geometry. We clarify its meaning by exhibiting the assignment sending a locally small category to its Isbell envelope as the action on objects of a pseudomonad on the 2-category of locally small categories; this is the *Isbell monad* of the title. We characterise the pseudoalgebras of the Isbell monad as categories equipped with a *cylinder factorisation system*; this notion, which appears to be new, is an extension of Freyd and Kelly’s notion of factorisation system [5] from orthogonal classes of arrows to orthogonal classes of cocones and cones.

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1. Introduction

One of the most fundamental constructions in category theory is that which assigns to a small category \mathcal{C} the Yoneda embedding $Y: \mathcal{C} \rightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ into its category of

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presheaves. As is well known, this embedding has the effect of exhibiting $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ as a free cocompletion of \mathcal{C} : the value at \mathcal{C} of a left biadjoint

$$\mathbf{COCTS} \begin{array}{c} \longleftarrow \\ \perp \\ \longrightarrow \end{array} \mathbf{CAT} \tag{1.1}$$

to the forgetful 2-functor from small-cocomplete categories and cocontinuous functors to locally small ones. At a \mathcal{C} which is not necessarily small, this left biadjoint still exists, but now has its unit $Y:\mathcal{C} \rightarrow \mathcal{PC}$ given by the Yoneda embedding into the subcategory $\mathcal{PC} \subset [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ of *small* presheaves: those which can be expressed as small colimits of representables. Composing the two biadjoints in (1.1) exhibits the process of free cocompletion as the functor part of a pseudomonad \mathcal{P} on \mathbf{CAT} , and it turns out that the \mathcal{P} -pseudoalgebras and algebra pseudomorphisms are once again the small-cocomplete categories and cocontinuous functors between them; which is to say that the biadjunction (1.1) is *pseudomonadic* [12].

Dually, we speak of *free completions* of categories, meaning the values of a left biadjoint to the forgetful 2-functor $\mathbf{CTS} \rightarrow \mathbf{CAT}$ from complete categories to locally small ones. The free completion of a small \mathcal{C} is witnessed by the dual Yoneda embedding $Y:\mathcal{C} \rightarrow [\mathcal{C}, \mathbf{Set}]^{\text{op}}$, while the general completion $Y:\mathcal{C} \rightarrow \mathcal{P}^\dagger\mathcal{C}$ is constructed as $\mathcal{P}^\dagger\mathcal{C} = \mathcal{P}(\mathcal{C}^{\text{op}})^{\text{op}} \subset [\mathcal{C}, \mathbf{Set}]^{\text{op}}$. As before, the biadjunction $\mathbf{CTS} \rightleftarrows \mathbf{CAT}$ induced by free completion is pseudomonadic, so that, as before, complete categories and continuous functors between them may be identified with \mathcal{P}^\dagger -pseudoalgebras and their pseudomorphisms.

In [7, §1.1], Isbell describes a construction that, in some sense, combines the processes of free completion and cocompletion; while Isbell calls this construction the “couple category”, we follow Lawvere in terming it the *Isbell envelope*. Given a locally small category \mathcal{C} , the objects of its Isbell envelope \mathcal{IC} are triples (X^+, X^-, ξ^X) where $X^+ \in \mathcal{PC}$ and $X^- \in \mathcal{P}^\dagger\mathcal{C}$ and $\xi_{ab}^X: X^-(b) \times X^+(a) \rightarrow \mathcal{C}(a, b)$ is a family of functions, natural in a and b ; while morphisms $(X^+, X^-, \xi^X) \rightarrow (Y^+, Y^-, \xi^Y)$ in \mathcal{IC} are pairs (f^+, f^-) , where $f^+: X^+ \rightarrow Y^+$ in \mathcal{PC} and $f^-: X^- \rightarrow Y^-$ in $\mathcal{P}^\dagger\mathcal{C}$ are such that each square

$$\begin{array}{ccc} Y^-(b) \times X^+(a) & \xrightarrow{1 \times f^+} & Y^-(b) \times Y^+(a) \\ f^- \times 1 \downarrow & & \downarrow \xi^Y \\ X^-(b) \times X^+(a) & \xrightarrow{\xi^X} & \mathcal{C}(a, b) \end{array} \tag{1.2}$$

commutes in \mathbf{Set} . There is a Yoneda embedding $Y:\mathcal{C} \rightarrow \mathcal{IC}$ into the Isbell envelope, whose value at an object c is given by:

$$(\mathcal{C}(-, c) \in [\mathcal{C}^{\text{op}}, \mathbf{Set}], \mathcal{C}(c, -) \in [\mathcal{C}, \mathbf{Set}], (\mathcal{C}(c, b) \times \mathcal{C}(a, c) \xrightarrow{\circ} \mathcal{C}(a, b))_{a,b}),$$

and it is related to the usual two Yoneda embeddings of \mathcal{C} through projection functors π_1 and π_2 fitting into a commuting diagram

$$\begin{array}{ccccc}
 & & \mathcal{C} & & \\
 & \swarrow Y & \downarrow Y & \searrow Y & \\
 \mathcal{P}\mathcal{C} & \xleftarrow{\pi_1} & \mathcal{I}\mathcal{C} & \xrightarrow{\pi_2} & \mathcal{P}^\dagger\mathcal{C}.
 \end{array} \tag{1.3}$$

Isbell envelopes have a range of applications. Isbell used them to study *normal completions* of categories [8] (the categorical correlate of Dedekind–MacNeille completions of posets); they are closely related to constructions in linear logic [3,13], due in part to the “self-duality” $\mathcal{I}\mathcal{C} \cong \mathcal{I}(\mathcal{C}^{\text{op}})^{\text{op}}$; in [16] they were used to study convenient categories of smooth spaces; and in future work we will see that they play a role in the *Reedy categories* [14] of abstract homotopy theory.¹ In this paper, however, our interest in Isbell envelopes stems from the following natural question: given that the two outside Yoneda embeddings in (1.3) are the units at \mathcal{C} of the pseudomonads for small-cocomplete and small-complete categories, is there a corresponding pseudomonad whose unit at \mathcal{C} is the central embedding? The main contribution of this paper is answer this question in the affirmative; the pseudomonad in question is the *Isbell monad* of the title, and we will characterise its pseudoalgebras as categories equipped with a *cylinder factorisation system*.

By a *cylinder* between small diagrams $D: \mathcal{I} \rightarrow \mathcal{C}$ and $E: \mathcal{J} \rightarrow \mathcal{C}$, we mean a family of maps $r = (r_{ij}: Di \rightarrow Ej)$ natural in i and j . A cylinder factorisation system provides a way of factorising each such cylinder in an essentially-unique way as a cocone followed by a cone; the unicity is assured by the requirement that the two parts of the factorisation should lie in suitably orthogonal classes \mathcal{E} of cocones and \mathcal{M} of cones. Cylinder factorisation systems are thus a generalisation of the *orthogonal factorisation systems* of [5] from single maps to small families of maps; while certain aspects of this generalisation are known in the literature, the complete definition appears to be new; we give it in Section 2.

Now our first main result, [Theorem 11](#), exhibits a biadjunction

$$\mathbf{CFS} \overset{\perp}{\rightleftarrows} \mathbf{CAT} \tag{1.4}$$

between categories and cylinder factorisation systems on categories, with as unit at \mathcal{C} the embedding $Y: \mathcal{C} \rightarrow \mathcal{I}\mathcal{C}$ of (1.3). Composing the biadjoints, we thus exhibit this embedding as the unit at \mathcal{C} of a pseudomonad on \mathbf{CAT} , which is the Isbell monad we seek. Our second main result, [Theorem 12](#), shows that the pseudoalgebras for the Isbell monad correspond with categories equipped with cylinder factorisation systems; in other words,

¹ Roughly speaking, if \mathcal{C} is a Reedy category, then an element of the Isbell envelope $\mathcal{I}\mathcal{C}$ is what one needs to extend \mathcal{C} to a Reedy category with one additional object.

we show that (1.4), like (1.1), is pseudomonadic. This generalises [11]’s characterisation of orthogonal factorisation systems as pseudoalgebras for the squaring monad $(-)^2$ on **CAT**.

Our third main result concerns morphisms of cylinder factorisation systems, of which we have said nothing so far. Given categories \mathcal{C} and \mathcal{D} equipped with cylinder factorisation systems, the morphisms between them in **CFS** are functors $F: \mathcal{C} \rightarrow \mathcal{D}$ preserving both the \mathcal{E} -cocones and the \mathcal{M} -cones; part of the pseudomonadicity result is that these correspond with the pseudomorphisms of Isbell pseudoalgebras. However, we also have the more general notion of *lax* and *colax* morphisms of pseudoalgebras; and Theorem 13 shows that these correspond to functors $F: \mathcal{C} \rightarrow \mathcal{D}$ preserving *only* \mathcal{M} -cones or \mathcal{E} -cocones respectively.

We conclude the paper by discussing variants of the notion of cylinder factorisation systems involving factorisations for only *certain* kinds of cylinders; our final main result, Theorem 14, exhibits these as the pseudoalgebras for certain variants of the Isbell monad, obtained by constraining the presheaves $X^+ \in \mathcal{PC}$ and $X^- \in \mathcal{P}^\dagger\mathcal{C}$ that constitute an object of \mathcal{IC} to lie in suitable *saturated classes* [2] of weights for colimits and limits.

2. Cylinder factorisation systems

Suppose that $D: \mathcal{I} \rightarrow \mathcal{C}$ and $E: \mathcal{J} \rightarrow \mathcal{C}$ are diagrams in a category \mathcal{C} . By a *cocone under D with vertex V* , we mean a natural transformation $p: D \rightarrow \Delta V$ into the constant functor at V , and by a *cone over E with vertex W* , a natural transformation $q: \Delta W \rightarrow E$. Given a map $f: V \rightarrow W$, we may postcompose p or precompose q with it to obtain a cocone $f \cdot p: D \rightarrow \Delta W$ or cone $q \cdot f: \Delta V \rightarrow E$. By a *cylinder from D to E* , written $r: D \rightsquigarrow E$, we mean a natural transformation

$$\begin{array}{ccc}
 & \mathcal{I} & \\
 \pi_1 \nearrow & & \searrow D \\
 \mathcal{I} \times \mathcal{J} & & \mathcal{C} \\
 \pi_2 \searrow & \Downarrow r & \nearrow E \\
 & \mathcal{J} &
 \end{array}$$

thus, a natural family of maps $(r_{ij}: Di \rightarrow Ej)_{i,j \in \mathcal{I} \times \mathcal{J}}$. For example, if $\mathcal{J} = 1$, then E picks out a single vertex and so a cylinder is simply a cocone; while if $\mathcal{I} = 1$ then a cylinder is just a cone. For a further example, if $p: D \rightarrow \Delta V$ is a cocone and $q: \Delta V \rightarrow E$ a cone, then we have a cylinder $q \cdot p: D \rightsquigarrow E$ whose components are the family of maps $(q_j \cdot p_i: Di \rightarrow V \rightarrow Ej)_{i,j \in \mathcal{I} \times \mathcal{J}}$.

Definition 1. A cocone $p: D \rightarrow \Delta V$ and a cone $q: \Delta W \rightarrow E$ are said to be *orthogonal*, written $p \perp q$, if for every diagram as in the solid part of

$$\begin{array}{ccc}
 D & \xrightarrow{p} & \Delta V \\
 \downarrow h & \swarrow j & \downarrow k \\
 \Delta W & \xrightarrow{q} & E,
 \end{array} \tag{2.1}$$

wherein h is a cocone, k is a cone, and $q \cdot h = k \cdot p: D \rightsquigarrow E$, there exists a unique map $j: V \rightarrow W$ as indicated making both triangles commute.

This definition generalises the classical notion of orthogonality of arrows in a category [5, §2.1]; it also generalises the notion of orthogonality of *discrete* cones and cocones—ones indexed by discrete categories—formulated in [9, §3], whose special case dealing with the orthogonality of an arrow to a discrete cone is already present in [5, §2.4].

The orthogonality of arrows underlies the notion of factorisation system introduced in [5, §2.2]; more generally, the orthogonality of arrows to discrete cones plays a role in [6]’s notion of (E, M) -category, in which E is a class of arrows, M an orthogonal class of discrete cones, and every discrete cone factors as an E -map followed by an M -cone. The following definition generalises these notions further to involve orthogonality of arbitrary small cocones and cones.

Definition 2. A *cylinder factorisation system* on a category \mathcal{C} comprises a class \mathcal{E} of small cocones—“small” meaning “indexed by a small category”—and a class \mathcal{M} of small cones, satisfying the following properties:

- (i) \mathcal{E} is closed under postcomposition with isomorphisms, and \mathcal{M} is closed under precomposition with isomorphisms;
- (ii) $p \perp q$ for all $p \in \mathcal{E}$ and $q \in \mathcal{M}$;
- (iii) Each small cylinder $r: D \rightsquigarrow E$ has a factorisation $r = q \cdot p$ with $p \in \mathcal{E}$ and $q \in \mathcal{M}$.

It follows that \mathcal{E} comprises all small cocones p such that $p \perp q$ for all $q \in \mathcal{M}$, and that \mathcal{M} comprises all small cones q such that $p \perp q$ for all $p \in \mathcal{E}$; and in fact these two conditions together with (iii) gives an alternate axiomatisation of cylinder factorisation systems. Every cylinder factorisation system $(\mathcal{E}, \mathcal{M})$ has an underlying orthogonal factorisation system $(\mathcal{E}_0, \mathcal{M}_0)$ —in the sense of [5]—obtained by restricting to cones and cocones over diagrams $1 \rightarrow \mathcal{C}$. The following result extends one of the basic facts in that theory to the cylinder setting.

Lemma 3. *Factorisations in a cylinder factorisation system are essentially unique: if the cylinder $r: D \rightsquigarrow E$ admits the $(\mathcal{E}, \mathcal{M})$ -factorisations $k \cdot p: D \rightarrow \Delta V \rightarrow E$ and $q \cdot h: D \rightarrow \Delta W \rightarrow E$, then the unique map $j: V \rightarrow W$ as in (2.1) is invertible.*

Proof. Mirroring (2.1) through the DE -axis and applying orthogonality again yields a filler $j': W \rightarrow V$; now both $j \cdot j'$ and 1_W fill the square $q \cdot h = q \cdot h$, and so must be equal; dually we have $j' \cdot j = 1_V$. \square

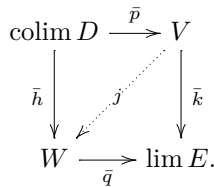
Examples 4.

- (a) If \mathcal{C} is complete, then it admits a cylinder factorisation system (small cocones, limit cones). Condition (i) is obvious, while (ii) is easy from the universality of a limiting cone. For (iii), we may factorise a cylinder $r: D \rightsquigarrow E$ as $p: D \rightarrow \Delta(\lim E)$ followed by $q: \Delta(\lim E) \rightarrow E$, where q is the limiting cone, and for each $i \in \mathcal{I}$, $p_i: Di \rightarrow \lim E$ is the unique map with $q_j \cdot p_i = r_{ij}$ for each $j \in \mathcal{J}$.
- (b) Dually, if \mathcal{C} is cocomplete, then it admits a cylinder factorisation system (colimit cocones, small cones).
- (c) Let \mathcal{C} be complete and cocomplete, and let $(\mathcal{E}_0, \mathcal{M}_0)$ be an orthogonal factorisation system on \mathcal{C} . We obtain a cylinder factorisation system $(\mathcal{E}, \mathcal{M})$ on \mathcal{C} by taking:

$$\mathcal{E} = \{ p: D \rightarrow \Delta V \text{ small} : \text{the induced } \bar{p}: \text{colim } D \rightarrow V \text{ is in } \mathcal{E}_0 \}$$

$$\mathcal{M} = \{ q: \Delta W \rightarrow E \text{ small} : \text{the induced } \bar{q}: W \rightarrow \lim E \text{ is in } \mathcal{M}_0 \}.$$

Axiom (i) is clear, while (ii) follows easily on observing that diagrams (2.1) correspond bijectively with squares in \mathcal{C} of the form:



As for (iii), given $r: D \rightsquigarrow E$, we first factorise as $q \cdot \ell: D \rightarrow \Delta(\lim E) \rightarrow E$ as in (a); then we factorise ℓ dually as $f \cdot p: D \rightarrow \Delta(\text{colim } D) \rightarrow \Delta(\lim E)$; then we factorise $f = e \cdot m: \text{colim } D \rightarrow V \rightarrow \lim E$ with $e \in \mathcal{E}_0$ and $m \in \mathcal{M}_0$; and finally take our desired factorisation to be $e \cdot p: D \rightarrow \Delta V$ followed by $q \cdot m: \Delta V \rightarrow E$. It is easy to see that *any* cylinder factorisation system on a complete and cocomplete category is induced in this way.

- (d) Let \mathcal{C} be a complete category which admits (strong epi, mono) factorisations and unions of small families of subobjects. Call a small cocone $p: D \rightarrow \Delta V$ *covering* if any monomorphism $V' \hookrightarrow V$ through which each p_i factors is invertible; and call a small cone *monic* if it is in \mathcal{M} as defined in (c) for \mathcal{M}_0 the class of monomorphisms. Now \mathcal{C} admits the cylinder factorisation system (covering cocones, mono cones). Axioms (i) and (ii) are straightforward. For (iii), given a cylinder $r: D \rightsquigarrow E$, we first factorise as $q \cdot p: D \rightarrow \Delta(\lim E) \rightarrow E$ as in (a); next we (strong epi, mono) factorise each p_i as $m_i \cdot e_i: Di \twoheadrightarrow Hi \hookrightarrow \lim E$; then we form the union $n: V \twoheadrightarrow \lim E$ of the

subobjects m_i with inclusions $h_i: Hi \rightarrow V$; finally, we obtain our desired factorisation as $h \cdot e: D \rightarrow H \rightarrow \Delta V$ followed by $q \cdot n: \Delta V \rightarrow \Delta(\lim E) \rightarrow E$. The only non-trivial point is showing that $h \cdot e: D \rightarrow \Delta V$ is covering. So suppose that each component $h_i \cdot e_i$ factors through some $g: V' \rightarrow V$. Because each e_i is strongly epic, this is equally to say that each h_i factors through g ; thus each $n \cdot h_i = m_i: Hi \rightarrow \lim E$ factors through $n \cdot g: V' \rightarrow \lim E$; but as n is the union of the m_i 's, g must be invertible as required.

- (e) If the *small* category \mathcal{C} bears a cylinder factorisation system, then all its \mathcal{E} -cocones must be jointly epimorphic, and all its \mathcal{M} -cones jointly monic, by an adaptation of an argument due to Freyd (though see also [1, Theorem 15.4]). Indeed, suppose that $k: \Delta V \rightarrow E$ is an \mathcal{M} -cone, and $f \neq g: W \rightarrow V$ with $k \cdot f = k \cdot g: \Delta W \rightarrow E$. Let D be the discrete diagram comprising $|\text{mor } \mathcal{C}|$ copies of W , let $r: D \rightsquigarrow E$ be the cylinder comprising $|\text{mor } \mathcal{C}|$ copies of the cocone $kf = kg$, and let $r = q \cdot p: D \rightarrow \Delta U \rightarrow E$ be an $(\mathcal{E}, \mathcal{M})$ -factorisation. Then in the diagram

$$\begin{array}{ccc}
 D & \xrightarrow{p} & \Delta U \\
 \ell \downarrow & & \downarrow q \\
 \Delta V & \xrightarrow{k} & E
 \end{array}$$

there are at least $2^{|\text{mor } \mathcal{C}|}$ distinct cones ℓ yielding commutativity; and so by orthogonality, at least $2^{|\text{mor } \mathcal{C}|}$ distinct maps $U \rightarrow V$ in \mathcal{C} , a contradiction.

We now define appropriate notions of morphism between categories equipped with cylinder factorisation systems. In considering cylinder factorisation systems on different categories, we will uniformly denote the classes of cocones and cones by \mathcal{E} and \mathcal{M} ; normally, context will make clear which \mathcal{E} and \mathcal{M} are intended, but where confusion seems possible, we will subscript them with the name of the category on which they reside.

Definition 5. We write **CFS** for the 2-category whose objects are locally small categories equipped with a cylinder factorisation system, whose 1-cells are functors $F: \mathcal{C} \rightarrow \mathcal{D}$ such that $F(\mathcal{E}) \subset \mathcal{E}$ and $F(\mathcal{M}) \subset \mathcal{M}$, and whose 2-cells are arbitrary natural transformations. We write **CFS $_{\mathcal{M}}$** and **CFS $_{\mathcal{E}}$** for the corresponding 2-categories wherein the morphisms are required only to preserve \mathcal{M} -cones, or only to preserve \mathcal{E} -cocones.

Examples 6.

- (a) If the complete \mathcal{C} and \mathcal{D} are equipped with the (all cocones, limit cones) cylinder factorisation system, then a functor $\mathcal{C} \rightarrow \mathcal{D}$ always preserves \mathcal{E} -cocones, and preserves \mathcal{M} -cones precisely when it is continuous. Dually, if the cocomplete \mathcal{C} and \mathcal{D} bear the (colimit cocones, all cones) cylinder factorisation systems, then a functor between

them always preserves \mathcal{M} -cones and preserves \mathcal{E} -cocones just when it is cocontinuous. It follows that **CFS** contains as full sub-2-categories both the 2-category **COCTS** of cocomplete categories and cocontinuous functors, and the 2-category **CTS** of complete categories and continuous functors.

- (b) If \mathcal{C} and \mathcal{D} are cocomplete, then the condition that a morphism $F: \mathcal{C} \rightarrow \mathcal{D}$ in **CFS** $_{\mathcal{E}}$ must satisfy can be reduced to the requirements that $F(\mathcal{E}_0) \subset \mathcal{E}_0$, and that F should preserve colimits “up to \mathcal{E}_0 ”; meaning that each canonical comparison $F \operatorname{colim} D \rightarrow \operatorname{colim} FD$ should be in \mathcal{E}_0 . In [10], Kelly calls this condition *preserving the \mathcal{E}_0 -tightness* of colimit cocones. Of course, we have a dual characterisation of morphisms of **CFS** $_{\mathcal{M}}$ between complete categories.
- (c) It is easy to see that if $F \dashv G: \mathcal{D} \rightarrow \mathcal{C}$, and p is a cocone in \mathcal{C} and q a cone in \mathcal{D} , then $Fp \perp q$ if and only if $p \perp Gq$. It follows that, if \mathcal{C} and \mathcal{D} are equipped with cylinder factorisation systems, then F preserves \mathcal{E} -cocones if and only if G preserves \mathcal{M} -cones.

We conclude this section with a technical result, necessary in the sequel, that gives an understanding of the effect of cylinder factorisation systems on cylinders which, though not small, are “essentially small” in a sense now to be described. Recall that a functor $K: \mathcal{J}' \rightarrow \mathcal{J}$ is called *initial* if, for each $j \in \mathcal{J}$, the comma category K/j is connected; which by the pointwise formula for Kan extensions, is equally to say that the triangle

$$\begin{array}{ccc}
 \mathcal{J}' & \xrightarrow{K} & \mathcal{J} \\
 \searrow & \Downarrow \cong & \swarrow \\
 & \mathbf{Set} &
 \end{array}$$

is a left Kan extension. The universal property of Kan extension now implies that, for each diagram $E: \mathcal{J} \rightarrow \mathcal{C}$ and $W \in \mathcal{C}$, precomposition with K induces a bijection

$$[\mathcal{J}, \mathbf{Set}](\Delta 1, \mathcal{C}(W, E-)) \cong [\mathcal{J}', \mathbf{Set}](\Delta 1, \mathcal{C}(W, EK-))$$

between cones $q: \Delta W \rightarrow E$ and cones $qK: \Delta W \rightarrow EK$; which in turn implies a bijection between cylinders $r: D \rightsquigarrow E$ and ones $r(1 \times K): D \rightsquigarrow EK$. Dually, a functor $H: \mathcal{I}' \rightarrow \mathcal{I}$ is called *final* if each comma category i/H is connected; which now implies a bijection between cocones $p: D \rightarrow \Delta V$ and ones $pH: DH \rightarrow \Delta V$, and between cylinders $r: D \rightsquigarrow E$ and ones $r(H \times 1): DH \rightsquigarrow E$. It immediately follows that:

Lemma 7. *If $H: \mathcal{I}' \rightarrow \mathcal{I}$ is final, $K: \mathcal{J}' \rightarrow \mathcal{J}$ is initial, $D: \mathcal{I} \rightarrow \mathcal{C}$ and $E: \mathcal{J} \rightarrow \mathcal{C}$, then for any cocone $p: D \rightarrow \Delta V$ and any cone $q: \Delta W \rightarrow E$, we have $p \perp q$ iff $pH \perp qK$.*

Let us now define a cylinder $r: D \rightsquigarrow E$ to be *essentially small* if the category \mathcal{I} indexing D admits a final functor from a small category, and the category \mathcal{J} indexing

E admits an initial functor from a small category. In particular, this gives a notion of essential-smallness for cocones and cones, on identifying these with degenerate cylinders.

For the nonce, we will call a structure as in Definition 2, but where “small” has everywhere been replaced by “essentially small”, an *extended cylinder factorisation system*. Restricting an extended cylinder factorisation system to its small cocones and cones yields a cylinder factorisation system; while in the other direction, we have:

Proposition 8. *Every cylinder factorisation system $(\mathcal{E}, \mathcal{M})$ on \mathcal{C} is the underlying cylinder factorisation system of a unique extended cylinder factorisation system $(\bar{\mathcal{E}}, \bar{\mathcal{M}})$; moreover, any morphism of cylinder factorisation systems $F: \mathcal{C} \rightarrow \mathcal{D}$ preserves these extended classes, in the sense that $F(\bar{\mathcal{E}}) \subset \bar{\mathcal{E}}$ and $F(\bar{\mathcal{M}}) \subset \bar{\mathcal{M}}$.*

Proof. Given $(\mathcal{E}, \mathcal{M})$, we define classes of essentially small cocones and cones by

$$\begin{aligned} \bar{\mathcal{E}} &= \{p: D \rightarrow \Delta V \mid pH \in \mathcal{E} \text{ for some final } H: \mathcal{I}' \rightarrow \mathcal{I}\} \\ \bar{\mathcal{M}} &= \{q: \Delta W \rightarrow E \mid qK \in \mathcal{E} \text{ for some initial } K: \mathcal{J}' \rightarrow \mathcal{J}\}. \end{aligned}$$

Clearly axiom (i) is satisfied, while (ii) is immediate from Lemma 7. This same lemma implies that $\bar{\mathcal{E}}$ comprises precisely those essentially small cocones orthogonal to every cone in \mathcal{M} , and vice versa, from which uniqueness of $(\bar{\mathcal{E}}, \bar{\mathcal{M}})$ follows easily. The final clause of the proposition is immediate from the definitions, and so it remains only to show axiom (iii): that each essentially small $r: D \rightsquigarrow E$ has an $(\bar{\mathcal{E}}, \bar{\mathcal{M}})$ -factorisation. Given such an r , choose a final $H: \mathcal{I}' \rightarrow \mathcal{I}$ and an initial $K: \mathcal{J}' \rightarrow \mathcal{J}$ with \mathcal{I}' and \mathcal{J}' small, let $r' = r(H \times K): DH \rightsquigarrow EK$, and form $q' \cdot p': DH \rightarrow \Delta V \rightarrow EK$ an $(\mathcal{E}, \mathcal{M})$ -factorisation of the small r' . Since H is final and K initial, there are unique $p: D \rightarrow \Delta V$ and $q: \Delta V \rightarrow E$ with $pH = p'$ and $qK = q'$, and clearly $p \in \bar{\mathcal{E}}$ and $q \in \bar{\mathcal{M}}$; finally, since $r(H \times K) = r' = q' \cdot p' = qK \cdot pH = (q \cdot p)(H \times K)$, we have by finality and initiality of H and K that $r = q \cdot p$, as desired. \square

Henceforth, then, there will be no explicit need to speak of extended cylinder factorisation systems; instead, we modify our notation by allowing \mathcal{E} and \mathcal{M} , which previously denoted the classes of small cocones and cones of a cylinder factorisation system, to denote instead the essentially small cocones and cones in the closures $\bar{\mathcal{E}}$ and $\bar{\mathcal{M}}$.

3. The free cylinder factorisation system

In this section, we give our first main result, showing that the Isbell envelope \mathcal{IC} is the free category with a cylinder factorisation system on \mathcal{C} . We begin by constructing the cylinder factorisation system in question.

Proposition 9. *For any category \mathcal{C} , the Isbell envelope \mathcal{IC} bears a cylinder factorisation system whose classes of small cocones and cones are given by:*

$$\mathcal{E} = \{ p: D \rightarrow \Delta V \mid \pi_1(p) \text{ is colimiting in } \mathcal{PC} \}$$

$$\mathcal{M} = \{ q: \Delta W \rightarrow E \mid \pi_2(q) \text{ is limiting in } \mathcal{P}^\dagger\mathcal{C} \},$$

where $\pi_1: \mathcal{IC} \rightarrow \mathcal{PC}$ and $\pi_2: \mathcal{IC} \rightarrow \mathcal{P}^\dagger\mathcal{C}$ are as in (1.3).

Proof. Axiom (i) is clear. For (ii), suppose given a diagram (2.1) in \mathcal{IC} with $p \in \mathcal{E}$ and $q \in \mathcal{M}$. Applying π_1 and π_2 we obtain diagrams

$$\begin{array}{ccc} D^+ & \xrightarrow{p^+} & \Delta(V^+) \\ h^+ \downarrow & \swarrow m^+ & \downarrow k^+ \\ \Delta(W^+) & \xrightarrow{q^+} & E^+ \end{array} \quad \text{and} \quad \begin{array}{ccc} D^- & \xrightarrow{p^-} & \Delta(V^-) \\ h^- \downarrow & \swarrow m^- & \downarrow k^- \\ \Delta(W^-) & \xrightarrow{q^-} & E^- \end{array}$$

in \mathcal{PC} and in $\mathcal{P}^\dagger\mathcal{C}$ respectively. Now p^+ is colimiting since $p \in \mathcal{E}$; it is thus orthogonal to any small cone, in particular to q^+ , and so there is a unique diagonal filler m^+ as on the left. Similarly, q^- is limiting since $q \in \mathcal{M}$, whence there is a unique diagonal filler m^- as on the right. We claim that $(m^+, m^-): V \rightarrow W$ is the required unique diagonal filler in \mathcal{IC} . The only point to check is that each square as on the left in

$$\begin{array}{ccc} W^-(b) \times V^+(a) & \xrightarrow{1 \times m^+} & W^-(b) \times W^+(a) \\ m^- \times 1 \downarrow & & \downarrow \xi^W \\ V^-(b) \times V^+(a) & \xrightarrow{\xi^V} & \mathcal{C}(a, b) \end{array} \quad \begin{array}{ccc} W^-(b) \times Di^+(a) & \xrightarrow{1 \times m^+ p_i^+} & W^-(b) \times W^+(a) \\ m^- \times p_i^+ \downarrow & & \downarrow \xi^W \\ V^-(b) \times V^+(a) & \xrightarrow{\xi^V} & \mathcal{C}(a, b) \end{array}$$

commutes. Now, evaluating the colimiting cocone p^+ at a yields a colimiting cocone $(p_i^+(a): Di^+(a) \rightarrow V^+(a))_{i \in \mathcal{I}}$; so by precomposing with these maps, it is enough to show commutativity of the squares on the right above. But by rewriting the bottom side using (1.2) for p_i^+ , this is equally to show that each square

$$\begin{array}{ccc} W^-(b) \times Di^+(a) & \xrightarrow{1 \times m^+ p_i^+} & W^-(b) \times W^+(a) \\ p_i^- m^- \times 1 \downarrow & & \downarrow \xi^W \\ Di^-(b) \times Di^+(a) & \xrightarrow{\xi^{Di}} & \mathcal{C}(a, b) \end{array}$$

commutes, which is so by (1.2) for $h = mp_i$.

This verifies (ii); and there remains only (iii). Given, then, a cylinder $r: D \rightsquigarrow E$ in \mathcal{IC} , we first apply π_1 and π_2 to obtain cylinders r^+ and r^- in the cocomplete \mathcal{PC} and complete $\mathcal{P}^\dagger\mathcal{C}$, which we then factor as in (a) and (b) of the preceding section as:

$$r^+ = D^+ \xrightarrow{p^+} \Delta V^+ \xrightarrow{q^+} E^+ \quad \text{and} \quad r^- = D^- \xrightarrow{p^-} \Delta V^- \xrightarrow{q^-} E^-$$

with p^+ colimiting and q^- limiting. We next define maps $\xi_{ab}^V: V^-(b) \times V^+(a) \rightarrow \mathcal{C}(a, b)$ making $V = (V^+, V^-, \xi^V)$ into an object of \mathcal{IC} . Evaluating the colimiting p^+ and limiting q^- at each object a and b yields colimiting cocones $(p_i^+(a): Di^+(a) \rightarrow V^+(a))_{i \in \mathcal{I}}$ and $(q_j^-(b): Ej^-(b) \rightarrow V^-(b))_{j \in \mathcal{J}}$ in **Set**; so to give the ξ_{ab}^V 's is equally to give their composites

$$\delta_{abij}: Ej^-(b) \times Di^+(a) \rightarrow \mathcal{C}(a, b)$$

with the components of these cocones: a family of maps natural in a, b, i, j . To obtain such, consider for each a, b, i, j the square (2.1) associated to the map $r_{ij}: Di \rightarrow Ej$ in \mathcal{IC} ; the common diagonal of the two sides gives the desired δ_{abij} 's, whose naturality in each variable is easily checked. The $(\mathcal{E}, \mathcal{M})$ -factorisation of r in \mathcal{IC} is now given by

$$D \xrightarrow{(p^+, p^-)} \Delta(V^+, V^-, \xi^V) \xrightarrow{(q^+, q^-)} E;$$

the only thing left to check is that the components (p_i^+, p_i^-) and (q_j^+, q_j^-) of the cocone and the cone are in fact maps of \mathcal{IC} . By duality, we need only check the former; thus, that each square as on the left in

$$\begin{array}{ccc} V^-(b) \times Di^+(a) & \xrightarrow{1 \times p_i^+} & V^-(b) \times V^+(a) & & Ej^-(b) \times Di^+(a) & \xrightarrow{q_j^- \times p_i^+} & V^-(b) \times V^+(a) \\ p_i^- \times 1 \downarrow & & \downarrow \xi^V & & p_i^- q_j^- \times 1 \downarrow & & \downarrow \xi^V \\ Di^-(b) \times Di^+(a) & \xrightarrow{\xi^{D_i}} & \mathcal{C}(a, b) & & Di^-(b) \times Di^+(a) & \xrightarrow{\xi^{D_i}} & \mathcal{C}(a, b) \end{array}$$

commutes. Precomposing with the colimit cocone $(q_j^-(b): Ej^-(b) \rightarrow V^-(b))_{j \in \mathcal{J}}$, this is equally to show that each square as on the right commutes. The upper side is, by definition of ξ^V , the common diagonal of the square (2.1) associated to r_{ij} ; but as $p_i^- q_j^- = r_{ij}^-$, the lower side of the above square is also the lower side of that selfsame (2.1); whence commutativity. \square

We are almost ready to give our first main result. First we need a preparatory lemma.

Lemma 10. *For each $X \in \mathcal{IC}$ and $a, b \in \mathcal{C}$, the action of the functors π_1 and π_2 on morphisms induce homset isomorphisms $\pi_1: \mathcal{IC}(Ya, X) \rightarrow \mathcal{PC}(Ya, X^+)$ and $\pi_2: \mathcal{IC}(X, Yb) \rightarrow \mathcal{P}^\dagger \mathcal{C}(X^-, Yb)$.*

Proof. To give a map $f: Ya \rightarrow X$ in \mathcal{IC} is to give $f^+: \mathcal{C}(-, a) \rightarrow X^+$ in \mathcal{PC} together with $f^-: \mathcal{C}(a, -) \rightarrow X^-$ in $\mathcal{P}^\dagger \mathcal{C}$ rendering commutative each diagram

$$\begin{array}{ccc}
 X^-(b) \times \mathcal{C}(a', a) & \xrightarrow{1 \times f^+} & X^-(b) \times X^+(a') \\
 f^- \times 1 \downarrow & & \downarrow \xi^X \\
 \mathcal{C}(a, b) \times \mathcal{C}(a', a) & \xrightarrow{\circ} & \mathcal{C}(a, b).
 \end{array} \tag{3.1}$$

This forces the components of f^- in **Set** to be given by $\xi_{ab}^X(-, x): X^-(b) \rightarrow \mathcal{C}(a, b)$, where $x = f^+(1_a) \in X^+(a)$. Thus $\pi_1: \mathcal{IC}(Ya, X) \rightarrow \mathcal{PC}(Ya, X^+)$ is injective; for surjectivity, given any $f^+ \in \mathcal{PC}(Ya, X^+)$, we may define f^- in the above manner, and verify naturality and commutativity in (3.1) using the Yoneda lemma. The case of π_2 is dual. \square

Theorem 11. *For any category \mathcal{C} , the Yoneda embedding $Y: \mathcal{C} \rightarrow \mathcal{IC}$ into the Isbell envelope exhibits \mathcal{IC} , equipped with the cylinder factorisation system of Proposition 9, as the value at \mathcal{C} of a left biadjoint to the forgetful 2-functor from **CFS** to **CAT**.*

Proof. We must show that, for any category \mathcal{D} equipped with a cylinder factorisation system, the functor

$$(-) \cdot Y: \mathbf{CFS}(\mathcal{IC}, \mathcal{D}) \rightarrow \mathbf{CAT}(\mathcal{C}, \mathcal{D}) \tag{3.2}$$

is an equivalence of categories. First we show full fidelity: thus, given morphisms of cylinder factorisation systems $F, G: \mathcal{IC} \rightarrow \mathcal{D}$ and a natural transformation $\alpha: FY \rightarrow GY$, we must find a unique $\beta: F \rightarrow G$ with $\beta Y = \alpha$. So given $X \in \mathcal{IC}$, form the category of elements $U: \text{el } X^+ \rightarrow \mathcal{C}$ and dually $V: \text{el } X^- \rightarrow \mathcal{C}$; by the Yoneda lemma, we have a colimit cocone $p^+: YU \rightarrow \Delta X^+$ in \mathcal{PC} —essentially small as X^+ is a small colimit of representables—and likewise an essentially small limit cone $q^-: \Delta X^- \rightarrow YV$ in $\mathcal{P}^i\mathcal{C}$. By Lemma 10, these lift to a cocone $p: YU \rightarrow \Delta X$ and cone $q: \Delta X \rightarrow YV$ in \mathcal{IC} , necessarily in \mathcal{E} and \mathcal{M} respectively. Now as $F(\mathcal{E}) \subset \mathcal{E}$ and $G(\mathcal{M}) \subset \mathcal{M}$, the diagram

$$\begin{array}{ccc}
 FYU & \xrightarrow{Fp} & \Delta FX \\
 Gp \cdot \alpha U \downarrow & \swarrow \beta_X \text{ (dotted)} & \downarrow \alpha V \cdot Fq \\
 \Delta GX & \xrightarrow{Gq} & GYV
 \end{array}$$

of cocones and cones in \mathcal{D} has top edge in \mathcal{E} and bottom edge in \mathcal{M} . The composites around the two sides agree by naturality of α , and so by orthogonality there is a unique diagonal filler β_X as shown making both triangles commute. If $\beta: F \rightarrow G$ is to extend α and be natural, then it must render these triangles commutative; so these β_X 's are the unique possible choice for an extension, and it remains only to show their naturality in X .

So let $f: X \rightarrow X'$ in \mathcal{IC} ; we have the \mathcal{E} -cocone p and \mathcal{M} -cone q as before, but now also $p': YU' \rightarrow \Delta X'$ and $q': \Delta X' \rightarrow YV'$. We also have functors $H = \text{el } f^+: \text{el } X^+ \rightarrow \text{el } Y^+$

and $K = \text{el } f^- : \text{el } Y^- \rightarrow \text{el } X^-$, satisfying $U'H = U$ and $VK = V'$, and, we claim, rendering commutative both triangles—and hence the outside—in:

$$\begin{array}{ccc}
 YU & \xrightarrow{p} & \Delta X \\
 p'H \downarrow & \swarrow f & \downarrow qK \\
 \Delta X' & \xrightarrow{q'} & YV'.
 \end{array} \tag{3.3}$$

To see this last claim, note that π_1 of the top triangle commutes in \mathcal{PC} by the Yoneda lemma and definition of H , and similarly π_2 of the bottom triangle commutes; now apply Lemma 10. Using this, we now show naturality of β at f ; thus that $Gf \cdot \beta_X = \beta_{X'} \cdot Ff$. By orthogonality it suffices to show equality after precomposition with the \mathcal{E} -cocone Fp and after postcomposition with the \mathcal{M} -cone Gq' . For the former, we have that $Gf \cdot \beta_X \cdot Fp = Gf \cdot Gp \cdot \alpha U = Gp'H \cdot \alpha U = Gp'H \cdot \alpha U'H = \beta_{X'} \cdot Fp'H = \beta_{X'} \cdot Ff \cdot Fp$; for the latter, $Gq' \cdot Gf \cdot \beta_X = GqK \cdot \beta_X = \alpha VK \cdot FqK = \alpha V' \cdot FqK = \alpha V' \cdot Fq' \cdot Ff = Gq' \cdot \beta_{X'} \cdot Ff$.

This proves that (3.2) is fully faithful; it remains to show essential surjectivity. Given $F: \mathcal{C} \rightarrow \mathcal{D}$, we must exhibit a map $G: \mathcal{IC} \rightarrow \mathcal{D}$ of cylinder factorisation systems and a natural isomorphism $GY \cong F$. For each $X \in \mathcal{IC}$, let $q \cdot p: YU \rightarrow \Delta X \rightarrow YV$ be its canonical essentially small cylinder, as above. Since Y is fully faithful, there is a unique cylinder $r: U \rightsquigarrow V$ with $Yr = q \cdot p$; now let $t \cdot s: FU \rightarrow \Delta GX \rightarrow FV$ be an $(\mathcal{E}, \mathcal{M})$ -factorisation in \mathcal{D} of the essentially small $Fr: FU \rightsquigarrow FV$. This defines G on objects. On morphisms, let $f: X \rightarrow X'$ in \mathcal{IC} , and let p, q, p', q', H and K be as in the preceding paragraph. We have by commutativity in (3.3) and full fidelity of Y that $r(1 \times K) = r'(H \times 1): U \rightsquigarrow V'$; whence in the diagram on the left in

$$\begin{array}{ccc}
 FU & \xrightarrow{s} & \Delta GX \\
 s'H \downarrow & \swarrow Gf & \downarrow tK \\
 \Delta GX' & \xrightarrow{t'} & FV'
 \end{array}
 \qquad
 \begin{array}{ccc}
 FU & \xrightarrow{s} & \Delta GX \\
 s''H_g H_f \downarrow & \swarrow G(gf) & \downarrow tK_f K_g \\
 \Delta GX'' & \xrightarrow{t''} & FV''
 \end{array}$$

the composite cylinders $Fr(1 \times K)$ and $Fr'(H \times 1)$ are equal. Since $s \in \mathcal{E}$ and $t' \in \mathcal{M}$, we induce by orthogonality a unique filler, as displayed; which gives the action of G on morphisms. Clearly, when $f = 1_X$, we have $H = K = 1$ and $s = s'$ and $t = t'$ and the unique filler $G1_X$ must be 1_{GX} . So G preserves identities; as for binary composition, given $f: X \rightarrow X'$ and $g: X' \rightarrow X''$, the map $G(gf)$ is the unique filler for the square on the right above; but since $Gg \cdot Gf \cdot s = Gg \cdot s'H_f = s''H_g H_f$ and $t'' \cdot Gg \cdot Gf = t'K_g \cdot Gf = tK_f K_g$, the map $Gg \cdot Gf$ is also a filler. So $G(gf) = Gg \cdot Gf$ and G is a functor.

To see that $GY \cong F: \mathcal{C} \rightarrow \mathcal{D}$, note that the canonical cylinder $r: U \rightsquigarrow V$ in \mathcal{C} associated to $YX \in \mathcal{IC}$ has $U: \mathcal{C}/c \rightarrow \mathcal{C}$ and $V: c/\mathcal{C} \rightarrow \mathcal{C}$ the forgetful functors from the slice and coslice, and $r_{f:a \rightarrow c, g:c \rightarrow b} = gf: a \rightarrow b$; so in particular, $r_{1_c, 1_c} = 1_c$. Consequently, the chosen factorisation $t \cdot s: FU \rightarrow \Delta GYc \rightarrow FV$ of Fr in \mathcal{D} involves maps $s_{1_c}: Fc \rightarrow GYc$

and $t_{1_c} : GYc \rightarrow Fc$ with $t_{1_c} \cdot s_{1_c} = 1_{Fc}$. Now as 1_c is terminal in \mathcal{C}/c , the functor $1 \rightarrow \mathcal{C}/c$ picking it out is final: whence by Lemma 7, s_{1_c} , like s , is in \mathcal{E} ; dually, t_{1_c} is in \mathcal{M} . So $t_{1_c} \cdot s_{1_c}$ is an $(\mathcal{E}, \mathcal{M})$ -factorisation of 1_{Fc} ; but so too is $1_{Fc} \cdot 1_{Fc}$, whence by Lemma 3, t_{1_c} is invertible, and provides the component at c of the natural isomorphism $GY \cong F$.

Finally, we must show that G is a map of cylinder factorisation systems. By duality, we need only show that $G(\mathcal{E}) \subset \mathcal{E}$. So let $w : D \rightarrow \Delta X$ be an \mathcal{E} -cocone in \mathcal{IC} ; we must show that $Gw : GD \rightarrow \Delta GX$ is an \mathcal{E} -cocone in \mathcal{D} . Consider the category $\text{el } D^+$ whose objects are triples $(i \in \mathcal{I}, a \in \mathcal{C}, d \in Di^+a)$ and whose morphisms $(i, a, d) \rightarrow (i', a', d')$ are pairs of $f : i \rightarrow i'$ in \mathcal{I} and $k : a \rightarrow a'$ such that $f \cdot d = d' \cdot k$. Clearly there is a functor $I : \text{el } D^+ \rightarrow \mathcal{I}$ sending (i, a, d) to i , but there is also a functor $W : \text{el } D^+ \rightarrow \text{el } X^+$ sending (i, a, d) to $(a, w_i^+(d))$ and sending (f, k) to k . We claim that W is final.

Indeed, for any $x \in X^+a$, the comma category $(a, x)/W$ has objects being triples of $i \in \mathcal{I}, h : a \rightarrow b$ in \mathcal{C} and $d \in Di^+b$ with $x = w_i^+(d) \cdot h$, and morphisms $(i, h, d) \rightarrow (i', h', d')$ being pairs $f : i \rightarrow i'$ and $k : b \rightarrow b'$ with $kh = h'$ and $d' \cdot k = f \cdot d$. We must show this category to be connected. Since any object (i, h, d) admits a map $(1_i, h)$ from one of the form $(i, 1_a, d')$, it's enough to show connectedness of the full subcategory on objects of this form. This subcategory is equally the full subcategory $\mathcal{A}_{a,x} \subset \text{el } (D-)^+a$ on those pairs $(i \in \mathcal{I}, d \in Di^+a)$ with $x = w_i^+(a)$. Now as w is an \mathcal{E} -cocone in \mathcal{IC} , its projection w^+ in \mathcal{PC} is colimiting, which is to say that each cocone $(w_i^+(a) : Di^+(a) \rightarrow X^+a)_{i \in \mathcal{I}}$ is colimiting; whence $\mathcal{A}_{a,x}$ is connected, $(a, x)/W$ is connected, and so W is final.

Now, let $\tau \cdot \sigma : FU \rightarrow \Delta GX \rightarrow FV$ be the factorisation defining GX , and for each $i \in \mathcal{I}$, let $t_i \cdot s_i : FU_i \rightarrow \Delta GDi \rightarrow FV_i$ be the corresponding factorisation for GDi . For each $i \in \mathcal{I}$, let $W_i : \text{el } Di^+ \rightarrow \text{el } X^+$ be the functor induced by w_i^+ ; note that we have $U_i = UW_i$ and commuting diagrams of cocones as on the left in

$$\begin{array}{ccc}
 FU_i & \xrightarrow{\sigma W_i} & \Delta GX \\
 s_i \downarrow & \nearrow Gw_i & \\
 \Delta GDi & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 FUW & \xrightarrow{\sigma W} & \Delta GX \\
 s \downarrow & \nearrow GwI & \\
 GDI & &
 \end{array}$$

It follows that the natural $s : FUW \rightarrow GDI$ whose component at $(i, a, d) \in \text{el } D^+$ is $(s_i)_{(a,d)} : Fa \rightarrow GDi$ fits into a commuting diagram as on the right above. We are now ready to prove that Gw is an \mathcal{E} -cocone. Suppose given an \mathcal{M} -cone v fitting into a diagram of cocones and cones in \mathcal{D} as on the left in

$$\begin{array}{ccc}
 GD & \xrightarrow{Gw} & \Delta GX \\
 h \downarrow & & \downarrow k \\
 \Delta W & \xrightarrow{v} & E
 \end{array}
 \qquad
 \begin{array}{ccc}
 FUW & \xrightarrow{GwI \cdot s} & \Delta GX \\
 hI \cdot s \downarrow & \nearrow m & \downarrow k \\
 \Delta W & \xrightarrow{v} & E
 \end{array}
 \qquad
 \begin{array}{ccc}
 FU_i & \xrightarrow{Gw_i \cdot s_i} & \Delta GX \\
 h_i \cdot s_i \downarrow & \nearrow m & \downarrow k \\
 \Delta W & \xrightarrow{v} & E.
 \end{array}$$

Whiskering the cocones with I and precomposing with s yields the commuting diagram in the centre. The top edge therein is σW which by Lemma 7 is in \mathcal{E} , since σ is so and

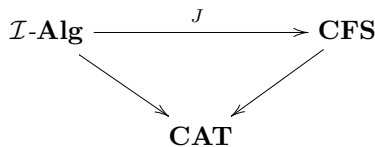
W is final. So by orthogonality there is a unique m as indicated making both triangles commute. This commutativity is equivalent to that of the two triangles on the right for every $i \in \mathcal{I}$; wherein the condition $m \cdot Gw_i \cdot s_i = h_i \cdot s_i$ for the top triangle, together with $v \cdot m \cdot Gw_i = k \cdot Gw_i = v \cdot h_i$, implies that $m \cdot Gw_i = h_i$, since $v \in \mathcal{M}$ and $s_i \in \mathcal{E}$. So, finally, m is unique such that $v \cdot m = k$ and $m \cdot Gw = h$, thus a unique filler for the left square, as required. \square

4. Pseudomonadicity

The preceding result shows that the embedding $Y: \mathcal{C} \rightarrow \mathcal{IC}$ into the Isbell envelope is the unit at \mathcal{C} of a biadjunction $\mathbf{CFS} \rightleftarrows \mathbf{CAT}$. This biadjunction induces a pseudomonad \mathcal{I} on \mathbf{CAT} , and a canonical comparison homomorphism $K: \mathbf{CFS} \rightarrow \mathcal{I}\text{-Alg}$, whose codomain is the 2-category of \mathcal{I} -pseudoalgebras, algebra pseudomorphisms and algebra 2-cells. Recall—for instance, from [17, §2]—that an \mathcal{I} -pseudoalgebra involves a morphism $A: \mathcal{IC} \rightarrow \mathcal{C}$ and invertible 2-cells $\theta: 1_{\mathcal{C}} \cong AY$ and $\pi: A \cdot \mu_{\mathcal{C}} \cong A \cdot \mathcal{I}A$ satisfying two coherence axioms; and that an algebra pseudomorphism $(\mathcal{C}, A) \rightarrow (\mathcal{D}, B)$ involves a morphism $F: \mathcal{C} \rightarrow \mathcal{D}$ and an invertible 2-cell $\varphi: B \cdot \mathcal{I}F \cong FA$, also satisfying two coherence axioms.

Our second main result states that the canonical comparison $K: \mathbf{CFS} \rightarrow \mathcal{I}\text{-Alg}$ is a biequivalence; in other words, that \mathbf{CFS} is pseudomonadic over \mathbf{CAT} . We could prove this using the pseudomonadicity theorem of [12], but it will be simpler and more illuminating to construct directly a biequivalence inverse.

Theorem 12. *The forgetful 2-functor $\mathcal{I}\text{-Alg} \rightarrow \mathbf{CAT}$ has a (strictly commuting) factorisation*



wherein J is a biequivalence 2-functor satisfying $JK = 1$; it follows that K is a biequivalence, and so that \mathbf{CFS} is pseudomonadic over \mathbf{CAT} .

Proof. We first introduce some terminology: given a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ and a cylinder factorisation system on \mathcal{C} , we say that a cocone $p: D \rightarrow \Delta V$ in \mathcal{C} is F -nearly in \mathcal{E} if, on forming an $(\mathcal{E}, \mathcal{M})$ -factorisation $p = t \cdot s: D \rightarrow \Delta W \rightarrow \Delta V$, the map t is inverted by F . It is easy to see that if $p \in \mathcal{E}$, then p is F -nearly in \mathcal{E} ; and that, if $G: \mathcal{B} \rightarrow \mathcal{C}$ is a map in $\mathbf{CFS}_{\mathcal{E}}$, then a cocone p in \mathcal{B} is FG -nearly in $\mathcal{E}_{\mathcal{B}}$ iff Gp is F -nearly in $\mathcal{E}_{\mathcal{C}}$. Of course, there is the dual notion of a cone being F -nearly in \mathcal{M} , with the corresponding dual results.

With this in place, we now define J on objects. Let $A: \mathcal{IC} \rightarrow \mathcal{C}$ be an \mathcal{I} -pseudoalgebra. We define classes of small cocones and cones in \mathcal{C} by:

$$\begin{aligned} \mathcal{E} &= \{ p: D \rightarrow \Delta V \mid Yp \text{ is } A\text{-nearly in } \mathcal{E}_{\mathcal{IC}} \} \\ \mathcal{M} &= \{ q: \Delta W \rightarrow E \mid Yq \text{ is } A\text{-nearly in } \mathcal{M}_{\mathcal{IC}} \}, \end{aligned} \tag{4.1}$$

and claim that this provides the required cylinder factorisation system on \mathcal{C} . As a first step, we prove that $A: \mathcal{IC} \rightarrow \mathcal{C}$ has $A(\mathcal{E}) \subset \mathcal{E}$ and $A(\mathcal{M}) \subset \mathcal{M}$; by duality we need only prove the first. So given $p \in \mathcal{E}_{\mathcal{IC}}$, we must show that YAp is A -nearly in $\mathcal{E}_{\mathcal{IC}}$. By pseudonaturality of the unit of \mathcal{I} , we have $YA \cong \mathcal{I}A \cdot Y$, so this is equally to show that $\mathcal{I}A \cdot Yp$ is A -nearly in $\mathcal{E}_{\mathcal{IC}}$. Since $\mathcal{I}A: \mathcal{IIC} \rightarrow \mathcal{IC}$ is a map of (free) cylinder factorisation systems, this is equally to show that Yp is $A \cdot \mathcal{I}A$ -nearly in $\mathcal{E}_{\mathcal{IIC}}$; but $A \cdot \mathcal{I}A \cong A \cdot \mu_{\mathcal{C}}$ since \mathcal{C} is a pseudoalgebra, and so this is equally to show that Yp is $A \cdot \mu_{\mathcal{C}}$ -nearly in $\mathcal{E}_{\mathcal{IIC}}$. Now as $\mu_{\mathcal{C}}$ is a map of cylinder factorisation systems, this is equally to show that $\mu_{\mathcal{C}} \cdot Yp$ is A -nearly in $\mathcal{E}_{\mathcal{IC}}$; finally, since $\mu_{\mathcal{C}} \cdot Y \cong 1$, this is equally to show that p is A -nearly in $\mathcal{E}_{\mathcal{IC}}$, which is certainly so if $p \in \mathcal{E}_{\mathcal{IC}}$.

We now show that the classes (4.1) verify the axioms (i)–(iii) for a cylinder factorisation system on \mathcal{C} . (i) is trivial; for (iii), given a small cylinder $r: D \rightsquigarrow E$ in \mathcal{C} , we form an $(\mathcal{E}, \mathcal{M})$ -factorisation $Yr = q \cdot p$ in \mathcal{IC} ; by the above, $AYr = Aq \cdot Ap$ is an $(\mathcal{E}, \mathcal{M})$ -factorisation in \mathcal{C} , and so conjugating by the isomorphism $\theta: 1_{\mathcal{C}} \cong AY$ (coming from the pseudoalgebra structure of \mathcal{C}) we obtain the desired factorisation $r = (\theta^{-1}E \cdot Aq) \cdot (Ap \cdot \theta D): D \rightarrow \Delta V \rightarrow E$. It remains to verify (ii). Let $p \in \mathcal{E}_{\mathcal{C}}$ and $q \in \mathcal{M}_{\mathcal{C}}$ and suppose given a square $q \cdot h = k \cdot p$ as in (2.1). In \mathcal{IC} we may form the diagram on the left

$$\begin{array}{ccc} YD & \xrightarrow{s} & \Delta X & \xrightarrow{t} & \Delta YV \\ Yh \downarrow & & \vdots \ell & & \downarrow Yk \\ \Delta YW & \xrightarrow{u} & \Delta Y & \xrightarrow{v} & YE \end{array} \qquad \begin{array}{ccc} AYD & \xrightarrow{AYp} & \Delta AYV \\ AYh \downarrow & \swarrow m & \downarrow AYk \\ \Delta AYW & \xrightarrow{AYq} & AYE, \end{array}$$

wherein both rows are $(\mathcal{E}, \mathcal{M})$ -factorisations and ℓ is the unique map induced by orthogonality of s and v . Since $p \in \mathcal{E}_{\mathcal{C}}$ and $q \in \mathcal{M}_{\mathcal{C}}$, applying A inverts u and t , and so we obtain a diagonal filler for the square on the right above by taking $m = (Au)^{-1} \cdot Al \cdot (At)^{-1}$; conjugating by $\theta: 1_{\mathcal{C}} \cong AY$ now yields the required filler $j = \theta_W^{-1} \cdot m \cdot \theta_V: V \rightarrow W$ for the original square (2.1). To show uniqueness of j , let $j': V \rightarrow W$ be another diagonal filler; then $u \cdot Yj' \cdot t: \Delta X \rightarrow \Delta Y$ fills the rectangle on the left above, and so by orthogonality must be ℓ ; whence $Al = Au \cdot AYj' \cdot At$, so that $m = AYj'$ and so finally $j = \theta_W^{-1} \cdot AYj' \cdot \theta_V = j'$.

This defines J on objects; since $\mathbf{CFS} \rightarrow \mathbf{CAT}$ is faithful on 1-cells and locally fully faithful, the definition on 1- and 2-cells is forced, and all that is required is to show that any pseudomorphism $F: (\mathcal{C}, A) \rightarrow (\mathcal{D}, B)$ of \mathcal{I} -pseudoalgebras preserves the classes

of the derived cylinder factorisation systems. So let p be a cocone in \mathcal{C} such that Yp is A -nearly in $\mathcal{E}_{\mathcal{I}\mathcal{C}}$; we must show that YFp is B -nearly in $\mathcal{E}_{\mathcal{I}\mathcal{D}}$. By naturality of Y , we have $YF \cong \mathcal{I}F \cdot Y$, so it's enough to show that $\mathcal{I}F \cdot Yp$ is B -nearly in $\mathcal{E}_{\mathcal{I}\mathcal{D}}$. Since $\mathcal{I}F$ is a map of cylinder factorisation systems, it's enough to show that Yp is $B \cdot \mathcal{I}F$ -nearly in $\mathcal{E}_{\mathcal{I}\mathcal{C}}$; but $B \cdot \mathcal{I}F \cong FA$ as F is a pseudomorphism, so it's enough to show that Yp is FA -nearly in $\mathcal{E}_{\mathcal{I}\mathcal{C}}$; which is so since Yp is A -nearly in $\mathcal{E}_{\mathcal{I}\mathcal{C}}$.

This completes the definition of J ; we next show that $JK = 1$. This is immediate on 1- and 2-cells, since J and K are both over **CAT** and **CFS** \rightarrow **CAT** is faithful on 1- and 2-cells. To show $JK = 1$ on objects, let \mathcal{C} be a category equipped with a cylinder factorisation system; then $K\mathcal{C}$ is the pseudoalgebra $A:\mathcal{I}\mathcal{C} \rightarrow \mathcal{C}$ whose structure map is obtained by extending the identity $\mathcal{C} \rightarrow \mathcal{C}$ using freeness of $\mathcal{I}\mathcal{C}$. Now $JK\mathcal{C}$ is the category \mathcal{C} equipped with the cylinder factorisation system $(\mathcal{E}', \mathcal{M}')$ where \mathcal{E}' comprises those cocones p such that Yp is A -nearly in $\mathcal{E}_{\mathcal{I}\mathcal{C}}$; but as $A:\mathcal{I}\mathcal{C} \rightarrow \mathcal{C}$ is a map of cylinder factorisation systems, these are equally the cocones p such that AYp is $1_{\mathcal{C}}$ -nearly in \mathcal{E} ; that is, the \mathcal{E} -cocones. Thus $\mathcal{E}' = \mathcal{E}$ and similarly $\mathcal{M}' = \mathcal{M}$, so that JK is the identity on objects as required.

Finally, we show that J is a biequivalence. Being a retraction, it is clearly surjective on objects; we claim that it also full on 1-cells and locally fully faithful. For the first claim, let (\mathcal{C}, A) and (\mathcal{D}, B) be \mathcal{I} -pseudoalgebras and $F: J(\mathcal{C}, A) \rightarrow J(\mathcal{D}, B)$ a map of induced cylinder factorisation systems. Then in the left square of

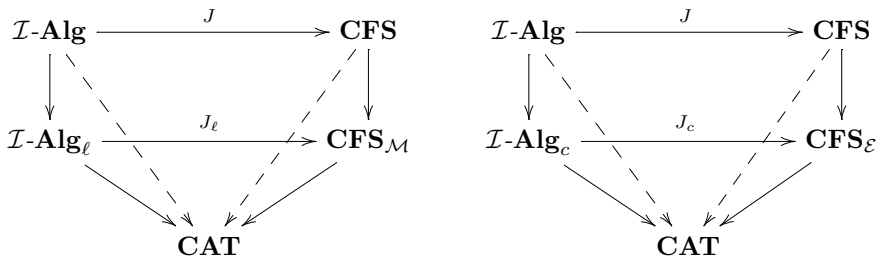
$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathcal{I}\mathcal{C} & \xrightarrow{\mathcal{I}F} & \mathcal{I}\mathcal{D} \\
 A \downarrow & & \downarrow B \\
 \mathcal{C} & \xrightarrow{F} & \mathcal{D}
 \end{array} & & \begin{array}{ccc}
 \mathcal{I}\mathcal{C} & \xrightarrow{\mathcal{I}F} & \mathcal{I}\mathcal{D} \\
 A \downarrow & \Downarrow \varphi_F & \downarrow B \\
 \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
 & \Downarrow \alpha & \\
 & G &
 \end{array} & = & \begin{array}{ccc}
 \mathcal{I}\mathcal{C} & \xrightarrow{\mathcal{I}F} & \mathcal{I}\mathcal{D} \\
 A \downarrow & \Downarrow \mathcal{I}\alpha & \downarrow B \\
 \mathcal{C} & \xrightarrow{\mathcal{I}G} & \mathcal{D} \\
 & \Downarrow \varphi_G & \\
 & G &
 \end{array}
 \end{array} \tag{4.2}$$

all four functors are maps of cylinder factorisation systems. Moreover, using the unit coherences for (\mathcal{C}, A) and (\mathcal{D}, B) and pseudonaturality of Y , we have an isomorphism $\alpha: B \cdot \mathcal{I}F \cdot Y \cong BYF \cong FAY$, and so, by full fidelity of (3.2), a unique invertible 2-cell $\varphi: B \cdot \mathcal{I}F \cong FA$ with $\varphi Y = \alpha$. This makes $(F, \varphi): (\mathcal{C}, A) \rightarrow (\mathcal{D}, B)$ into an algebra pseudomorphism with $J(F, \varphi) = F$; the first coherence axiom follows immediately from $\varphi Y = \alpha$, while the second one, equating two parallel morphisms in **CAT**($\mathcal{I}\mathcal{I}\mathcal{C}, \mathcal{D}$), follows by fidelity of (3.2) on observing these morphisms to reside in **CFS**($\mathcal{I}\mathcal{I}\mathcal{C}, \mathcal{D}$) and to have the same precomposite with $Y: \mathcal{I}\mathcal{C} \rightarrow \mathcal{I}\mathcal{I}\mathcal{C}$. It remains to show local full fidelity of J ; thus, that for any pair of algebra pseudomorphisms $(F, \varphi_F), (G, \varphi_G): (\mathcal{C}, A) \rightarrow (\mathcal{D}, B)$ and any 2-cell $\alpha: F \Rightarrow G$, the pasting equality above right holds. This follows, again, by observing these pastings to describe parallel morphisms in **CFS**($\mathcal{I}\mathcal{C}, \mathcal{D}$) which coincide on precomposition with $Y: \mathcal{C} \rightarrow \mathcal{I}\mathcal{C}$. \square

5. Lax and colax morphisms

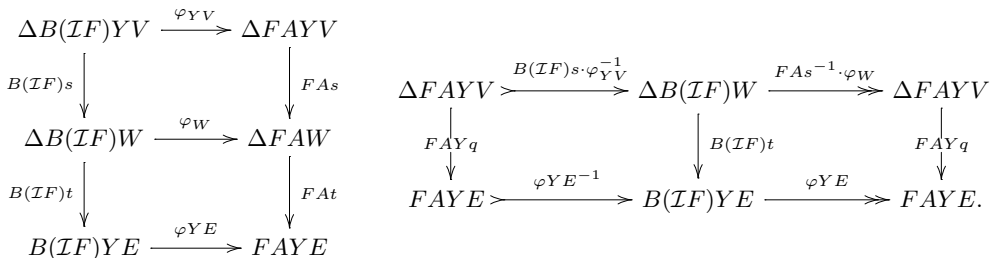
As well as the 2-category $\mathcal{I}\text{-Alg}$, we also have the larger 2-categories $\mathcal{I}\text{-Alg}_\ell$ and $\mathcal{I}\text{-Alg}_c$ whose objects are again pseudoalgebras, but whose 1- and 2-cells are now the *lax* or *colax* algebra morphisms and the algebra 2-cells between them. A lax algebra morphism $(\mathcal{C}, A) \rightsquigarrow (\mathcal{D}, B)$ comprises a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ and a potentially non-invertible 2-cell $\varphi: B \cdot \mathcal{I}F \Rightarrow FA$ satisfying two coherence axioms; a *colax* morphism is similar, but with the orientation of the non-invertible φ now reversed. Our final result identifies the lax and colax \mathcal{I} -algebra morphisms as the functors preserving only \mathcal{M} -cones and only \mathcal{E} -cocones respectively. As in the preceding section, we could proceed by applying a general theorem, in this case the two-dimensional monadicity theorem of [4]; but as in the preceding section, it will be simpler and more illuminating to give the constructions directly.

Theorem 13. *The factorisation of $\mathcal{I}\text{-Alg} \rightarrow \text{CAT}$ through CFS extends to a factorisation of $\mathcal{I}\text{-Alg}_\ell \rightarrow \text{CAT}$ through $\text{CFS}_\mathcal{M}$ and to one of $\mathcal{I}\text{-Alg}_c \rightarrow \text{CAT}$ through $\text{CFS}_\mathcal{E}$:*



wherein J_ℓ and J_c are biequivalences.

Proof. By duality, we consider only the lax case. First we extend J to J_ℓ ; of course, J_ℓ must agree with J on objects, and as before the definition is forced on 1- and 2-cells; so the only work is showing that, if $(F, \varphi): (\mathcal{C}, A) \rightsquigarrow (\mathcal{D}, B)$ is a lax algebra map, then F sends $\mathcal{M}_\mathcal{C}$ -cones to $\mathcal{M}_\mathcal{D}$ -cones. So let $q: \Delta V \rightarrow E$ be an $\mathcal{M}_\mathcal{C}$ -cone; we must show $Fq \in \mathcal{M}_\mathcal{D}$. Let $Yq = t \cdot s: \Delta YV \rightarrow \Delta W \rightarrow YE$ be an $(\mathcal{E}, \mathcal{M})$ -factorisation in $\mathcal{I}\mathcal{C}$, and consider the commuting diagram on the left in



To say $q \in \mathcal{M}_{\mathcal{C}}$ is to say that Yq is A -nearly in $\mathcal{M}_{\mathcal{I}\mathcal{C}}$: so FAs is invertible, and by the unit coherence axiom for a lax morphism so too are φ_{YV} and φ_{YE} . Moreover $B(\mathcal{I}F)t \in \mathcal{M}_{\mathcal{D}}$ since $t \in \mathcal{M}_{\mathcal{I}\mathcal{C}}$ and $B \cdot \mathcal{I}F$ is a map of cylinder factorisation systems. Thus the diagram on the right exhibits $FAYq$ as being a retract of an $\mathcal{M}_{\mathcal{D}}$ -cone and so, by an easy argument, itself an $\mathcal{M}_{\mathcal{D}}$ -cone; finally, since $AY \cong 1$, we have $Fq \cong FAYq$ an $\mathcal{M}_{\mathcal{D}}$ -cone as required.

This completes the definition of \mathcal{J}_{ℓ} , and it remains to show that it is a biequivalence. Of course, it is surjective on objects, since J is; we claim it is also full on 1-cells and locally fully faithful. We use the fact—generalising full fidelity of (3.2)—that for any $F \in \mathbf{CFS}_{\mathcal{E}}(\mathcal{I}\mathcal{C}, \mathcal{D})$ and $G \in \mathbf{CFS}_{\mathcal{M}}(\mathcal{I}\mathcal{C}, \mathcal{D})$, the function

$$(-) \cdot Y : \mathbf{CFS}_{\mathcal{M}}(\mathcal{I}\mathcal{C}, \mathcal{D})(F, G) \rightarrow \mathbf{CAT}(\mathcal{C}, \mathcal{D})(F, G) \tag{5.1}$$

is invertible; the proof is precisely the first two paragraphs of the proof of Theorem 11, noting that there we only needed that $F(\mathcal{E}) \subset \mathcal{E}$ and that $G(\mathcal{M}) \subset \mathcal{M}$. To show J_{ℓ} is full on 1-cells, let (\mathcal{C}, A) and (\mathcal{D}, B) be \mathcal{I} -pseudoalgebras and let $F: J_{\ell}(\mathcal{C}, A) \rightarrow J_{\ell}(\mathcal{D}, B)$ in $\mathbf{CFS}_{\mathcal{M}}$; then in the left square of (4.2), the maps along the upper side are in \mathbf{CFS} , and those along the lower side in $\mathbf{CFS}_{\mathcal{M}}$; so by invertibility of (5.1), the isomorphism $\alpha: B \cdot \mathcal{I}F \cdot Y \cong BYF \cong FAY$ induces a unique 2-cell $\varphi: B \cdot \mathcal{I}F \Rightarrow FA$ with $\varphi Y = \alpha$. Using injectivity of (5.1) and arguing as in the final paragraph of Theorem 12, we may show that this makes $(F, \varphi): (\mathcal{C}, A) \rightsquigarrow (\mathcal{D}, B)$ into a lax algebra morphism with $J_{\ell}(F, \varphi) = F$; so J_{ℓ} is full on 1-cells. In a similar manner, the argument showing local full fidelity of J generalises using (5.1) to one showing local full fidelity of J_{ℓ} . \square

6. (Φ, Ψ) -cylinder factorisation systems

The definition of cylinder factorisation system involves factorisations for all *small* cylinders—ones indexed by small categories. However, we could equally well have required factorisations only for finite cylinders, say, or only for discrete ones. In this final section, we exhibit such variant notions as the pseudoalgebras for corresponding variants of the Isbell monad, obtained by replacing the pseudomonads \mathcal{P} and \mathcal{P}^{\dagger} used in its construction by suitable *full submonads* thereof.

By a full submonad \mathcal{S} of a pseudomonad \mathcal{T} on \mathbf{CAT} , we mean the choice, for each category \mathcal{C} , of a full subcategory $\mathcal{S}\mathcal{C} \subset \mathcal{T}\mathcal{C}$, with these choices being closed under the pseudomonad structure of \mathcal{T} in an obvious sense. In the case of \mathcal{P} and \mathcal{P}^{\dagger} , full submonads $\Phi \subset \mathcal{P}$ and $\Psi \subset \mathcal{P}^{\dagger}$ correspond to *saturated classes of weights* for colimits or limits in the sense of [2] (there called *closed classes*); the corresponding Φ - or Ψ -pseudoalgebras are categories admitting all Φ -weighted colimits or all Ψ -weighted limits, respectively. Relative to a choice of full submonads $\Phi \subset \mathcal{P}$ and $\Psi \subset \mathcal{P}^{\dagger}$, we may construct a modified Isbell envelope whose value at a category \mathcal{C} is obtained as a pullback

$$\begin{array}{ccc}
 \mathcal{I}_{\Phi, \Psi}(\mathcal{C}) & \longrightarrow & \mathcal{IC} \\
 \downarrow & & \downarrow (\pi_1, \pi_2) \\
 \Phi\mathcal{C} \times \Psi\mathcal{C} & \longrightarrow & \mathcal{PC} \times \mathcal{P}^\dagger\mathcal{C}.
 \end{array} \tag{6.1}$$

Note that each $\mathcal{I}_{\Phi, \Psi}(\mathcal{C}) \rightarrow \mathcal{IC}$ may be taken to be the inclusion of a full subcategory; if we do so, then it is easy to see that these full inclusions assemble together to yield a full submonad $\mathcal{I}_{\Phi, \Psi} \subset \mathcal{I}$ —whose pseudoalgebras we now characterise.

A diagram $D: \mathcal{I} \rightarrow \mathcal{C}$ will be called a Φ -*diagram* if it admits a factorisation as on the left below for some $\varphi \in \Phi\mathcal{C}$. Dually, $E: \mathcal{J} \rightarrow \mathcal{C}$ is a Ψ -*diagram* if for some $\psi \in \Psi\mathcal{C}$ it admits a factorisation as on the right:

$$\begin{array}{ccc}
 \mathcal{I} & \xrightarrow{H \text{ final}} & \text{el } \varphi \\
 \searrow D & & \swarrow \pi \\
 & \mathcal{C} &
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{J} & \xrightarrow{K \text{ initial}} & \text{el } \psi. \\
 \searrow E & & \swarrow \pi \\
 & \mathcal{C} &
 \end{array} \tag{6.2}$$

A (Φ, Ψ) -*cylinder factorisation system* is now defined identically to a cylinder factorisation system, except that the cones, cocones and cylinders appearing in the definition are restricted to those whose domains and codomains are Φ - and Ψ -diagrams respectively. Categories equipped with (Φ, Ψ) -cylinder factorisation systems are the objects of a 2-category $\mathbf{CFS}_{\Phi, \Psi}$, whose maps are, as before, functors preserving the cocones and cones.

The proof of the following result follows precisely the arguments of the preceding sections, but with Φ and Ψ everywhere replacing \mathcal{P} and \mathcal{P}^\dagger , and with Φ -weighted colimits and Ψ -weighted limits replacing arbitrary colimits and limits. There is also an analogue of [Theorem 13](#), which we do not trouble to state, characterising the lax and oplax algebra morphisms in terms of maps preserving only cones or only cocones.

Theorem 14. *Given full submonads $\Phi \subset \mathcal{P}$ and $\Psi \subset \mathcal{P}^\dagger$, we have a pseudomonadic adjunction*

$$\mathbf{CFS}_{\Phi, \Psi} \xrightleftharpoons{\perp} \mathbf{CAT}$$

whose unit at \mathcal{C} may be taken to be the restricted Yoneda embedding $Y: \mathcal{C} \rightarrow \mathcal{I}_{\Phi, \Psi}(\mathcal{C})$.

In practice, the notions of Φ -diagram and Ψ -diagram tend to encompass slightly more than we would intuitively expect. For example, when $\Phi = 1_{\mathbf{CAT}}$, the Φ -diagrams are those $D: \mathcal{I} \rightarrow \mathcal{C}$ which admit an absolute colimit in \mathcal{C} , rather than simply those $D: 1 \rightarrow \mathcal{C}$ indexed by the terminal category. Towards rectifying this, we define a class \mathcal{A} of Φ -diagrams to be *generating* if, for every $\varphi \in \Phi\mathcal{C}$, there is some $D \in \mathcal{A}$ fitting into a diagram as to the left of (6.2); we define a generating class \mathcal{B} of Ψ -diagrams dually. If \mathcal{A}

and \mathcal{B} are generating classes, then by using [Lemma 7](#) and arguing as in [Proposition 8](#), we may show that a (Φ, Ψ) -cylinder factorisation system is completely and uniquely determined by its cocones, cones, and cylinder factorisations with respect to diagrams in \mathcal{A} and \mathcal{B} .

Examples 15.

- (a) Let $\Phi = \mathcal{P}$ and let $\Psi = \mathcal{F}$ be the pseudomonad for *finite* limits—for which \mathcal{FC} is given by the closure of the representables under finite limits in $\mathcal{P}^\dagger\mathcal{C}$ —with as generating class of Ψ -diagrams all diagrams indexed by a finite category. In this case, a (Φ, Ψ) -cylinder factorisation system involves factorisations for all cylinders with *finite* codomain. For example, any regular category with pullback-stable unions of subobjects admits a (Φ, Ψ) -cylinder factorisation system given by (covering cocones, jointly monic cones).
- (b) Let $\Phi = \Psi = 1_{\mathbf{CAT}}$, and take as generating classes of Φ - and Ψ -diagrams just those indexed by the terminal category 1. Then a (Φ, Ψ) -cylinder factorisation system is precisely an orthogonal factorisation system; moreover, $\mathcal{I}_{\Phi, \Psi}(\mathcal{C})$ is the arrow category \mathcal{C}^2 , and a short calculation shows the pseudomonad structure of $\mathcal{I}_{\Phi, \Psi}$ to be that of the “squaring” monad $(-)^2$ of [\[11\]](#). Thus we reconstruct the main result of *ibid.*, identifying orthogonal factorisation systems with $(-)^2$ -pseudoalgebras.
- (c) Let $\Phi = \mathbf{Fam}_\Sigma$ and $\Psi = \mathbf{Fam}_\Pi$ be the pseudomonads whose components at \mathcal{C} comprise the coproducts, respectively products, of representables in \mathcal{PC} and $\mathcal{P}^\dagger\mathcal{C}$, and take as generating classes of Φ - and Ψ -diagrams just those indexed by discrete categories. A (Φ, Ψ) -cylinder factorisation system now involves factorisations of small discrete cylinders—*arrays* in the terminology of [\[15\]](#)—into discrete cones and discrete cocones, and the notion of orthogonality involved is precisely that of [\[9, §3\]](#). In this case, the fact that $\mathcal{I}_{\Phi, \Psi}(\mathcal{C})$ is the free (Φ, Ψ) -cylinder factorisation system is quite palpable, since its objects are *precisely* the small discrete cylinders in \mathcal{C} .
- (d) Let $\Phi = 1_{\mathbf{CAT}}$ and $\Psi = \mathbf{Fam}_\Pi$, with generating classes of Φ - and Ψ -diagrams as before. In this case, a (Φ, Ψ) -cylinder factorisation system involves factorisations of discrete cones into \mathcal{E} -maps followed by \mathcal{M} -cones; it is thus a *factorisation structure for small sources* in the sense of [\[1, Exercise 15J\]](#). As in the preceding example, $\mathcal{I}_{\Phi, \Psi}(\mathcal{C})$ has a simple description as the category of all small discrete cones in \mathcal{C} .
- (e) Let $\Phi = (-)_\perp$ be the pseudomonad which freely adjoins an initial object, with as generating class of Φ -diagrams precisely those indexed by 0 or 1; and let $\Psi = 1_{\mathbf{CAT}}$, with generating class as before. In this case, a (Φ, Ψ) -cylinder factorisation system is an orthogonal factorisation system in which, additionally, every object admits an \mathcal{M} -map from an object orthogonal to every \mathcal{M} -map. As in [Examples 2.1\(c\)](#), this second condition follows automatically from the first in the presence of an initial object; but there are important cases where initial objects do not exist. For exam-

ple, a category \mathcal{C} admits a (Φ, Ψ) -cylinder factorisation system with \mathcal{M} the class of *all* maps just when every $A \in \mathcal{C}$ admits a map from a *strict generic* [18]—an object G such that, for every $X \in \mathcal{C}$, the action of $\text{Aut}(G)$ on $\mathcal{C}(G, X)$ is free and transitive.

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