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Monads and theories ☆

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ABSTRACT

Given a locally presentable enriched category $\mathcal{E}$ together with a small dense full subcategory $\mathcal{A}$ of *arities*, we study the relationship between monads on $\mathcal{E}$ and identity-on-objects functors out of $\mathcal{A}$, which we call $\mathcal{A}$-pretheories. We show that the natural constructions relating these two kinds of structure form an adjoint pair. The fixpoints of the adjunction are characterised on the one side as the $\mathcal{A}$-*nervous monads*—those for which the conclusions of Weber's *nerve theorem* hold—and on the other, as the $\mathcal{A}$-*theories* which we introduce here.

The resulting equivalence between $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories is best possible in a precise sense, and extends almost all previously known monad–theory correspondences. It also establishes some completely new correspondences, including one which captures the globular theories defining Grothendieck weak ω-groupoids.

Besides establishing our general correspondence and illustrating its reach, we study good properties of $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories that allow us to recognise and construct them with ease. We also compare them with the monads with arities and theories with arities introduced and studied by Berger, Melliès and Weber.

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1. Introduction

Category theory provides two approaches to classical universal algebra. On the one hand, we have finitary monads on **Set** and on the other hand, we have Lawvere theories. Relating the two approaches we have Linton's result [26], which shows that the category of finitary monads on **Set** is equivalent to the category of Lawvere theories. An essential feature of this equivalence is that it respects semantics, in the sense that the algebras for a finitary monad coincide up to equivalence over **Set** with the models of the associated theory, and vice versa.

There have been a host of generalisations of the above story, each dealing with algebraic structure borne by objects more general than sets. In many of these [32,31,22,23], one starts on one side with the monads on a given category that preserve a specified class of colimits. This class specifies, albeit indirectly, the arities of operations that may arise in the algebraic structures encoded by such monads, and from this one may define, on the other side, corresponding notions of theory and model. These are subtler than in the classical setting, but once the correct definitions have been found, the equivalence with the given class of monads, and the compatibility with semantics, follows much as before.

The most general framework for a monad–theory correspondence to date involves the notions of *monad with arities* and *theory with arities*. In this setting, the permissible arities of operations are part of the basic data, given as a small, dense, full subcategory of the base category. The monads with arities were introduced first, in [35], as a setting for an abstract nerve theorem. Particular cases of this theorem include the classical nerve theorem, identifying categories with simplicial sets satisfying the Segal condition of [33], and also Berger's nerve theorem [8] for the globular higher categories of [7]. More saliently, when Weber's nerve theorem is specialised to the settings appropriate to the monad–theory correspondences listed above, it becomes exactly the fact that the functor sending the algebras for a monad to the models of the associated theory is an equivalence. This observation led [29] and [9] to introduce *theories with arities*, and to prove, by using Weber's nerve theorem, their equivalence with the monads with arities. The monad–theory correspondence obtained in this way is general enough to encompass all of the instances from [32,31,22,23].

Our own work in this paper has two motivations: one abstract and one concrete. Our abstract motivation is a desire to explain the apparently *ad hoc* design choices involved in the monad–theory correspondences outlined above. For indeed, while these choices must be carefully balanced in order to obtain an equivalence, there is no reason to believe that different careful choices might not yield more general or more expressive results.

Our concrete motivation comes from the study of the Grothendieck weak ω-groupoids introduced by Maltsiniotis [27], which, by definition, are models of a globular theory in the sense of Berger [8]. Globular theories describe algebraic structure on globular sets with arities drawn from the dense subcategory of *globular cardinals*; see Example 8(v) below. However, globular theories are not necessarily theories with arities, and in particular, those capturing higher groupoidal structures are not. As such, they do not appear to one side of any of the monad–theory correspondences described above.

The first goal of this paper is to describe a new schema for monad–theory correspondences which addresses the gaps in our understanding noted above. In this schema, once we have fixed the process by which a theory is associated to a monad, everything else is forced. This addresses our first, abstract motivation. The correspondence obtained in this way is in fact *best possible*, in the sense that any other monad–theory correspondence for the same kind of algebraic structure must be a restriction of this particular one. In many cases, this best possible correspondence coincides with one in the literature, but in others, our correspondence goes beyond what already exists. In particular, an instance of our schema will identify the globular theories of [8] with a suitable class of monads on the category of globular sets. This addresses our second, concrete motivation.

The further goal of this paper is to study the classes of monads and theories that arise from our correspondence-schema. We do so both at a general level, where we will see that both the monads and the theories are closed under essentially all the constructions one could hope for; and also at a practical level, where we will see how these general constructions allow us to give expressive and intuitive *presentations* for the structure captured by a monad or theory.

To give a fuller account of our results, we must first describe how a typical monad–theory correspondence arises. As in [35], the basic setting for such a correspondence can be encapsulated by a pair consisting of a category $\mathcal{E}$ and a small, full, dense subcategory $K\colon \mathcal{A} \hookrightarrow \mathcal{E}$. For example, the Lawvere theory–finitary monad correspondence for finitary algebraic structure on sets is associated to the choice of $\mathcal{E} = \mathbf{Set}$ and $\mathcal{A} = \mathbb{F}$ the full subcategory of finite cardinals.

Given $\mathcal{E}$ and $K\colon \mathcal{A} \hookrightarrow \mathcal{E}$, the goal is to establish an equivalence between a suitable category of $\mathcal{A}$-*monads* and a suitable category of $\mathcal{A}$-*theories*. The $\mathcal{A}$-monads will be a certain class of monads on $\mathcal{E}$; while the $\mathcal{A}$-theories will be a certain class of identity-on-objects functors out of $\mathcal{A}$. We are being deliberately vague about the conditions on each side, as they are among the seemingly *ad hoc* design choices we spoke of earlier. But regardless of this, the monad–theory correspondence itself always arises through application of the following two constructions.

Construction A. For an $\mathcal{A}$-monad T on $\mathcal{E}$, the associated $\mathcal{A}$-theory $\Phi(\mathsf{T})$ is the identity-on-objects functor $J_\mathsf{T}\colon \mathcal{A} \to \mathcal{A}_\mathsf{T}$ arising from the (identity-on-objects, fully faithful) factorisation

$$\mathcal{A} \xrightarrow{J_\mathsf{T}} \mathcal{A}_\mathsf{T} \xrightarrow{V_\mathsf{T}} \mathcal{E}_\mathsf{T} \tag{1.1}$$

of the composite $F_\mathsf{T}K\colon \mathcal{A} \to \mathcal{E} \to \mathcal{E}_\mathsf{T}$. Here F_T is the free functor into the Kleisli category $\mathcal{E}_\mathsf{T}$, so $\mathcal{A}_\mathsf{T}$ is equally the full subcategory of $\mathcal{E}_\mathsf{T}$ with objects those of $\mathcal{A}$.

Construction B. For an $\mathcal{A}$-theory $J\colon \mathcal{A} \to \mathcal{T}$, the associated $\mathcal{A}$-monad $\Psi(\mathcal{T})$ is obtained from the category of *concrete $\mathcal{T}$-models*, which is by definition the pullback

$$\begin{tikzcd}
\mathbf{Mod}_c(\mathcal{T}) \arrow[r] \arrow[d, "U^{\mathcal{T}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{T}^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] \\
\mathcal{E} \arrow[r, "N_K = \mathcal{E}(K-,1)"] & {[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]}\ .
\end{tikzcd} \tag{1.2}$$

Since $U^{\mathcal{T}}$ is a pullback of the strictly monadic $[J^{\mathrm{op}}, 1]$, it will be strictly monadic so long as it has a left adjoint. The assumption that $\mathcal{E}$ is locally presentable ensures that this is the case, and so we can take $\Psi(\mathcal{T})$ to be the monad whose algebras are the concrete $\mathcal{T}$-models.

There remains the problem of choosing the appropriate conditions on a monad or theory for it to be an $\mathcal{A}$-monad or $\mathcal{A}$-theory. Of course, these must be carefully balanced so as to obtain an equivalence, but this still seems to leave too many degrees of freedom; one might hope that everything could be determined from $\mathcal{E}$ and $\mathcal{A}$ alone. The main result of this paper shows that this is so: there are notions of $\mathcal{A}$-monad and $\mathcal{A}$-theory which require no further choices to be made, and which rather than being plucked from the air, may be derived in a principled manner.

The key observation is that Constructions A and B make sense when given as input *any* monad on $\mathcal{E}$, or *any* "$\mathcal{A}$-pretheory"—by which we mean simply an identity-on-objects functor out of $\mathcal{A}$. When viewed in this greater generality, these constructions yield an adjunction

$$\begin{tikzcd}
\mathbf{Mnd}(\mathcal{E}) \arrow[r, bend right, "\Phi"'] \arrow[r, phantom, "\perp"] & \mathbf{Preth}_{\mathcal{A}}(\mathcal{E}) \arrow[l, bend right, "\Psi"']
\end{tikzcd} \tag{1.3}$$

between the category of monads on $\mathcal{E}$ and the category of $\mathcal{A}$-pretheories. Like any adjunction, this restricts to an equivalence between the objects at which the counit is invertible, and the objects at which the unit is invertible. Thus, if we *define* the $\mathcal{A}$-monads and $\mathcal{A}$-theories to be the objects so arising, then we obtain a monad–theory equivalence. By construction, it will be the *largest* possible equivalence whose two directions are given by Constructions A and B.

Having defined the $\mathcal{A}$-monads and $\mathcal{A}$-theories abstractly, it behoves us to give tractable concrete characterisations. In fact, we give a number of these, allowing us to relate our correspondence to existing ones in the literature. We also investigate further aspects of the general theory, and provide a wide range of examples illustrating the practical utility of our results.

Before getting started, we conclude this introduction with a more detailed outline of the paper's contents. In Section 2, we begin by introducing our basic setting and notions. We then construct, in Theorem 6, the adjunction (1.3) between monads and pretheories. In Section 3, with this abstract result in place, we introduce a host of running examples of our basic setting. To convince the reader of the expressive power of our notions, we construct, via colimit presentations, specific pretheories for a variety of mathematical structures.

In Section 4 we obtain our main result by characterising the fixpoints of the monad–theory adjunction: the $\mathcal{A}$-monads and $\mathcal{A}$-theories described above. The $\mathcal{A}$-monads are characterised as what we term the *$\mathcal{A}$-nervous monads*, since they are precisely those monads for which Weber's nerve theorem holds. The $\mathcal{A}$-theories turn out to be precisely those $\mathcal{A}$-pretheories for which each representable is a model; in the motivating case where $\mathcal{E} = \mathbf{Set}$ and $\mathcal{A} = \mathbb{F}$, they are exactly the Lawvere theories. With these characterisations in place, we obtain our main Theorem 19, which describes the "best possible" equivalence between $\mathcal{A}$-theories and $\mathcal{A}$-nervous monads.

Section 5 develops some of the general results associated to our correspondence-schema. We begin by showing that our monad–theory correspondence commutes, to within isomorphism, with the taking of semantics on each side. We also prove that the functors taking semantics are valued in monadic right adjoint functors between locally presentable categories. The final important result of this section states that colimits of $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories are algebraic, meaning that the semantics functors send them to limits.

Section 6 is devoted to exploring what the $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories amount to in our running examples. In order to understand the $\mathcal{A}$-nervous monads, we prove the important result that they are equally the colimits, amongst all monads, of free monads on $\mathcal{A}$-signatures. We also introduce the notion of a *saturated class of arities* as a setting in which, like in [32,31,22,23], the $\mathcal{A}$-nervous monads can be characterised in terms of a colimit-preservation property. With these results in place, we are able to exhibit many of these existing monad–theory correspondences as instances of our general framework.

In Section 7, we examine the relationship between the monads and theories of our correspondence, and the *monads with arities* and *theories with arities* of [35,29,9]. In particular, we see that every monad with arities $\mathcal{A}$ is an $\mathcal{A}$-nervous monad but that the converse implication need not be true: so $\mathcal{A}$-nervous monads are *strictly* more general. Of course, the same is also true on the theory side. We also exhibit a further important point of difference: colimits of monads with arities, unlike those of nervous monads, need not be algebraic. This means that there is no good notion of *presentation* for monads or theories with arities.

Finally, in Section 8, we give a number of proofs deferred from Section 6.

2. Monads and pretheories

2.1. The setting

In this section we construct the monad–pretheory adjunction

$$\mathbf{Mnd}(\mathcal{E}) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathbf{Preth}_{\mathcal{A}}(\mathcal{E}) \; . \tag{2.1}$$

The setting for this, and the rest of the paper, involves two basic pieces of data:

(i) A locally presentable $\mathcal{V}$-category $\mathcal{E}$ with respect to which we will describe the monad–pretheory adjunction; and
(ii) A notion of *arities* given by a small, full, dense sub-$\mathcal{V}$-category $K\colon \mathcal{A} \hookrightarrow \mathcal{E}$.

We will discuss examples in Section 2.1 below, but for now let us clarify some of the terms appearing above. While in the introduction, we focused on the unenriched context, we now work in the context of category theory enriched over a symmetric monoidal closed category $\mathcal{V}$ which is *locally presentable* as in [13]. In this context, a *locally presentable $\mathcal{V}$-category* [18] is one which is cocomplete as a $\mathcal{V}$-category, and whose underlying ordinary category is locally presentable.

We recall also some notions pertaining to density. Given a $\mathcal{V}$-functor $K\colon \mathcal{A} \to \mathcal{E}$ with small domain, the *nerve functor* $N_K\colon \mathcal{E} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is defined by $N_K(X) = \mathcal{E}(K\text{–}, X)$. We call a presheaf in the essential image of N_K a *K-nerve*, and we write $K\text{-}\mathbf{Ner}(\mathcal{V})$ for the full sub-$\mathcal{V}$-category of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ determined by these.

We say that K is *dense* if N_K is fully faithful; whereupon N_K induces an equivalence of categories $\mathcal{E} \simeq K\text{-}\mathbf{Ner}(\mathcal{V})$. Finally, we call a small sub-$\mathcal{V}$-category $\mathcal{A}$ of a $\mathcal{V}$-category $\mathcal{E}$ *dense* if its inclusion functor $K\colon \mathcal{A} \hookrightarrow \mathcal{E}$ is so.

2.2. Monads

We write $\mathbf{Mnd}(\mathcal{E})$ for the (ordinary) category whose objects are $\mathcal{V}$-monads on $\mathcal{E}$, and whose maps $\mathsf{S} \to \mathsf{T}$ are $\mathcal{V}$-transformations $\alpha\colon S \Rightarrow T$ compatible with unit and multiplication. For each $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$ we have the $\mathcal{V}$-category of algebras $U^{\mathsf{T}}\colon \mathcal{E}^{\mathsf{T}} \to \mathcal{E}$ over $\mathcal{E}$, but also the *Kleisli $\mathcal{V}$-category* $F_{\mathsf{T}}\colon \mathcal{E} \to \mathcal{E}_{\mathsf{T}}$ under $\mathcal{E}$, arising from an (identity-on-objects, fully faithful) factorisation

$$\begin{tikzcd} & \mathcal{E} \arrow[dl, "F_{\mathsf{T}}"'] \arrow[dr, "F^{\mathsf{T}}"] & \\ \mathcal{E}_{\mathsf{T}} \arrow[rr, "W_{\mathsf{T}}"] & & \mathcal{E}^{\mathsf{T}} \end{tikzcd}$$

of the free $\mathcal{V}$-functor $F^{\mathsf{T}}\colon \mathcal{E} \to \mathcal{E}^{\mathsf{T}}$; concretely, we may take $\mathcal{E}_{\mathsf{T}}$ to have objects those of $\mathcal{E}$, hom-objects $\mathcal{E}_{\mathsf{T}}(A, B) = \mathcal{E}(A, TB)$, and composition and identities derived using the monad structure of T. Each monad morphism $\alpha\colon \mathsf{S} \to \mathsf{T}$ induces, functorially in α, $\mathcal{V}$-functors α^* and $\alpha_!$ fitting into diagrams

$$\begin{tikzcd} \mathcal{E}^{\mathsf{T}} \arrow[rr, "\alpha^*"] \arrow[dr, "U^{\mathsf{T}}"'] & & \mathcal{E}^{\mathsf{S}} \arrow[dl, "U^{\mathsf{S}}"] \\ & \mathcal{E} & \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{E} \arrow[dl, "F_{\mathsf{S}}"'] \arrow[dr, "F_{\mathsf{T}}"] & \\ \mathcal{E}_{\mathsf{S}} \arrow[rr, "\alpha_!"] & & \mathcal{E}_{\mathsf{T}} \end{tikzcd} ;$$

here α^* sends an algebra $a\colon TA \to A$ to $a \circ \alpha_A\colon SA \to A$ and is the identity on homs, while $\alpha_!$ is the identity on objects and has action on homs given by the postcomposition maps $\alpha_B \circ (-)\colon \mathcal{E}_{\mathsf{S}}(A,B) \to \mathcal{E}_{\mathsf{T}}(A,B)$. In fact, every $\mathcal{V}$-functor $\mathcal{E}^{\mathsf{T}} \to \mathcal{E}^{\mathsf{S}}$ over $\mathcal{E}$ or $\mathcal{V}$-functor $\mathcal{E}_{\mathsf{S}} \to \mathcal{E}_{\mathsf{T}}$ under $\mathcal{E}$ is of the form α^* or $\alpha_!$ for a unique map of monads α—see, for example, [30]—and in this way, we obtain *fully faithful* functors

$$\mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\mathrm{Alg}} \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E} \qquad \text{and} \qquad \mathbf{Mnd}(\mathcal{E}) \xrightarrow{\mathrm{Kl}} \mathcal{E}/\mathcal{V}\text{-}\mathbf{CAT}\ . \tag{2.2}$$

2.3. Pretheories

An $\mathcal{A}$-*pretheory* is an identity-on-objects $\mathcal{V}$-functor $J\colon \mathcal{A} \to \mathcal{T}$ with domain $\mathcal{A}$. We write $\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$ for the ordinary category whose objects are $\mathcal{A}$-pretheories and whose morphisms are $\mathcal{V}$-functors commuting with the maps from $\mathcal{A}$. While the $\mathcal{A}$-pretheory is only fully specified by both pieces of data $\mathcal{T}$ and J, we will often, by abuse of notation, leave J implicit and refer to such a pretheory simply as $\mathcal{T}$.

Just as any $\mathcal{V}$-monad has a $\mathcal{V}$-category of algebras, so any $\mathcal{A}$-pretheory has a $\mathcal{V}$-category of models. Generalising (1.2), we define the $\mathcal{V}$-category of *concrete* $\mathcal{T}$*-models* $\mathbf{Mod}_c(\mathcal{T})$ by a pullback of $\mathcal{V}$-categories as below left; so a concrete $\mathcal{T}$-model is an object $X \in \mathcal{E}$ together with a chosen extension of $\mathcal{E}(K-, X)\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ along $J^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{T}^{\mathrm{op}}$. The reason for the qualifier “concrete” will be made clear in Section 5.2 below, where we will identify a more general notion of model.

$$\begin{tikzcd} \mathbf{Mod}_c(\mathcal{T}) \arrow[r, "P_{\mathcal{T}}"] \arrow[d, "U_{\mathcal{T}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] \\ \mathcal{E} \arrow[r, "N_K"] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Mod}_c(\mathcal{S}) \arrow[r, "P_{\mathcal{S}}"] \arrow[d, "U_{\mathcal{S}}"'] & {[\mathcal{S}^{\mathrm{op}}, \mathcal{V}]} \arrow[r, "{[H^{\mathrm{op}},1]}"] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] \\ \mathcal{E} \arrow[rr, "N_K"] & & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} \end{tikzcd} \tag{2.3}$$

Remark 1. Avery considers a notion very similar to our $\mathcal{A}$-pretheories under the name *prototheories* [4, Definition 4.1.1]. The differences are that Avery's prototheories $\mathcal{A} \to \mathcal{T}$ are not enriched, and the hom-sets of $\mathcal{T}$ need not be small. He also defines a category of (concrete) models for a prototheory, relative to a given functor $\mathcal{E} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{C}]$ called an *aritation*. When this functor is the nerve $N_K\colon \mathcal{E} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, his category of models agrees with our $\mathbf{Mod}_c(\mathcal{T})$.

Any $\mathcal{A}$-pretheory map $H\colon \mathcal{T} \to \mathcal{S}$ gives a functor $H^*\colon \mathbf{Mod}_c(\mathcal{S}) \to \mathbf{Mod}_c(\mathcal{T})$ over $\mathcal{E}$ by applying the universal property of the pullback on the left of (2.3) to the commuting square on the right. In this way, we obtain a semantics functor:

$$\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\mathrm{Mod}_c} \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}\ . \tag{2.4}$$

However, unlike (2.2), this is *not* always fully faithful. Indeed, in Example 10 below, we will see that non-isomorphic pretheories can have isomorphic categories of concrete models over $\mathcal{E}$.

2.4. Monads to pretheories

We now define the functor $\Phi\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$ in (2.1). As in Construction A of the introduction, this will take the $\mathcal{V}$-monad T to the $\mathcal{A}$-pretheory $J_{\mathsf{T}}\colon \mathcal{A} \to \mathcal{A}_{\mathsf{T}}$ arising as the first part of an (identity-on-objects, fully faithful) factorisation of $F_{\mathsf{T}}K\colon \mathcal{A} \to \mathcal{E}_{\mathsf{T}}$, as to the left in:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "J_{\mathsf{T}}"] \arrow[d, "K"'] & \mathcal{A}_{\mathsf{T}} \arrow[d, "V_{\mathsf{T}}"] \\ \mathcal{E} \arrow[r, "F_{\mathsf{T}}"] & \mathcal{E}_{\mathsf{T}} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, "J_{\mathsf{T}}"] \arrow[d, "K"'] & \mathcal{A}_{\mathsf{T}} \arrow[d, "K_{\mathsf{T}}"] \\ \mathcal{E} \arrow[r, "F^{\mathsf{T}}"] & \mathcal{E}^{\mathsf{T}} \end{tikzcd}\ . \tag{2.5}$$

Since the comparison $W_{\mathsf{T}}\colon \mathcal{E}_{\mathsf{T}} \to \mathcal{E}^{\mathsf{T}}$ is fully faithful, we can also view J_{T} as arising from an (identity-on-objects, fully faithful) factorisation as above right; the relationship between the two is that $K_{\mathsf{T}} = W_{\mathsf{T}} \circ V_{\mathsf{T}}$. Both perspectives will be used in what follows, with the functor $K_{\mathsf{T}}\colon \mathcal{A}_{\mathsf{T}} \to \mathcal{E}^{\mathsf{T}}$ of particular importance.

To define Φ on morphisms, we make use of the *orthogonality* of identity-on-objects $\mathcal{V}$-functors to fully faithful ones; this asserts that any commuting square of $\mathcal{V}$-functors as below, with F identity-on-objects and G fully faithful, admits a unique diagonal filler J making both triangles commute.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "H"] \arrow[d, "F"'] & \mathcal{C} \arrow[d, "G"] \\ \mathcal{B} \arrow[r, "K"'] \arrow[ur, dashed, "J"] & \mathcal{D} \end{tikzcd}$$

Explicitly, J is given on objects by $Ja = Ha$, and on homs by

$$\mathcal{B}(a,b) \xrightarrow{K_{a,b}} \mathcal{D}(Ka,Kb) = \mathcal{D}(GHa,GHb) \xrightarrow{(G_{Ha,Hb})^{-1}} \mathcal{C}(Ha,Hb)\ .$$

In particular, given a map $\alpha\colon \mathsf{S} \to \mathsf{T}$ of $\mathbf{Mnd}(\mathcal{E})$, this orthogonality guarantees the existence of a diagonal filler in the diagram below, whose upper triangle we take to be the map $\Phi(\alpha)\colon \Phi(\mathsf{S}) \to \Phi(\mathsf{T})$ in $\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "J_{\mathsf{T}}"] \arrow[d, "J_{\mathsf{S}}"'] & & \mathcal{A}_{\mathsf{T}} \arrow[d, "V_{\mathsf{T}}"] \\ \mathcal{A}_{\mathsf{S}} \arrow[r, "V_{\mathsf{S}}"'] \arrow[urr, dashed] & \mathcal{E}_{\mathsf{S}} \arrow[r, "\alpha_!"'] & \mathcal{E}_{\mathsf{T}} \end{tikzcd}\ .$$

2.5. Pretheories to monads

Thus far we have not exploited the local presentability of $\mathcal{E}$. It *will* be used in the next step, that of constructing the left adjoint to $\Phi\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$. We first state a general result which, independent of local presentability, gives a sufficient condition for an individual pretheory to have a reflection along Φ. Here, by a *reflection* of an object $c \in \mathcal{C}$ along a functor $U\colon \mathcal{B} \to \mathcal{C}$, we mean a representation for the functor $\mathcal{C}(c, U–)\colon \mathcal{B} \to \mathbf{Set}$.

Theorem 2. *A pretheory $J\colon \mathcal{A} \to \mathcal{T}$ admits a reflection along Φ whenever the forgetful functor $U_{\mathcal{T}}\colon \mathbf{Mod}_c(\mathcal{T}) \to \mathcal{E}$ from the category of concrete models has a left adjoint $F_{\mathcal{T}}$. In this case, the reflection $\Psi\mathcal{T}$ is characterised by an isomorphism $\mathcal{E}^{\Psi\mathcal{T}} \cong \mathbf{Mod}_c(\mathcal{T})$ over $\mathcal{E}$, or equally, by a pullback square*

$$\begin{tikzcd} \mathcal{E}^{\Psi\mathcal{T}} \arrow[r] \arrow[d, "U^{\Psi\mathcal{T}}"'] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] \\ \mathcal{E} \arrow[r, "N_K"] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} \end{tikzcd} \quad . \tag{2.6}$$

To prove this result, we will need a preparatory lemma, relating to the notion of *discrete isofibration*: this is a $\mathcal{V}$-functor $U\colon \mathcal{D} \to \mathcal{C}$ such that, for each $f\colon c \cong Ud$ in $\mathcal{C}$, there is a unique $f'\colon c' \cong d$ in $\mathcal{D}$ with $U(f') = f$.

Example 3. For any $\mathcal{V}$-monad T on $\mathcal{C}$, the forgetful $\mathcal{V}$-functor $U^{\mathsf{T}}\colon \mathcal{C}^{\mathsf{T}} \to \mathcal{C}$ is a discrete isofibration. Indeed, if $x\colon Ta \to a$ is a T-algebra and $f\colon b \cong a$ in $\mathcal{C}$, then the composite $y = f^{-1} \circ x \circ Tf\colon Tb \to b$ is the unique algebra structure on b for which $f\colon (b, y) \to (a, x)$ belongs to $\mathcal{C}^{\mathsf{T}}$. In particular, for any identity-on-objects $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ between small $\mathcal{V}$-categories, the functor $[F, 1]\colon [\mathcal{B}, \mathcal{V}] \to [\mathcal{A}, \mathcal{V}]$ has a left adjoint and strictly creates colimits, whence is strictly monadic. It is therefore a discrete isofibration by the above argument.

Lemma 4. *Let $U\colon \mathcal{A} \to \mathcal{B}$ be a discrete isofibration and $\alpha\colon F \Rightarrow G\colon \mathcal{X} \to \mathcal{B}$ an invertible $\mathcal{V}$-transformation. The displayed projections give isomorphisms between liftings of F through U, liftings of α through U, and liftings of G through U:*

$$\begin{tikzcd} & \mathcal{A} \arrow[d, "U"] \\ \mathcal{X} \arrow[ur, dashed, "\bar{F}"] \arrow[r, "F"'] & \mathcal{B} \end{tikzcd} \xleftarrow{\ \mathrm{dom}\ } \begin{tikzcd} & \mathcal{A} \arrow[d, "U"] \\ \mathcal{X} \arrow[ur, dashed, bend left, "\bar{F}"] \arrow[ur, dashed, bend right, "\bar{G}"'] \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[ur, phantom, "\Downarrow\bar{\alpha}"] & \mathcal{B} \end{tikzcd} \xrightarrow{\ \mathrm{cod}\ } \begin{tikzcd} & \mathcal{A} \arrow[d, "U"] \\ \mathcal{X} \arrow[ur, dashed, "\bar{G}"] \arrow[r, "G"'] & \mathcal{B} \end{tikzcd} .$$

Proof. Given $\bar{G}\colon \mathcal{X} \to \mathcal{A}$ as to the right, there is for each $x \in \mathcal{X}$ a unique lifting of the isomorphism $\alpha_x\colon Fx \cong U\bar{G}x$ to one $\bar{\alpha}_x\colon \bar{F}x \cong \bar{G}x$. There is now a unique way of extending $x \mapsto \bar{F}x$ to a $\mathcal{V}$-functor $\bar{F} \colon \mathcal{X} \to \mathcal{A}$ so that $\bar{\alpha} \colon \bar{F} \cong \bar{G}$; namely, by taking the action on homs to be $\bar{F}_{x,y} = \mathcal{A}(\bar{\alpha}_x, \bar{\alpha}_y{}^{-1}) \circ \bar{G}_{x,y} \colon X(x,y) \to \mathcal{A}(\bar{F}x, \bar{F}y)$. In this way, we have found a unique lifting of α through U whose codomain is the given lifting of G through U. So the right-hand projection is invertible; the argument for the left-hand one is the same on replacing α by α^{-1}. □

We can now give:

Proof of Theorem 2. $U_{\mathcal{T}}$ has a left adjoint by assumption, and—as a pullback of the strictly monadic $[J^{\mathrm{op}}, 1] \colon [\mathcal{T}^{\mathrm{op}}, \mathcal{V}] \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$—strictly creates coequalisers for $U_{\mathcal{T}}$-absolute pairs. It is therefore strictly monadic. Taking $\Psi\mathcal{T} = U_{\mathcal{T}} F_{\mathcal{T}}$ to be the induced monad, we thus have an isomorphism $\mathcal{E}^{\Psi\mathcal{T}} \cong \mathbf{Mod}_c(\mathcal{T})$ over $\mathcal{E}$.

It remains to exhibit isomorphisms $\mathbf{Mnd}(\mathcal{E})(\Psi\mathcal{T}, \mathsf{S}) \cong \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})(\mathcal{T}, \Phi\mathsf{S})$ natural in S. We do so by chaining together the following sequence of natural bijections. Firstly, by full fidelity in (2.2), monad maps $\alpha_0 \colon \Psi\mathcal{T} \to \mathsf{S}$ correspond naturally to functors $\alpha_1 \colon \mathcal{E}^{\mathsf{S}} \to \mathcal{E}^{\Psi\mathcal{T}}$ rendering commutative the left triangle in

$$
\begin{tikzcd}
\mathcal{E}^{\mathsf{S}} \arrow[rr, "\alpha_1"] \arrow[dr, "U^{\mathsf{S}}"'] & & \mathcal{E}^{\Psi\mathcal{T}} \arrow[dl, "U^{\Psi\mathcal{T}}"] \\
& \mathcal{E} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}^{\mathsf{S}} \arrow[r, "\alpha_2"] \arrow[d, "U^{\mathsf{S}}"'] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] \\
\mathcal{E} \arrow[r, "N_K"] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]}
\end{tikzcd}
\tag{2.7}
$$

Since $\mathcal{E}^{\Psi\mathcal{T}}$ is defined by the pullback (2.6), such functors α_1 correspond naturally to functors α_2 rendering commutative the square above right. Next, we observe that there is a natural isomorphism in the triangle below left

$$
\begin{tikzcd}
\mathcal{E}^{\mathsf{S}} \arrow[rr, "U^{\mathsf{S}}"] \arrow[dr, "N_{F^{\mathsf{S}}K}"'] & \cong & \mathcal{E} \arrow[dl, "N_K"] \\
& {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}^{\mathsf{S}} \arrow[rr, "\alpha_3"] \arrow[dr, "N_{F^{\mathsf{S}}K}"'] & & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[dl, "{[J^{\mathrm{op}},1]}"] \\
& {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} &
\end{tikzcd}
\tag{2.8}
$$

with components the adjointness isomorphisms $\mathcal{E}(Ka, U^{\mathsf{S}}b) \cong \mathcal{E}^{\mathsf{S}}(F^{\mathsf{S}}Ka, b)$. Since J^{op} is identity-on-objects, $[J^{\mathrm{op}}, 1]$ is a discrete isofibration by Example 3, whence by Lemma 4 there is a natural bijection between functors α_2 as in (2.7) and ones α_3 as in (2.8). We should now like to transpose this last triangle through the following natural isomorphisms (taking $\mathcal{X} = \mathcal{A}, \mathcal{T}$):

$$\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{E}^{\mathsf{S}}, [\mathcal{X}^{\mathrm{op}}, \mathcal{V}]) \cong \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{X}^{\mathrm{op}}, [\mathcal{E}^{\mathsf{S}}, \mathcal{V}]) \; . \tag{2.9}$$

However, since $\mathcal{E}^{\mathsf{S}}$ is large, the functor category $[\mathcal{E}^{\mathsf{S}}, \mathcal{V}]$ will not always exist as a $\mathcal{V}$-category, and so (2.9) is ill-defined. To resolve this, note that $N_{F^{\mathsf{S}}K}$ is, by its definition, *pointwise representable*; whence so too is α_3, since J is identity-on-objects. We may thus transpose the right triangle of (2.8) through the legitimate isomorphisms

$$\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{E}^{\mathsf{S}}, [\mathcal{X}^{\mathrm{op}}, \mathcal{V}])_{\mathrm{pwr}} \cong \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{X}^{\mathrm{op}}, [\mathcal{E}^{\mathsf{S}}, \mathcal{V}]_{\mathrm{rep}}) \tag{2.10}$$

where on the left we have the category of pointwise representable $\mathcal{V}$-functors, and on the right, the *legitimate* $\mathcal{V}$-category of representable $\mathcal{V}$-functors $\mathcal{E}^{\mathsf{S}} \to \mathcal{V}$. In this way, we establish a natural bijection between functors α_3 and functors α_4 rendering commutative the left square in:

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{op}} \arrow[r, "J^{\mathrm{op}}"] \arrow[d, "(F^{\mathsf{S}}K)^{\mathrm{op}}"'] & \mathcal{T}^{\mathrm{op}} \arrow[d, "\alpha_4"] \arrow[dl, dotted, "(\alpha_5)^{\mathrm{op}}"'] \\
(\mathcal{E}^{\mathsf{S}})^{\mathrm{op}} \arrow[r, "Y"'] & {[\mathcal{E}^{\mathsf{S}}, \mathcal{V}]_{\mathrm{rep}}}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{A} \arrow[r, "J"] \arrow[d, "J_{\mathsf{S}}"'] & \mathcal{T} \arrow[d, "\alpha_5"] \arrow[dl, dotted, "\alpha_6"'] \\
\mathcal{A}_{\mathsf{S}} \arrow[r, "K_{\mathsf{S}}"'] & \mathcal{E}^{\mathsf{S}} .
\end{tikzcd}$$

Now orthogonality of the identity-on-objects J^{op} and the fully faithful Y draws the correspondence between functors α_4 and functors α_5 satisfying $\alpha_5 \circ J = F^{\mathsf{S}}K$ as left above. Finally, since $\mathcal{A}_{\mathsf{S}}$ fits in to an (identity-on-objects, fully faithful) factorisation of $F^{\mathsf{S}}K$, orthogonality also gives the correspondence, as right above, between functors α_5 and functors α_6 satisfying $\alpha_6 \circ J = J_{\mathsf{S}}$, as required. $\square$

We now show that the assumed local presentability of $\mathcal{E}$ ensures that every pretheory has a reflection along $\Phi\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$, which consequently has a left adjoint. The key result about locally presentable categories enabling this is the following lemma.

Lemma 5. *Consider a pullback square of $\mathcal{V}$-categories*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "U"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d, "V"] \\
\mathcal{C} \arrow[r, "G"] & \mathcal{D}
\end{tikzcd} \tag{2.11}$$

in which G and V are right adjoints between locally presentable $\mathcal{V}$-categories and V is strictly monadic. Then U and F are right adjoints between locally presentable $\mathcal{V}$-categories and U is strictly monadic.

Proof. Since V is strictly monadic, it is a discrete isofibration, and so its pullback against G is, by [14, Corollary 1], also a bipullback. By [10, Theorem 6.11] the 2-category of locally presentable $\mathcal{V}$-categories and right adjoint functors is closed under bilimits in $\mathcal{V}$-$\mathbf{CAT}$, so that both U and F are right adjoints between locally presentable categories. Finally, since U is a pullback of the strictly monadic V, it strictly creates coequalisers for U-absolute pairs. Since it is already known to be a right adjoint, it is therefore also strictly monadic. $\square$

With this in place, we can now prove:

Theorem 6. *Let $\mathcal{E}$ be locally presentable. Then $\Phi\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$ has a left adjoint $\Psi\colon \mathbf{Preth}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Mnd}(\mathcal{E})$, whose value at the pretheory $J\colon \mathcal{A} \to \mathcal{T}$ is characterised by an isomorphism $\mathcal{E}^{\Psi(\mathcal{T})} \cong \mathbf{Mod}_c(\mathcal{T})$ over $\mathcal{E}$, or equally, by a pullback square*

$$\begin{tikzcd} \mathcal{E}^{\Psi(\mathcal{T})} \arrow[r] \arrow[d, "U^{\Psi(\mathcal{T})}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] \\ \mathcal{E} \arrow[r, "N_K"] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]}\ . \end{tikzcd} \tag{2.12}$$

Proof. Let $J\colon \mathcal{A} \to \mathcal{T}$ be a pretheory. The pullback square (2.3) defining $\mathbf{Mod}_c(\mathcal{T})$ is a pullback of a right adjoint functor between locally presentable categories along a strictly monadic one: so it follows from Lemma 5 that $U_{\mathcal{T}}\colon \mathbf{Mod}_c(\mathcal{T}) \to \mathcal{E}$ is a right adjoint, whence the result follows from Theorem 2. $\square$

Remark 7. In Avery's study of prototheories, he establishes a *structure–semantics* adjunction [4, Theorem 4.4.8] of the form $\mathbf{Proto}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \leftrightarrows \mathbf{CAT}/\mathcal{E}$, where here $\mathbf{CAT}$ is the category of *large* categories. By restricting to the locally small prototheories to the left and to the strictly monadic functors to the right of this adjunction, one can recover, via (2.2), the unenriched case of our adjunction (2.1).

3. Pretheories as presentations

In the next section, we will describe how the monad–pretheory adjunction (2.1) restricts to an equivalence between suitable subcategories of $\mathcal{A}$-*theories* and of $\mathcal{A}$-*nervous monads*. However, the results we have so far are already practically useful. The notion of $\mathcal{A}$-pretheory provides a tool for presenting certain kinds of algebraic structure, by exhibiting them as categories of concrete $\mathcal{T}$-models for a suitable pretheory in a manner reminiscent of the theory of *sketches* [6]. Equivalently, via the functor Ψ, we can see $\mathcal{A}$-pretheories as a way of presenting certain monads on $\mathcal{E}$.

3.1. Examples of the basic setting

Before giving examples of algebraic structures presented by pretheories, we first describe a range of examples of the basic setting of Section 2.1 above.

Examples 8. We begin by considering the unenriched case where $\mathcal{V} = \mathbf{Set}$.

(i) Taking $\mathcal{E} = \mathbf{Set}$ and $\mathcal{A} = \mathbb{F}$ the full subcategory of finite cardinals captures the classical case of *finitary* algebraic structure borne by *sets*; so examples like groups, rings, lattices, Lie algebras, and so on.

(ii) Taking $\mathcal{E}$ a locally finitely presentable category and $\mathcal{A} = \mathcal{E}_f$ a skeleton of the full subcategory of finitely presentable objects, we capture *finitary* algebraic structure borne by *$\mathcal{E}$-objects*. Examples when $\mathcal{E} = \mathbf{Cat}$ include *finite product*, *finite colimit*, and *monoidal closed* structure; for $\mathcal{E} = \mathbf{CRng}$, we have *commutative k-algebra*, *differential ring* and *reduced ring* structure.

(iii) We can replace "finitary" above by "λ-ary" for any regular cardinal λ. For example, when $\lambda = \aleph_1$, this allows for the structure of poset with joins of ω-chains [28] when $\mathcal{E} = \mathbf{Set}$, and for *countable product* structure when $\mathcal{E} = \mathbf{Cat}$. When $\mathcal{E} = [\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ for some space X, and λ is suitably chosen, it also permits *sheaf* or *sheaf of rings* structure.

(iv) Let $\mathbb{G}_1$ be the category freely generated by the graph $0 \rightrightarrows 1$, so that $\mathcal{E} = [\mathbb{G}_1^{\mathrm{op}}, \mathbf{Set}]$ is the category of directed multigraphs, and let $\mathcal{A} = \Delta_0$ be the full subcategory of $[\mathbb{G}_1^{\mathrm{op}}, \mathbf{Set}]$ on graphs of the form

$$[n] := 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n \qquad\qquad \text{for } n > 0.$$

Δ_0 is dense in $[\mathbb{G}_1^{\mathrm{op}}, \mathbf{Set}]$ because it contains the representables $[0]$ and $[1]$. This example captures structure borne by graphs in which the operations build vertices and arrows from *paths* of arrows: for example, the structures of *categories*, *involutive categories*, and *groupoids*.

(v) The *globe category* $\mathbb{G}$ is freely generated by the graph

$$\begin{tikzcd} 0 \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & 1 \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & 2 \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & \cdots \end{tikzcd}$$

subject to the coglobular relations $\sigma\sigma = \sigma\tau$ and $\tau\sigma = \tau\tau$. This means that for each $m > n$, there are precisely two maps $\sigma^{m-n}, \tau^{m-n} \colon n \rightrightarrows m$, which by abuse of notation we will write simply as σ and τ.

The category $\mathcal{E} = [\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ is the category of *globular sets*; it has a dense subcategory $\mathcal{A} = \Theta_0$, first described by Berger [8], whose objects have been termed *globular cardinals* by Street [34]. The globular cardinals include the representables—the n-globes Yn for each n—but also shapes such as the globular set with distinct cells as depicted below.

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r, bend left=50, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=50, ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=M] \arrow[Rightarrow, from=M, to=D] \end{tikzcd} \tag{3.1}$$

The globular cardinals can be parametrised in various ways, for instance using trees [7,8]; following [27], we will use *tables of dimensions*—sequences $\vec{n} = (n_1, \ldots, n_k)$

of natural numbers of odd length with $n_{2i-1} > n_{2i} < n_{2i+1}$. Given such a table $\vec{n}$ and a functor $D\colon \mathbb{G} \to \mathcal{C}$, we obtain a diagram

$$\begin{tikzcd}[column sep=small] & Dn_2 \arrow[dl, "D\tau"'] \arrow[dr, "D\sigma"] & & Dn_4 \arrow[dl, "D\tau"'] \arrow[dr, "D\sigma"] & & \cdots & & Dn_{k-1} \arrow[dl, "D\tau"'] \arrow[dr, "D\sigma"] & \\ Dn_1 & & Dn_3 & & Dn_5 & & Dn_{k-2} & & Dn_k \end{tikzcd}$$

whose colimit in $\mathcal{C}$, when it exists, will be written as $D(\vec{n})$, and called the *D-globular sum indexed by $\vec{n}$*. Taking $D = Y\colon \mathbb{G} \to [\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$, the category Θ_0 of globular cardinals is now defined as the full subcategory of $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ spanned by the Y-globular sums. For example, the globular cardinal in (3.1) corresponds to the Y-globular sum $Y(1, 0, 2, 1, 2)$.

This example captures algebraic structures on globular sets in which the operations build globes out of diagrams with shapes like (3.1); these include *strict ω-categories* and *strict ω-groupoids*, but also the (globular) *weak ω-categories* and *weak ω-groupoids* studied in [7,25,3].

We now turn to examples over enriched bases.

(vi) Let $\mathcal{V}$ be a locally finitely presentable symmetric monoidal category whose finitely presentable objects are closed under the tensor product (cf. [18]). By taking $\mathcal{E} = \mathcal{V}$ and $\mathcal{A} = \mathcal{V}_f$ a skeleton of the full sub-$\mathcal{V}$-category of finitely presentable objects, we capture *$\mathcal{V}$-enriched* finitary algebraic structure on $\mathcal{V}$-objects as studied in [32]. When $\mathcal{V} = \mathbf{Cat}$ this means structure on categories $\mathcal{C}$ built from functors and natural transformations $\mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ for finitely presentable $\mathcal{I}$: which includes *symmetric monoidal* or *finite limit* structure, but not *symmetric monoidal closed* or *factorisation system* structure. Similarly, when $\mathcal{V} = \mathbf{Ab}$, it includes *$A$-module* structure but not *commutative ring* structure.

(vii) Taking $\mathcal{V}$ as before, taking $\mathcal{E}$ to be any locally finitely presentable $\mathcal{V}$-category [18] and taking $\mathcal{A} = \mathcal{E}_f$ a skeleton of the full subcategory of finitely presentable objects in $\mathcal{E}$, we capture $\mathcal{V}$-enriched finitary algebraic structure on $\mathcal{E}$-objects as studied in [31]. As before, there is the obvious generalisation from finitary to λ-ary structure.

(viii) This example builds on [23]. Let $\mathcal{V}$ be a locally presentable symmetric monoidal closed category, and consider a class of $\mathcal{V}$-enriched limit-types Φ with the property that the free Φ-completion of a small $\mathcal{V}$-category is again small. A $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{V}$ with small domain is called *Φ-flat* if its cocontinuous extension $\mathrm{Lan}_y F\colon [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \to \mathcal{V}$ preserves Φ-limits, and $a \in \mathcal{V}$ is *Φ-presentable* if $\mathcal{V}(a, –)\colon \mathcal{V} \to \mathcal{V}$ preserves colimits by Φ-flat weights.

Suppose that *if $\mathcal{C}$ is small and Φ-complete, then every Φ-continuous $F\colon \mathcal{C} \to \mathcal{V}$ is Φ-flat*; this is Axiom A of [23]. Then by Proposition 3.4 and §7.1 of [23], we obtain an instance of our setting on taking $\mathcal{E} = \mathcal{V}$ and $\mathcal{A} = \mathcal{V}_\Phi$ a skeleton of the full sub-$\mathcal{V}$-category of Φ-presentable objects.

A key example takes $\mathcal{V} = \mathcal{E} = \mathbf{Cat}$ and Φ the class of finite products; whereupon $\mathcal{V}_\Phi$ is the subcategory $\mathbb{F}$ of finite cardinals, seen as discrete categories. This example captures *strongly finitary* [19] structure on categories involving functors and transformations $\mathcal{C}^n \to \mathcal{C}$; this includes *monoidal* or *finite product* structure, but not *finite limit* structure.

(ix) More generally, we can take $\mathcal{E} = \Phi\text{-}\mathbf{Cts}(\mathcal{C}, \mathcal{V})$, the $\mathcal{V}$-category of Φ-continuous functors $\mathcal{C} \to \mathcal{V}$ for some small Φ-complete $\mathcal{C}$, and take $\mathcal{A}$ to be the full image of the Yoneda embedding $Y : \mathcal{C}^{\mathrm{op}} \to \Phi\text{-}\mathbf{Cts}(\mathcal{C}, \mathcal{V})$. This example is appropriate to the study of "Φ-ary algebraic structure on $\mathcal{E}$-objects"—subsuming most of the preceding examples.

3.2. Pretheories as presentations

We will now describe examples of pretheories and their models in various contexts; in doing so, it will be useful to avail ourselves of the following constructions. Given a pretheory $\mathcal{A} \to \mathcal{T}$ and objects $a, b \in \mathcal{T}$, to *adjoin a morphism* $f : a \to b$ is to form the $\mathcal{V}$-category $\mathcal{T}[f]$ in the pushout square to the left of:

$$\begin{tikzcd} 2 \arrow[r, "\langle a,b\rangle"] \arrow[d, "\iota"'] & \mathcal{T} \arrow[d, "\bar\iota"] \\ \mathbf{2} \arrow[r, "f"] & \mathcal{T}[f] \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{2} +_2 \mathbf{2} \arrow[r, "\langle f,g\rangle"] \arrow[d, "\langle \mathrm{id},\mathrm{id}\rangle"'] & \mathcal{T} \arrow[d, "\bar\iota"] \\ \mathbf{2} \arrow[r, "f=g"] & \mathcal{T}[f=g] \end{tikzcd} \; . \tag{3.2}$$

Here, $\iota : 2 \to \mathbf{2}$ is the inclusion of the free $\mathcal{V}$-category on the set $\{0, 1\}$ into the free $\mathcal{V}$-category $\mathbf{2} = \{0 \to 1\}$ on an arrow. Since ι is identity-on-objects, its pushout $\bar\iota$ may also be chosen thus, so that we may speak of adjoining an arrow to a pretheory $J : \mathcal{A} \to \mathcal{T}$ to obtain the pretheory $J[f] = \bar\iota \circ J : \mathcal{A} \to \mathcal{T}[f]$.

Recall from (2.3) that a concrete $\mathcal{T}$-model comprises $X \in \mathcal{E}$ and $F \in [\mathcal{T}^{\mathrm{op}}, \mathcal{V}]$ for which $F \circ J^{\mathrm{op}} = \mathcal{E}(K{-}, X) : \mathcal{A} \to \mathcal{V}$. Thus, by the universal property of the pushout (3.2), a concrete $\mathcal{T}[f]$-model is the same as a concrete $\mathcal{T}$-model (X, F) together with a map $[f] : \mathcal{E}(Kb, X) \to \mathcal{E}(Ka, X)$ in $\mathcal{V}$.

Similarly given parallel morphisms $f, g : a \rightrightarrows b$ in the underlying category of $\mathcal{T}$ we can form the pushout above right. In this way we may speak of adjoining an equation $f = g$ to a pretheory $J : \mathcal{A} \to \mathcal{T}$ to obtain the pretheory $J[f = g] = \bar\iota \circ J : \mathcal{A} \to \mathcal{T}[f = g]$. In this case, we see that a concrete $\mathcal{T}[f = g]$-model is a concrete $\mathcal{T}$-model (X, F) such that $Ff = Fg : \mathcal{E}(Kb, X) \to \mathcal{E}(Ka, X)$.

Example 9. In the context of Examples 8(i) appropriate to classical finitary algebraic theories—so $\mathcal{V} = \mathcal{E} = \mathbf{Set}$ and $\mathcal{A} = \mathbb{F}$—we will construct a pretheory $J : \mathbb{F} \to \mathcal{M}$ whose category of concrete models is the category of monoids.

We start from the initial pretheory $\mathrm{id} : \mathbb{F} \to \mathbb{F}$ whose concrete models are simply sets, and construct from it a pretheory $J_1 : \mathbb{F} \to \mathcal{M}_1$ by adjoining morphisms

$$m\colon 1 \to 2 \qquad \text{and} \qquad i\colon 1 \to 0 \tag{3.3}$$

representing the monoid multiplication and unit operations, and also morphisms

$$m1, 1m\colon 2 \rightrightarrows 3 \qquad \text{and} \qquad i1, 1i\colon 2 \rightrightarrows 1 \tag{3.4}$$

which will be necessary later to express the monoid equations. Note that our directional conventions mean that the input arity of these operations is in the *codomain* rather than the domain. It follows from the preceding remarks that a concrete $\mathcal{M}_1$-model is a set X equipped with functions

$$[m]\colon X^2 \to X \ , \quad [i]\colon 1 \to X \ , \quad [m1], [1m]\colon X^3 \rightrightarrows X^2 \ , \quad [i1], [1i]\colon 1 \rightrightarrows X$$

interpreting the morphisms adjoined above. We now adjoin to $\mathcal{M}_1$ the eight equations necessary to render commutative the following squares in $\mathcal{M}_1$:

$$\begin{tikzcd}
1 \arrow[r, "m"] \arrow[d, "\iota_1"'] & 2 \arrow[d, "\iota_1"] \\
1+1 \arrow[r, "m1"] & 2+1 \\
1 \arrow[r, "\mathrm{id}"] \arrow[u, "\iota_2"] & 1 \arrow[u, "\iota_2"']
\end{tikzcd}
\quad
\begin{tikzcd}
1 \arrow[r, "\mathrm{id}"] \arrow[d, "\iota_1"'] & 1 \arrow[d, "\iota_1"] \\
1+1 \arrow[r, "1m"] & 1+2 \\
1 \arrow[r, "m"] \arrow[u, "\iota_2"] & 2 \arrow[u, "\iota_2"']
\end{tikzcd}
\quad
\begin{tikzcd}
1 \arrow[r, "i"] \arrow[d, "\iota_1"'] & 0 \arrow[d, "!"] \\
1+1 \arrow[r, "i1"] & 1 \\
1 \arrow[r, "\mathrm{id}"] \arrow[u, "\iota_2"] & 1 \arrow[u, "\mathrm{id}"']
\end{tikzcd}
\quad
\begin{tikzcd}
1 \arrow[r, "\mathrm{id}"] \arrow[d, "\iota_1"'] & 1 \arrow[d, "\iota_1"] \\
1+1 \arrow[r, "1i"] & 1 \\
1 \arrow[r, "i"] \arrow[u, "\iota_2"] & 0 \arrow[u, "!"']
\end{tikzcd}
\tag{3.5}$$

where ι_1, ι_2 and ! are the images under J_1 of the relevant coproduct injections or maps from 0 in $\mathbb{F}$; together with three equations which render commutative:

$$\begin{tikzcd}
1 \arrow[r, "m"] \arrow[d, "m"'] & 2 \arrow[d, "1m"] \\
2 \arrow[r, "m1"] & 3
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "m"] \arrow[dr, "1"'] & 2 \arrow[d, "i1"] \\
 & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "m"] \arrow[dr, "1"'] & 2 \arrow[d, "1i"] \\
 & 1 \ .
\end{tikzcd}
\tag{3.6}$$

A concrete model for the resulting theory $J\colon \mathbb{F} \to \mathcal{M}$ is a concrete $\mathcal{M}_1$-model (X, F) for which $F^{\mathrm{op}}\colon \mathcal{M}_1 \to \mathbf{Set}^{\mathrm{op}}$ sends each diagram in (3.5) and (3.6) to a commuting one. Commutativity in (3.5) forces $[m1] = [m] \times \mathrm{id}\colon X^3 \to X^2$ and so on; whereupon commutativity of (3.6) expresses precisely the monoid axioms, so that concrete $\mathcal{M}$-models are monoids, as desired. Extending this analysis to morphisms we see that $\mathbf{Mod}_c(\mathcal{M})$ is isomorphic to the category of monoids and monoid homomorphisms.

Example 10. In the same way we can describe $\mathbb{F}$-pretheories modelling any of the categories of classical universal algebra—groups, rings and so on. Note that the same structure can be presented by distinct pretheories: for instance, we could extend the pretheory $\mathcal{M}$ of the preceding example by adjoining a further morphism $m11\colon 3 \to 4$ and two equations forcing it to become $[m] \times 1 \times 1\colon X^4 \to X^3$ in any model; on doing so, we would not change the category of concrete models. However, in $\mathcal{M}$, all of the maps $3 \to 4$ belong to $\mathbb{F}$ while in the new pretheory, $m11$ does not. This non-canonicity will be rectified by the *theories* introduced in Section 4 below; in particular, Corollary 24 implies that, to within isomorphism, there is at most one $\mathbb{F}$-theory which captures a given type of structure.

Example 11. In the situation of Examples 8(iv), where $\mathcal{E} = [\mathbb{G}_0^{\mathrm{op}}, \mathbf{Set}]$ is the category of directed graphs and $\mathcal{A} = \Delta_0$, we will describe a pretheory $\Delta_0 \to \mathcal{C}$ whose concrete models are categories. The construction is largely identical to the example of monoids above. Starting from the initial Δ_0-pretheory, we adjoin composition and unit maps $m\colon [1] \to [2]$ and $i\colon [1] \to [0]$ as well as the morphisms $1m, m1\colon [2] \rightrightarrows [3]$ and $i1, 1i\colon [2] \rightrightarrows [1]$ required to describe the category axioms.

We now adjoin the necessary equations. First, we have four equations ensuring that composition and identities interact appropriately with source and target:

$$
\begin{tikzcd}
{[0]} \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & {[1]} \arrow[d, "m"] \\
{[1]} \arrow[r, "\iota_1"] & {[2]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[0]} \arrow[r, "\tau"] \arrow[d, "\tau"'] & {[1]} \arrow[d, "m"] \\
{[1]} \arrow[r, "\iota_2"] & {[2]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[0]} \arrow[r, "\sigma"] \arrow[dr, "\mathrm{id}"'] & {[1]} \arrow[d, "i"] \\
& {[0]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[0]} \arrow[r, "\tau"] \arrow[dr, "\mathrm{id}"'] & {[1]} \arrow[d, "i"] \\
& {[0]}
\end{tikzcd}
$$

where here we write $\sigma, \tau\colon [0] \rightrightarrows [1]$ for the two endpoint inclusions, and ι_1, ι_2 for the two colimit injections into $[1]\ {}_{\tau}{+}_{\sigma}\ [1] = [2]$. We also require analogues of the eight equations of (3.5) and three equations of (3.6). The modifications are minor: replace n by $[n]$, the coproduct inclusions $\iota_1\colon n \to n + m \leftarrow m\colon \iota_2$ by the pushout inclusions $\iota_1\colon [n] \to [n]\ {}_{\tau}{+}_{\sigma}\ [m] \leftarrow [m]\colon \iota_2$, the first appearance of $!\colon 0 \to 1$ by $\sigma\colon [0] \to [1]$ and its second appearance by $\tau\colon [0] \to [1]$. After adjoining these six morphisms and fifteen equations, we find that the concrete models of the resulting pretheory $\Delta_0 \to \mathcal{C}$ are precisely *small categories*.

We can extend this pretheory to one for groupoids. To do so, we adjoin a morphism $c\colon [1] \to [1]$ modelling the inversion plus the further maps $1c\colon [2] \to [2]$ and $c1\colon [2] \to [2]$ required for the axioms. Now four equations must be adjoined to force the correct interpretation of $1c$ and $c1$, plus the two equations for left and right inverses. On doing so, the resulting pretheory $\Delta_0 \to \mathcal{G}$ has as its concrete models the *small groupoids*.

Example 12. In the situation of Examples 8(v), where $\mathcal{E}$ is the category of globular sets and $\mathcal{A} = \Theta_0$ is the full subcategory of globular cardinals, one can similarly construct pretheories whose concrete models are strict ω-categories or strict ω-groupoids. For instance, one encodes binary composition of n-cells along a k-cell boundary (for $k < n$) by adjoining morphisms $m_{n,k}\colon Y(n) \to Y(n,k,n)$ to Θ_0. In fact, all of the standard flavours of globular *weak* ω-category and *weak* ω-groupoid can also be encoded using Θ_0-pretheories; see Examples 44(v) below.

Example 13. Consider the case of Examples 8(viii) where $\mathcal{V} = \mathcal{E} = \mathbf{Cat}$ and $\mathcal{A} = \mathbb{F}$, the full subcategory of finite cardinals (seen as discrete categories). We will describe an $\mathbb{F}$-pretheory capturing the structure of a monoidal category. In doing so, we exploit the fact that our pretheories are no longer mere categories, but 2-categories; so we may speak not only of adjoining morphisms and equations between such, but also of *adjoining an (invertible) 2-cell*—by taking a pushout of the inclusion $\mathbf{2} +_2 \mathbf{2} \to D_2$ of the parallel pair 2-category into the free 2-category on an (invertible) 2-cell—and similarly of *adjoining an equation between 2-cells.*

To construct a pretheory for monoidal categories, we start essentially as for monoids: freely adjoining the usual maps $m, i, m1, 1m, i1, 1i$ to the initial pretheory, but now also morphisms $m11, 1m1, 11m \colon 3 \to 4$ and $1i1 \colon 3 \to 2$ needed for the monoidal category coherence axioms; thus, ten morphisms in all.

We now add the $8 \times 2 = 16$ equations asserting that each of the morphisms beyond m and i has the expected interpretation in a model, plus[1] the equation $1m \circ m11 = m1 \circ 11m \colon 2 \to 4$. This being done, we next adjoin invertible 2-cells

$$
\begin{tikzcd}
1 \arrow[r, "m"] \arrow[d, "m"'] \arrow[dr, phantom, "\Downarrow\alpha"] & 2 \arrow[d, "m1"] \\
2 \arrow[r, "1m"'] & 3
\end{tikzcd}
\qquad
\begin{tikzcd}
& 2 \arrow[d, "i1"] \\
1 \arrow[ur, bend left, "m"] \arrow[r, "1"'] \arrow[ur, phantom, "\Downarrow\lambda"] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
& 2 \arrow[d, "1i"] \\
1 \arrow[ur, bend left, "m"] \arrow[r, "1"'] \arrow[ur, phantom, "\Uparrow\rho"] & 1
\end{tikzcd}
$$

expressing the associativity and unit coherences, as well as the invertible 2-cells

$$
\begin{tikzcd}
2 \arrow[r, "m1"] \arrow[d, "m1"'] \arrow[dr, phantom, "\Downarrow\alpha 1"] & 3 \arrow[d, "m11"] \\
3 \arrow[r, "1m1"'] & 4
\end{tikzcd}
\qquad
\begin{tikzcd}
2 \arrow[r, "1m"] \arrow[d, "1m"'] \arrow[dr, phantom, "\Downarrow 1\alpha"] & 3 \arrow[d, "1m1"] \\
3 \arrow[r, "11m"'] & 4
\end{tikzcd}
\qquad
\begin{tikzcd}
& 3 \arrow[d, "1i1"] \\
2 \arrow[ur, bend left, "1m"] \arrow[r, "1"'] \arrow[ur, phantom, "\Downarrow 1\lambda"] & 2
\end{tikzcd}
\qquad
\begin{tikzcd}
& 3 \arrow[d, "1i1"] \\
2 \arrow[ur, bend left, "m1"] \arrow[r, "1"'] \arrow[ur, phantom, "\Uparrow \rho 1"] & 2
\end{tikzcd}
$$

which will be needed to express the coherence axioms. Finally, we must adjoin equations between 2-cells: the $2 \times 4 = 8$ equations ensuring that $\alpha 1$, 1α, 1λ and $\rho 1$ have the intended interpretation in any model, plus two equations expressing the coherence axioms:

$$
\begin{tikzcd}
& 2 \arrow[r, "m1"] \arrow[d, phantom, "\Downarrow\alpha"] & 3 \arrow[dr, "m11"] & \\
1 \arrow[ur, "m"] \arrow[r, "m"] \arrow[dr, "m"'] & 2 \arrow[ur, "1m"'] \arrow[dr, "m1"] \arrow[d, phantom, "\Downarrow\alpha"] & & 4 \\
& 2 \arrow[r, "1m"'] & 3 \arrow[ur, "11m"'] &
\end{tikzcd}
=
\begin{tikzcd}
& 2 \arrow[r, "m1"] \arrow[dr, "m1"] & 3 \arrow[dr, "m11"] \arrow[d, phantom, "\Downarrow\alpha 1"] & \\
1 \arrow[ur, "m"] \arrow[dr, "m"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & 3 \arrow[r, "1m1"] \arrow[d, phantom, "\Downarrow 1\alpha"] & 4 \\
& 2 \arrow[r, "1m"'] \arrow[ur, "1m"] & 3 \arrow[ur, "11m"'] &
\end{tikzcd}
\tag{3.7}
$$

[1] It may be prima facie unclear why this is necessary; after all, if $1m$, $m11$, $m1$ and $11m$ have the intended interpretations in a model, then it is certainly the case that they will verify this equality. Yet this equality is not forced to hold *in the pretheory*, and we need it to do so in order for (3.7) to type-check.

$$\begin{tikzcd} & 2 \arrow[dr, "m1"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[dr, phantom, shift left=4, "\Downarrow\rho 1"] & & \\ 1 \arrow[ur, "m"] \arrow[dr, "m"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & 3 \arrow[r, "1i1"] & 2 \\ & 2 \arrow[ur, "1m"] \arrow[urr, bend right, "\mathrm{id}"'] \arrow[ur, phantom, shift right=4, "\Downarrow 1\lambda"] & & \end{tikzcd} \quad = \quad \begin{tikzcd} & 2 \arrow[dr, bend left, "\mathrm{id}"] & \\ 1 \arrow[ur, "m"] \arrow[dr, "m"'] \arrow[rr, phantom, "\Downarrow\mathrm{id}"] & & 2\,. \\ & 2 \arrow[ur, bend right, "\mathrm{id}"'] & \end{tikzcd}$$

All told, we have adjoined ten morphisms, seventeen equations between morphisms, seven invertible 2-cells, and nine equations between 2-cells to obtain a pretheory $J \colon \mathbb{F} \to \mathcal{MC}$ whose concrete models are precisely monoidal categories.

4. The monad–theory correspondence

In this section, we return to the general theory and establish our "best possible" monad–theory correspondence. This will be obtained by restricting the adjunction (2.1) to its *fixpoints*: the objects on the left and right at which the counit and the unit are invertible. The categories of fixpoints are the *largest* subcategories on which the adjunction becomes an adjoint equivalence, and it is in this sense that our monad–theory correspondence is the best possible.

4.1. A pullback lemma

The following lemma will be crucial in characterising the fixpoints of (2.1) on each side. Note that the force of (2) below is in the "if" direction; the "only if" is always true.

Lemma 14. *A commuting square in* $\mathcal{V}$-**CAT**

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "H"'] & \mathcal{B} \arrow[d, "K"] \\ \mathcal{C} \arrow[r, "G"] & \mathcal{D} \end{tikzcd}$$

with G fully faithful and H, K discrete isofibrations is a pullback just when:

(1) F is fully faithful; and
(2) An object $b \in \mathcal{B}$ is in the essential image of F if and only if Kb is in the essential image of G.

Proof. If the square is a pullback, then F is fully faithful as a pullback of G. As for (2), if $Kb \cong Gc$ in $\mathcal{D}$ then since K is an isofibration we can find $b \cong b'$ in $\mathcal{B}$ with $Kb' = Gc$; now by the pullback property we induce $a \in \mathcal{A}$ with $Fa = b'$ so that $b \cong Fa$ as required. Suppose conversely that (1) and (2) hold. We form the pullback $\mathcal{P}$ of K along G and the induced map L as below.

$$\begin{tikzcd}
\mathcal{A} \arrow[dr, dashed, "L"] \arrow[drr, bend left, "F"] \arrow[ddr, bend right, "H"'] & & \\
& \mathcal{P} \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{B} \arrow[d, "K"] \\
& \mathcal{C} \arrow[r, "G"] & \mathcal{D}
\end{tikzcd} \tag{4.1}$$

P is fully faithful as a pullback of G, and F is so by assumption; whence by standard cancellativity properties of fully faithful functors, L is also fully faithful.

In fact, discrete isofibrations are also stable under pullback, and also have the same cancellativity property; this follows from the fact that they are exactly the maps with the unique right lifting property against the inclusion of the free $\mathcal{V}$-category on an object into the free $\mathcal{V}$-category on an isomorphism. Consequently, in (4.1), Q is a discrete isofibration as a pullback of K, and H is so by assumption; whence by cancellativity, L is also a discrete isofibration.

If we can now show L is also essentially surjective, we will be done: for then L is a discrete isofibration and an equivalence, whence invertible. So let $(b, c) \in \mathcal{P}$. Since $Kb = Gc$, by (2) we have that b is in the essential image of F. So there is $a \in \mathcal{A}$ and an isomorphism $\beta \colon b \cong Fa$. Now $K\beta \colon Gc = Kb \cong KFa = GHa$ so by full fidelity of G there is $\gamma \colon c \cong Ha$ with $G\gamma = K\beta$; and so we have $(\beta, \gamma) \colon (b, c) \cong La$ exhibiting (b, c) as in the essential image of L, as required. $\square$

4.2. $\mathcal{A}$-theories

We first use the pullback lemma to describe the fixpoints of (2.1) on the pretheory side.

Definition 15. An $\mathcal{A}$-pretheory $J \colon \mathcal{A} \to \mathcal{T}$ is said to be an $\mathcal{A}$-*theory* if each $\mathcal{T}(J-, a) \in [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is a K-nerve. We write $\mathbf{Th}_{\mathcal{A}}(\mathcal{E})$ for the full subcategory of $\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$ on the $\mathcal{A}$-theories.

In the language of Section 5.2 below, a pretheory $\mathcal{T}$ is an $\mathcal{A}$-theory just when each representable $\mathcal{T}(-, a) \colon \mathcal{T}^{\mathrm{op}} \to \mathcal{V}$ is a (*non-concrete*) $\mathcal{T}$-model. When $\mathcal{V} = \mathcal{E} = \mathbf{Set}$ and $\mathcal{A} = \mathbb{F}$, an $\mathcal{A}$-pretheory is an $\mathcal{A}$-theory precisely when it is a Lawvere theory; see Examples 44(i) below.

Theorem 16. *An $\mathcal{A}$-pretheory $J \colon \mathcal{A} \to \mathcal{T}$ is an $\mathcal{A}$-theory if and only if the unit component $\eta_{\mathcal{T}} \colon \mathcal{T} \to \Phi\Psi\mathcal{T}$ of* (2.1) *is invertible.*

Proof. The unit $\eta_{\mathcal{T}} \colon \mathcal{T} \to \Phi\Psi\mathcal{T}$ is obtained by starting with $\alpha_0 = 1 \colon \Psi\mathcal{T} \to \Psi\mathcal{T}$ and chasing through the bijections of Theorem 6 to obtain $\alpha_6 = \eta_{\mathcal{T}}$. Doing this, we quickly arrive at α_2 equal to P, the projection in the depicted pullback square

$$\begin{tikzcd} \mathcal{E}^{\Psi\mathcal{T}} \arrow[r, "P"] \arrow[d, "U^{\Psi\mathcal{T}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] \\ \mathcal{E} \arrow[r, "N_K"] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}^{\mathrm{op}} \arrow[rrr, "J^{\mathrm{op}}"] \arrow[d, "(J_{\Psi\mathcal{T}})^{\mathrm{op}}"'] & & & \mathcal{T}^{\mathrm{op}} \arrow[d, "\alpha_4"] \arrow[lll, dotted, "\alpha_6^{\mathrm{op}}"', start anchor=south west, end anchor=north east, shift left=0] \arrow[dll, dotted, "{\alpha_5}^{\mathrm{op}}"] \\ (\mathcal{A}_{\Psi\mathcal{T}})^{\mathrm{op}} \arrow[r, "(K_{\Psi\mathcal{T}})^{\mathrm{op}}"'] & (\mathcal{E}^{\Psi\mathcal{T}})^{\mathrm{op}} \arrow[rr, "Y"'] & & {[\mathcal{E}^{\Psi\mathcal{T}}, \mathcal{V}]_{\mathrm{rep}}} \end{tikzcd} \tag{4.2}$$

defining $\mathcal{E}^{\Psi\mathcal{T}}$. Now $\alpha_3 \colon \mathcal{E}^{\Psi\mathcal{T}} \to [\mathcal{T}^{\mathrm{op}}, \mathcal{V}]$ is obtained by lifting an isomorphism through $[J^{\mathrm{op}}, 1]$ and so we have $\alpha_3 \cong P$. We obtain α_4 by transposing α_3 through the isomorphism $(-)^t \colon \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{E}^{\Psi\mathcal{T}}, [\mathcal{T}^{\mathrm{op}}, \mathcal{V}])_{\mathrm{pwr}} \cong \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{T}^{\mathrm{op}}, [\mathcal{E}^{\Psi\mathcal{T}}, \mathcal{V}]_{\mathrm{rep}})$ displayed in (2.10). The relationships between α_4, α_5 and the unit component $\eta_{\mathcal{T}} = \alpha_6$ are depicted in the commutative diagram above right.

The identity-on-objects unit $\eta_{\mathcal{T}} = \alpha_6$ will be invertible just when it is fully faithful which, since $K_{\Psi\mathcal{T}}$ is fully faithful, will be so just when α_5 is fully faithful. Now, since $P \cong \alpha_3 = (\alpha_4)^t = (Y \circ \alpha_5^{\mathrm{op}})^t = N_{\alpha_5}$, and P is fully faithful, as the pullback of the fully faithful N_K, it follows that $N_{\alpha_5} \colon \mathcal{E}^{\Psi\mathcal{T}} \to [\mathcal{T}^{\mathrm{op}}, \mathcal{V}]$ is also fully faithful. As a consequence, α_5 is fully faithful just when there exists a factorisation to within isomorphism:

$$Y \cong N_{\alpha_5} \circ G \colon \mathcal{T} \to \mathcal{E}^{\Psi\mathcal{T}} \to [\mathcal{T}^{\mathrm{op}}, \mathcal{V}] \; . \tag{4.3}$$

Indeed, in one direction, if α_5 is fully faithful then the canonical natural transformation $Y \Rightarrow N_{\alpha_5} \circ \alpha_5$ is invertible. In the other, given a factorisation as displayed, G is fully faithful since N_{α_5} and Y are. Moreover we have isomorphisms

$$\mathcal{E}^{\Psi\mathcal{T}}(\alpha_5 b, -) \cong [\mathcal{T}^{\mathrm{op}}, \mathcal{V}](Yb, N_{\alpha_5}-) \cong [\mathcal{T}^{\mathrm{op}}, \mathcal{V}](N_{\alpha_5}Gb, N_{\alpha_5}-) \cong \mathcal{E}^{\Psi\mathcal{T}}(Gb, -)$$

natural in b. So by Yoneda, $\alpha_5 \cong G$ and so α_5 is fully faithful since G is so.

This shows that η_T is invertible just when there is a factorisation (4.3). Since N_{α_5} is fully faithful this in turn is equivalent to asking that each $Yb = \mathcal{T}(-, b)$ lies in the essential image of α_5, or equally in the essential image of the isomorphic P. As the left square of (4.2) is a pullback, Lemma 14 asserts that this is, in turn, equivalent to each $[J^{\mathrm{op}}, 1](Yb) = \mathcal{T}(J-, b)$ being in the essential image of N_K; which is precisely the condition that J is an $\mathcal{A}$-theory. $\square$

4.3. $\mathcal{A}$-nervous monads

We now characterise the fixpoints on the monad side. In the following definition, $\mathcal{A}_{\mathsf{T}}$, J_{T} and K_{T} are as in (2.5).

Definition 17. A $\mathcal{V}$-monad T on $\mathcal{E}$ is called $\mathcal{A}$-*nervous* if

(i) The fully faithful $K_{\mathsf{T}} \colon \mathcal{A}_{\mathsf{T}} \to \mathcal{E}^{\mathsf{T}}$ is dense;
(ii) A presheaf $X \in [{\mathcal{A}_{\mathsf{T}}}^{\mathrm{op}}, \mathcal{V}]$ is a K_{T}-nerve if and only if $X \circ {J_{\mathsf{T}}}^{\mathrm{op}}$ is a K-nerve.

We write $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ for the full subcategory of $\mathbf{Mnd}(\mathcal{E})$ on the $\mathcal{A}$-nervous monads.

Note that the adjointness isomorphisms $\mathcal{E}^{\mathsf{T}}(K_{\mathsf{T}}J_{\mathsf{T}}X, Y) = \mathcal{E}^{\mathsf{T}}(F^{\mathsf{T}}KX, Y) \cong \mathcal{E}(KX, U^{\mathsf{T}}Y)$ for the adjunction $F^{\mathsf{T}} \dashv U^{\mathsf{T}}$ give a pseudo-commutative square

$$\begin{tikzcd} \mathcal{E}^{\mathsf{T}} \arrow[r, "N_{K_{\mathsf{T}}}"] \arrow[d, "U^{\mathsf{T}}"'] \arrow[dr, phantom, "\cong"] & {[\mathcal{A}_{\mathsf{T}}{}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J_{\mathsf{T}}{}^{\mathrm{op}}, 1]}"] \\ \mathcal{E} \arrow[r, "N_K"'] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} \ ; \end{tikzcd} \tag{4.4}$$

as a result of which, $[J_{\mathsf{T}}^{\mathrm{op}}, 1]$ maps K_{T}-nerves to K-nerves. Thus the force of clause (ii) of the preceding definition lies in the *if* direction.

Theorem 18. *The counit component $\varepsilon_{\mathsf{T}} \colon \Psi\Phi\mathsf{T} \to \mathsf{T}$ of (2.1) at a monad T on $\mathcal{E}$ is invertible if and only if T is $\mathcal{A}$-nervous.*

Proof. ε_{T} is obtained by taking $\alpha_6 = 1 \colon J_{\mathsf{T}} \to J_{\mathsf{T}}$ and proceeding in reverse order through the series of six natural isomorphisms in the proof of Theorem 6. Doing this, we quickly reach $\alpha_3 = N_{K_{\mathsf{T}}}$. Then $\alpha_2 \colon \mathcal{E}^{\mathsf{T}} \to [(\mathcal{A}_{\mathsf{T}})^{\mathrm{op}}, \mathcal{V}]$ is obtained by lifting the natural isomorphism φ of (4.4) through the discrete isofibration $[J_{\mathsf{T}}^{\mathrm{op}}, 1]$, yielding a commutative square as left below.

$$\begin{tikzcd} \mathcal{E}^{\mathsf{T}} \arrow[r, "\alpha_2"] \arrow[d, "U^{\mathsf{T}}"'] & {[(\mathcal{A}_{\mathsf{T}})^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J_{\mathsf{T}}^{\mathrm{op}}, 1]}"] \\ \mathcal{E} \arrow[r, "N_K"'] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}^{\mathsf{T}} \arrow[r, "\alpha_1"] \arrow[dr, "U^{\mathsf{T}}"'] & \mathcal{E}^{\Psi\Phi\mathsf{T}} \arrow[d, "U^{\Psi\Phi\mathsf{T}}"] \\ & \mathcal{E}. \end{tikzcd} \tag{4.5}$$

The map $\alpha_1 \colon \mathcal{E}^{\mathsf{T}} \to \mathcal{E}^{\Psi\Phi\mathsf{T}}$ is the unique map to the pullback, and $\alpha_0 = \varepsilon_{\mathsf{T}}$ the corresponding morphism of monads. It follows that ε_{T} is invertible if and only if the square to the left of (4.5) is a pullback. Both vertical legs are discrete isofibrations and N_K is fully faithful, so by Lemma 14 this happens just when, firstly, α_2 is fully faithful, and, secondly, $X \in [\mathcal{A}_{\mathsf{T}}^{\mathrm{op}}, \mathcal{V}]$ is in the essential image of α_2 if and only if XJ_{T} is a K-nerve. But as $\alpha_2 \cong N_{K_{\mathsf{T}}}$, and natural isomorphism does not change either full fidelity or essential images, this happens just when T is $\mathcal{A}$-nervous. $\square$

4.4. The monad–theory equivalence

Putting together the preceding results now yields the main result of this paper.

Theorem 19. *The adjunction* (2.1) *restricts to an adjoint equivalence*

$$\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) \underset{\Phi}{\overset{\Psi}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Th}_{\mathcal{A}}(\mathcal{E}) \tag{4.6}$$

between the category of $\mathcal{A}$-nervous monads and the category of $\mathcal{A}$-theories.

Proof. Any adjunction restricts to an adjoint equivalence between the objects with invertible unit and counit components respectively, and Theorems 16 and 18 identify these objects as the $\mathcal{A}$-theories and the $\mathcal{A}$-nervous monads. □

Note that there is an asymmetry between the conditions found on each side. On the one hand, the condition characterising the $\mathcal{A}$-theories among the $\mathcal{A}$-pretheories is intrinsic, and easy to check in practice. On the other hand, the condition defining an $\mathcal{A}$-nervous monad refers to the associated pretheory, and is non-trivial to check in practice. Indeed, one of the main points of [35,9] is to provide a general set of *sufficient* conditions under which a monad can be shown to be $\mathcal{A}$-nervous.

In the sections which follow, we will provide a number of more tractable characterisations of the $\mathcal{A}$-theories and $\mathcal{A}$-nervous monads; the crucial fact which drives all of these is that the adjunction (2.1) is in fact idempotent. Recall that an adjunction $L \dashv R \colon \mathcal{D} \to \mathcal{C}$ is *idempotent* if the monad RL on $\mathcal{C}$ is idempotent, and that this is equivalent to asking that the comonad LR is idempotent, or that any one of the natural transformations $R\varepsilon$, εL, ηR and $L\eta$ is invertible.

Theorem 20. *The adjunction* (2.1) *is idempotent.*

Proof. We show for each $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$ that the unit $\eta_{\Phi\mathsf{T}} \colon \Phi\mathsf{T} \to \Phi\Psi\Phi\mathsf{T}$ is invertible. By Theorem 16, this is equally to show that $J_{\mathsf{T}} \colon \mathcal{A} \to \mathcal{A}_{\mathsf{T}}$ is an $\mathcal{A}$-theory, i.e., that each $\mathcal{A}_{\mathsf{T}}(J_{\mathsf{T}}-, J_{\mathsf{T}}a) \in [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is a K-nerve. But $\mathcal{A}_{\mathsf{T}}(J_{\mathsf{T}}-, J_{\mathsf{T}}a) \cong \mathcal{E}^{\mathsf{T}}(F^{\mathsf{T}}K-, F^{\mathsf{T}}Ka) \cong \mathcal{E}(K-, U^{\mathsf{T}}F^{\mathsf{T}}Ka) = \mathcal{E}(K-, TKa)$ as required. □

Exploiting the alternative characterisations of idempotent adjunctions listed above, we immediately obtain the following result, which tells us in particular that a monad T is $\mathcal{A}$-nervous if and only if it can be presented by some $\mathcal{A}$-pretheory.

Corollary 21. *A monad T on $\mathcal{E}$ is $\mathcal{A}$-nervous if and only if $\mathsf{T} \cong \Psi\mathcal{T}$ for some $\mathcal{A}$-pretheory $J \colon \mathcal{A} \to \mathcal{T}$; while an $\mathcal{A}$-pretheory $J \colon \mathcal{A} \to \mathcal{T}$ is an $\mathcal{A}$-theory if and only if $\mathcal{T} \cong \Phi\mathsf{T}$ for some monad T on $\mathcal{E}$.*

The next result also follows directly from the definition of idempotent adjunction.

Corollary 22. *The full subcategory $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) \subseteq \mathbf{Mnd}(\mathcal{E})$ is coreflective via $\Psi\Phi$, while the full subcategory $\mathbf{Th}_{\mathcal{A}}(\mathcal{E}) \subseteq \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$ is reflective via $\Phi\Psi$.*

5. Semantics

In the next section, we will explicitly identify the $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories for the examples listed in Section 2.1. Before doing this, we study further aspects of the general theory, namely those related to the taking of semantics.

5.1. Interaction with the semantics functors

We begin by examining the interaction of our monad–theory correspondence with the semantics functors of Section 2. In fact, we begin at the level of the monad–pretheory adjunction (2.1).

Proposition 23. *There is a natural isomorphism θ as on the left in:*

$$\begin{tikzcd} \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[rr, "\Psi^{\mathrm{op}}"] \arrow[dr, "\mathrm{Mod}_c"'] & \arrow[d, phantom, "\overset{\theta}{\Leftarrow}"] & \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \arrow[dl, "\mathrm{Alg}"] \\ & \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E} & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \arrow[rr, "\Phi^{\mathrm{op}}"] \arrow[dr, "\mathrm{Alg}"'] & \arrow[d, phantom, "\overset{\bar\theta}{\Rightarrow}"] & \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[dl, "\mathrm{Mod}_c"] \\ & \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E} & \end{tikzcd} .$$

Let $\bar\theta$ be its mate under the adjunction $\Phi^{\mathrm{op}} \dashv \Psi^{\mathrm{op}}$, as right above. The component of $\bar\theta$ at $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$ is invertible if and only if T is $\mathcal{A}$-nervous.

Proof. For the first claim, Theorem 6 provides the necessary natural isomorphisms $\theta_{\mathcal{T}} \colon \mathcal{E}^{\Psi\mathcal{T}} \to \mathbf{Mod}_c(\mathcal{T})$ over $\mathcal{E}$. For the second, if we write as before $\varepsilon_{\mathsf{T}} \colon \Psi\Phi\mathsf{T} \to \mathsf{T}$ for the counit component of (2.1) at $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$, then the T-component of $\bar\theta$ is the composite $\theta_{\Psi\mathsf{T}} \circ (\varepsilon_{\mathsf{T}})^* \colon \mathcal{E}^{\mathsf{T}} \to \mathcal{E}^{\Psi\Phi\mathsf{T}} \to \mathbf{Mod}_c(\Phi\mathcal{T})$ over $\mathcal{E}$. Since $\theta_{\Psi\mathcal{T}}$ is invertible and since Alg is fully faithful, $\bar\theta_{\mathsf{T}}$ will be invertible just when ε_{T} is so; that is, by Theorem 18, just when T is $\mathcal{A}$-nervous. $\square$

From this and the fact that each monad $\Psi\mathcal{T}$ is $\mathcal{A}$-nervous, it follows that *an $\mathcal{A}$-pretheory $\mathcal{T}$ and its associated theory $\Phi\Psi\mathcal{T}$ have isomorphic categories of concrete models*. By contrast, the passage from a monad T to its $\mathcal{A}$-nervous coreflection $\Psi\Phi\mathsf{T}$ may well change the category of algebras. For example, the power-set monad on **Set**, whose algebras are complete lattices, has its $\mathbb{F}$-nervous coreflection given by the finite-power-set monad, whose algebras are $\vee$-semilattices. However, if we restrict to $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories, then the situation is much better behaved.

Theorem 24. *The monad–theory equivalence* (4.6) *commutes with the semantics functors; that is, we have natural isomorphisms:*

$$\begin{tikzcd}
\mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[rr, "\Psi^{\mathrm{op}}"] \arrow[dr, "\mathrm{Mod}_c"'] & {} \arrow[d, phantom, "\overset{\theta}{\Leftarrow}"] & \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[dl, "\mathrm{Alg}"] \\
& \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[rr, "\Phi^{\mathrm{op}}"] \arrow[dr, "\mathrm{Alg}"'] & {} \arrow[d, phantom, "\overset{\bar\theta}{\Rightarrow}"] & \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[dl, "\mathrm{Mod}_c"] \\
& \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E} \; . &
\end{tikzcd} \tag{5.1}$$

Moreover, both $\mathrm{Mod}_c\colon \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ *and* $\mathrm{Alg}\colon \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ *are fully faithful functors.*

Proof. The first statement follows from Proposition 23. For the second, note that the functor $\mathrm{Alg}\colon \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ is obtained by restricting the fully faithful $\mathrm{Alg}\colon \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ along a full embedding, and so is itself fully faithful. It follows that $\mathrm{Mod}_c \cong \mathrm{Alg} \circ \Psi^{\mathrm{op}}\colon \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ is also fully faithful. $\square$

Full fidelity of $\mathrm{Mod}_c\colon \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ means that an $\mathcal{A}$-theory is determined to within isomorphism by its category of concrete models over $\mathcal{E}$. This rectifies the non-uniqueness of pretheories noted in Example 10 above.

5.2. Non-concrete models

In Section 2.3 we defined a *concrete model* of an $\mathcal{A}$-pretheory $\mathcal{T}$ to be an object $X \in \mathcal{E}$ endowed with an extension of $\mathcal{E}(K-, X)\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ to a functor $\mathcal{T}^{\mathrm{op}} \to \mathcal{V}$. In the literature, one often encounters a looser notion of model for a theory, in which an underlying object in $\mathcal{E}$ is not provided. In our setting, this notion becomes the following one: by an (unqualified) $\mathcal{T}$-*model*, we mean a functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{V}$ whose restriction $FJ^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is a K-nerve.

The $\mathcal{T}$-models span a full sub-$\mathcal{V}$-category $\mathbf{Mod}(\mathcal{T})$ of $[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]$. Recalling from Section 2.1 that $K\text{-}\mathbf{Ner}(\mathcal{V})$ denotes the full sub-$\mathcal{V}$-category of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ on the K-nerves, we may also express $\mathbf{Mod}(\mathcal{T})$ as a pullback as to the right in:

$$\begin{tikzcd}
\mathbf{Mod}_c(\mathcal{T}) \arrow[r, dashed] \arrow[rr, bend left, "P_{\mathcal{T}}"] \arrow[d, "U_{\mathcal{T}}"'] & \mathbf{Mod}(\mathcal{T}) \arrow[r, hook] \arrow[d, "W_{\mathcal{T}}"'] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] \\
\mathcal{E} \arrow[r, "N_K"] & K\text{-}\mathbf{Ner}(\mathcal{V}) \arrow[r, hook] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} \; .
\end{tikzcd} \tag{5.2}$$

On the other hand, $\mathbf{Mod}_c(\mathcal{T})$ is the pullback around the outside, and so there is a canonical induced map $\mathbf{Mod}_c(\mathcal{T}) \to \mathbf{Mod}(\mathcal{T})$ as displayed. By the usual cancellativity properties, the left square above is now also a pullback. Moreover, $W_{\mathcal{T}}$ is an isofibration, as a pullback of the discrete isofibration $[J^{\mathrm{op}}, 1]$, and $N_K\colon \mathcal{E} \to K\text{-}\mathbf{Ner}(\mathcal{V})$ is an equivalence. Since equivalences are stable under pullback along isofibrations, we conclude that:

Proposition 25. *The comparison* $\mathbf{Mod}_c(\mathcal{T}) \to \mathbf{Mod}(\mathcal{T})$ *in* (5.2) *is an equivalence.*

Taking non-concrete models gives rise to a semantics functor landing in the category $\mathcal{V}\text{-}\mathbf{CAT}/K\text{-}\mathbf{Ner}(\mathcal{V})$ which, like before, is *not* fully faithful on $\mathcal{A}$-pretheories, but is so on the subcategory of $\mathcal{A}$-theories. Note that the "underlying K-nerve" of a $\mathcal{T}$-model is more natural than it might seem, being the special case of the functor $\mathbf{Mod}(\mathcal{T}) \to \mathbf{Mod}(\mathcal{S})$ induced by a morphism of $\mathcal{A}$-pretheories for which $\mathcal{S}$ is the initial pretheory. However, in the following result, for simplicity, we view the semantics functors for $\mathcal{T}$-models as landing simply in $\mathcal{V}\text{-}\mathbf{CAT}$.

Theorem 26. *The monad–theory equivalence* (4.6) *commutes with the non-concrete semantics functors in the sense that we have natural transformations*

$$\begin{tikzcd}[column sep=small] \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[rr, "\Psi^{\mathrm{op}}"] \arrow[dr, "\mathrm{Mod}"'] & \arrow[d, phantom, "\overset{\theta}{\Leftarrow}"] & \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[dl, "\mathrm{Alg}"] \\ & \mathcal{V}\text{-}\mathbf{CAT} & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[rr, "\Phi^{\mathrm{op}}"] \arrow[dr, "\mathrm{Alg}"'] & \arrow[d, phantom, "\overset{\bar\theta}{\Rightarrow}"] & \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[dl, "\mathrm{Mod}"] \\ & \mathcal{V}\text{-}\mathbf{CAT} & \end{tikzcd}$$

whose components are equivalences in $\mathcal{V}\text{-}\mathbf{CAT}$.

Proof. First postcompose the natural isomorphisms (5.1) with the forgetful functor $\mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E} \to \mathcal{V}\text{-}\mathbf{CAT}$. Then paste the resulting natural isomorphisms with the natural transformation $\mathrm{Mod}_c \Rightarrow \mathrm{Mod}\colon \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}$ coming from the previous proposition. $\square$

5.3. Local presentability and algebraic left adjoints

Next in this section, we consider the categorical properties of the $\mathcal{V}$-categories and $\mathcal{V}$-functors in the image of the semantics functors. We begin with the case of pretheories.

Proposition 27.

(i) If $J\colon \mathcal{A} \to \mathcal{T}$ *is an* $\mathcal{A}$*-pretheory then* $\mathbf{Mod}_c(\mathcal{T})$ *is locally presentable and* $U_{\mathcal{T}}\colon \mathbf{Mod}_c(\mathcal{T}) \to \mathcal{E}$ *is a strictly monadic right adjoint.*

(ii) If $H\colon \mathcal{T} \to \mathcal{S}$ *is a map of* $\mathcal{A}$*-pretheories, then* $H^*\colon \mathbf{Mod}_c(\mathcal{S}) \to \mathbf{Mod}_c(\mathcal{T})$ *is a strictly monadic right adjoint.*

Proof. (i) follows from Lemma 5 and the description in (2.3) of $\mathbf{Mod}_c(\mathcal{T}) \to \mathcal{E}$ as a pullback. For (ii), applying the standard cancellativity properties to the pullbacks defining $\mathbf{Mod}_c(\mathcal{S})$ and $\mathbf{Mod}_c(\mathcal{T})$ yields a pullback square

$$\begin{tikzcd}
\mathbf{Mod}_c(\mathcal{S}) \arrow[r, "P_\mathcal{S}"] \arrow[d, "H^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{S}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[H^{\mathrm{op}},1]}"] \\
\mathbf{Mod}_c(\mathcal{T}) \arrow[r, "P_\mathcal{T}"] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \ .
\end{tikzcd}$$

Since $[H^{\mathrm{op}}, 1]$ is strictly monadic and $P_\mathcal{T}$ is a right adjoint between locally presentable categories, the result follows again from Lemma 5. □

Composing with the equivalence $\mathbf{Mod}_c(\mathcal{T}) \simeq \mathbf{Mod}(\mathcal{T})$ of Proposition 25, this result immediately implies the local presentability of the category $\mathbf{Mod}(\mathcal{T})$ of non-concrete models. Likewise, in the non-concrete setting, the analogue of Proposition 27 remains true on replacing "strict monadicity" by "monadicity" throughout. On the other hand, taken together with Proposition 23, the preceding result immediately implies the corresponding one for nervous monads. We state this here as:

Proposition 28.

(i) If T is an $\mathcal{A}$-nervous monad then $\mathcal{E}^\mathsf{T}$ is locally presentable, and $U^\mathsf{T} : \mathcal{E}^\mathsf{T} \to \mathcal{E}$ is a strictly monadic right adjoint.

(ii) If $\alpha : \mathsf{T} \to \mathsf{S}$ is a map of $\mathcal{A}$-nervous monads, then $\alpha^* : \mathcal{E}^\mathsf{S} \to \mathcal{E}^\mathsf{T}$ is a strictly monadic right adjoint.

5.4. Algebraic colimits of monads and theories

To conclude this section, we examine the interaction of the semantics functors with colimits. We begin with the more-or-less classical case of the semantics functor for monads $\mathrm{Alg} : \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$.

In general, $\mathbf{Mnd}(\mathcal{E})$ need not be cocomplete. Indeed, when $\mathcal{V} = \mathcal{E} = \mathbf{Set}$, it does not even have all binary coproducts; see [5, Proposition 6.10]. However many colimits of monads do exist, and an important point about these is that, in the terminology of [16], they are *algebraic*. That is, they are sent to limits by the semantics functor $\mathrm{Alg} : \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$.

To prove this, we use the following lemma, which is a mild variant of the standard result that right adjoints preserve limits.

Lemma 29. *Let $\mathcal{C}$ be a complete (ordinary) category with a strongly generating class of objects X and consider a functor $U : \mathcal{A} \to \mathcal{C}$. If each $x \in X$ admits a reflection along U then U preserves any limits that exist in $\mathcal{A}$.*

Proof. As X is a strong generator, the functors $\mathcal{C}(x, –)$ with $x \in X$ jointly reflect isomorphisms, and so jointly reflect limits. Accordingly U preserves any limits that are preserved by $\mathcal{C}(x, U-)$ for each $x \in X$. But each $\mathcal{C}(x, U-)$ is representable and so preserves all limits; whence U preserves any limits that exist. □

In the setting of **Set**-enriched categories the following result, which expresses the algebraicity of colimits of monads, is a special case of Proposition 26.3 of [16].

Proposition 30. Alg: $\mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ *preserves limits.*

Proof. The $\mathcal{V}$-functors $F : \mathcal{X} \to \mathcal{E}$ with small domain form a strong generator for $\mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$. Moreover, it is shown in [12, Theorem II.1.1] that each such F has a reflection along Alg: $\mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ given by its codensity monad $\mathrm{Ran}_F(F) : \mathcal{E} \to \mathcal{E}$. The result thus follows from Lemma 29. □

We now adapt the above results concerning $\mathbf{Mnd}(\mathcal{E})$ to the cases of $\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$, $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ and $\mathbf{Th}_{\mathcal{A}}(\mathcal{E})$. In Theorem 38 below, we will see that these categories are locally presentable; in particular, and by contrast with $\mathbf{Mnd}(\mathcal{E})$, they are cocomplete. It is also not difficult to prove the cocompleteness directly.

Proposition 31. *Each of the semantics functors* Alg: $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$, $\mathrm{Mod}_c : \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ *and* $\mathrm{Mod}_c : \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ *preserves limits.*

Proof. These three functors are isomorphic to the respective composites:

$$\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\mathrm{incl}^{\mathrm{op}}} \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\mathrm{Alg}} \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E} \tag{5.3}$$

$$\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\Psi^{\mathrm{op}}} \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\mathrm{Alg}} \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E} \tag{5.4}$$

$$\mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\Psi^{\mathrm{op}}} \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\mathrm{Alg}} \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}\ ; \tag{5.5}$$

for (5.3) this is clear, while for (5.4) and (5.5) it follows from Proposition 23. The common second functor in each composite is limit-preserving by Proposition 30, while the first functor is limit-preserving in each case since it is the opposite of a left adjoint functor—by Corollary 22, Theorem 6 and Theorem 19 (taken together with Corollary 22) respectively. □

We leave it to the reader to formulate this result also for non-concrete models.

6. The monad–theory correspondence in practice

In this section, we return to the examples of our general setting described in Section 2.1, with the goal of describing as explicitly as possible what the $\mathcal{A}$-nervous monads, the $\mathcal{A}$-theories, and the corresponding models look like in each case. By way of these descriptions, we will re-find many of the monad–theory correspondences existing in the literature as instances of our main Theorem 19.

To obtain our explicit descriptions, we will require some further results which characterise $\mathcal{A}$-theories and $\mathcal{A}$-nervous monads in particular situations. We begin this section by describing these results.

6.1. Theories in the presheaf context

A number of the examples of our basic setting described in Section 3.1 arise in the following manner. We take $\mathcal{E} = [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ a presheaf category, and take $\mathcal{A}$ to be any full subcategory of $\mathcal{E}$ containing the representables. In this situation, we then have a factorisation

$$\mathcal{C} \xrightarrow{I} \mathcal{A} \xrightarrow{K} [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] = \mathcal{E} \tag{6.1}$$

of the Yoneda embedding. The Yoneda lemma implies that $Y \colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is dense, whereupon by Theorem 5.13 of [17], both I and K are too. In particular, K provides an instance of our basic setting; we will call this the *presheaf context*. Each of Examples 8(i), (iv), (v), (vi), and (viii) arise in this way.

Lemma 32. *In the presheaf context, we have $N_I \cong K$ and $N_K \cong \mathrm{Ran}_{I^{\mathrm{op}}}$. Moreover, a functor $F \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is a K-nerve just when it is the right Kan extension of its restriction along $I^{\mathrm{op}} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$.*

Proof. For the first isomorphism we calculate that

$$N_I(x) = \mathcal{A}(I-, x) \cong [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](KI-, Kx) = [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](Y-, Kx) \cong Kx \tag{6.2}$$

by full fidelity of K and the Yoneda lemma. For the second, since $\mathrm{Lan}_Y K \dashv N_K$ and $[I^{\mathrm{op}}, 1] \dashv \mathrm{Ran}_{I^{\mathrm{op}}}$ it suffices to show $\mathrm{Lan}_Y K \cong [I^{\mathrm{op}}, 1] \colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$. Since both are cocontinuous, it suffices to show $(\mathrm{Lan}_Y K)Y \cong [I^{\mathrm{op}}, 1]Y$, which follows since $(\mathrm{Lan}_Y K)Y \cong K \cong N_I = [I^{\mathrm{op}}, 1]Y$ using full fidelity of Y and (6.2). Finally, since I^{op} is fully faithful, $F \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is a right Kan extension along I^{op} just when it is the right Kan extension of its own restriction. Thus the final claim follows using the isomorphism $N_K \cong \mathrm{Ran}_{I^{\mathrm{op}}}$. $\square$

In this setting, we have practically useful characterisations of the $\mathcal{A}$-theories and their (non-concrete) models.

Proposition 33. *Let $J \colon \mathcal{A} \to \mathcal{T}$ be an $\mathcal{A}$-pretheory in the presheaf context (6.1).*

(i) A functor $F \colon \mathcal{T}^{\mathrm{op}} \to \mathcal{V}$ is a $\mathcal{T}$-model just when $FJ^{\mathrm{op}} \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is the right Kan extension of its restriction along $I^{\mathrm{op}} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$;
(ii) $J \colon \mathcal{A} \to \mathcal{T}$ is itself an $\mathcal{A}$-theory just when it is the pointwise left Kan extension of its restriction along $I \colon \mathcal{C} \to \mathcal{A}$.

Proof. (i) follows immediately from Lemma 32 since, by definition, F is a $\mathcal{T}$-model just when FJ^{op} is a K-nerve. For (ii), note that by Proposition 4.46 of [17], $J\colon \mathcal{A} \to \mathcal{T}$ is the pointwise left Kan extension of its restriction along I just when, for each $x \in \mathcal{T}$, the functor $\mathcal{T}(J-, x)\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is the right Kan extension of its restriction along I^{op}. By Lemma 32, this happens just when each $\mathcal{T}(J-, x)$ is a K-nerve—that is, just when J is a $\mathcal{A}$-theory. □

We can sharpen these results using Day's notion of *density presentation* [11]. The density of an ordinary functor $K\colon \mathcal{C} \to \mathcal{D}$ is often introduced as the assertion that each object of $\mathcal{D}$ is the colimit of a certain diagram in the image of K. It is this perspective that the notion of density presentation generalises.

A *family of colimits* Φ in the ordinary category $\mathcal{D}$ is a class of diagrams $(D_i\colon \mathcal{J}_i \to \mathcal{D})_{i \in I}$ each of which has a colimit in $\mathcal{D}$. In the enriched case, a *family of colimits* Φ in the $\mathcal{V}$-category $\mathcal{D}$ is a class of pairs $(W_i \in [\mathcal{J}_i^{\mathrm{op}}, \mathcal{V}], D_i\colon \mathcal{J}_i \to \mathcal{D})_{i \in I}$ such that each weighted colimit $W_i \star D_i$ exists in $\mathcal{D}$. In either case, a full replete subcategory $\mathcal{B}$ of $\mathcal{D}$ is *closed in $\mathcal{D}$ under Φ-colimits* if it contains the (weighted) colimit of any D_i in Φ whenever it contains each vertex of D_i. We say that $\mathcal{D}$ is the *closure of $\mathcal{B}$ under Φ-colimits* if the only replete full subcategory of $\mathcal{D}$ containing $\mathcal{B}$ and closed under Φ-colimits is $\mathcal{D}$ itself.

Now given a fully faithful $K\colon \mathcal{C} \to \mathcal{D}$, we say that a colimit in $\mathcal{D}$ is *K-absolute* if it is preserved by N_K, or equivalently, by each representable $\mathcal{D}(Kx, -)\colon \mathcal{D} \to \mathcal{V}$. If $\mathcal{D}$ is the closure of $\mathcal{C}$ under a family Φ of K-absolute colimits then Φ is said to be a *density presentation* for K. The nomenclature is justified by Theorem 5.19 of [17], which, among other things, says that the fully faithful K has a density presentation just when it is dense.

We will make use of density presentations in the presheaf context (6.1) with respect not to the dense K, but to the dense I. By Lemma 32 we have $N_I \cong K$, and so the I-absolute colimits are in this case those preserved by $K\colon \mathcal{A} \to \mathcal{E}$. We will see numerous instances of this situation in Section 6.3 below; we give a couple of examples now to clarify the ideas.

Examples 34.

(i) Example 8(i) corresponds to the presheaf context

$$1 \xrightarrow{I} \mathbb{F} \xrightarrow{K} \mathbf{Set} ,$$

and here I has a density presentation given by all *finite copowers of* $1 \in \mathbb{F}$; these are I-absolute since K preserves them. In fact, $\mathbb{F}$ has all finite coproducts and these are preserved by K, so that there is a larger density presentation given by *all finite coproducts* in $\mathbb{F}$.

(ii) Example 8(iv) yields the presheaf context below, wherein I has a density presentation given by the wide pushouts $[n] \cong [1] +_{[0]} [1] +_{[0]} \ldots +_{[0]} [1]$:

$$\mathbb{G}_1 \xrightarrow{I} \Delta_0 \xrightarrow{K} [\mathbb{G}_1{}^{\mathrm{op}}, \mathbf{Set}] \ .$$

The reason we care about density presentations is the following result, which comprises various parts of Theorem 5.29 of [17].

Proposition 35. *Let $K\colon \mathcal{C} \to \mathcal{D}$ be fully faithful and dense. The following are equivalent:*

(i) $F\colon \mathcal{D} \to \mathcal{E}$ is the pointwise left Kan extension of its restriction along K;
(ii) F sends Φ-colimits to colimits for any density presentation Φ of K;
(iii) F sends K-absolute colimits to colimits.

Combined with Proposition 33, this yields the desired sharper characterisation of the $\mathcal{A}$-theories and their models.

Theorem 36. *Let $J\colon \mathcal{A} \to \mathcal{T}$ be an $\mathcal{A}$-pretheory in the presheaf context* (6.1), *and let Φ be a density presentation for I.*

(i) A functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{V}$ is a $\mathcal{T}$-model just when $FJ^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ sends Φ-colimits in $\mathcal{A}$ to limits in $\mathcal{V}$;
(ii) $J\colon \mathcal{A} \to \mathcal{T}$ is an $\mathcal{A}$-theory just when it sends Φ-colimits to colimits.

6.2. Nervous monads, signatures and saturated classes

We now turn from characterisations for $\mathcal{A}$-theories to characterisations for $\mathcal{A}$-nervous monads. We know from Corollary 21 that a monad is $\mathcal{A}$-nervous just when it is isomorphic to $\Psi\mathcal{T}$ for some $\mathcal{A}$-pretheory $J\colon \mathcal{A} \to \mathcal{T}$, and the examples in Section 3 make it an intuitively reasonable idea that these are the monads which can be "presented by operations and equations with arities from $\mathcal{A}$".

Our first characterisation result makes this idea precise by exhibiting the category of $\mathcal{A}$-nervous monads as monadic over a category of *signatures*. We defer the proof of this result to Section 8.

Definition 37. The category $\mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ of *signatures* is the category $\mathcal{V}$-$\mathbf{CAT}(\mathrm{ob}\,\mathcal{A}, \mathcal{E})$. We write $V\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ for the functor sending T to $(Ta)_{a\in\mathcal{A}}$.

Theorem 38. *$V\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ has a left adjoint $F\colon \mathbf{Sig}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Mnd}(\mathcal{E})$ taking values in $\mathcal{A}$-nervous monads. Moreover:*

(i) The restricted functor $V\colon \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ is monadic;
(ii) A monad $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$ is $\mathcal{A}$-nervous if and only if it is a colimit in $\mathbf{Mnd}(\mathcal{E})$ of monads in the image of F;
(iii) Each of $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$, $\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$ and $\mathbf{Th}_{\mathcal{A}}(\mathcal{E})$ is locally presentable.

The idea behind this result originates in [20]. A signature $\Sigma \in \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ specifies for each $a \in \mathcal{A}$ an $\mathcal{E}$-object Σa of "operations of input arity a". The free monad $F\Sigma$ on this signature has as its algebras the Σ-*structures*: objects $X \in \mathcal{E}$ endowed with a function $\mathcal{E}(a, X) \to \mathcal{E}(\Sigma a, X)$ for each $a \in \mathcal{A}$. The above result implies that a monad $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$ is $\mathcal{A}$-nervous just when it admits a presentation as a coequaliser $F\Gamma \rightrightarrows F\Sigma \twoheadrightarrow \mathsf{T}$—that is, a presentation by a signature Σ of basic operations together with a family Γ of equations between derived operations.

We now turn to our second characterisation result for $\mathcal{A}$-nervous monads. This is motivated by the fact, noted in the introduction, that in many monad–theory correspondences the class of monads can be characterised by a colimit-preservation property. To reproduce this result in our setting, we require a closure property of the arities in the subcategory $\mathcal{A}$ which, roughly speaking, says that substituting $\mathcal{A}$-ary operations into $\mathcal{A}$-ary operations again yields $\mathcal{A}$-ary operations.

Definition 39. An endo-$\mathcal{V}$-functor $F : \mathcal{E} \to \mathcal{E}$ is called $\mathcal{A}$-*induced* if it is the pointwise left Kan extension of its restriction along K. We call $\mathcal{A}$ a *saturated class of arities* if $\mathcal{A}$-induced endofunctors of $\mathcal{E}$ are closed under composition.

Example 40. In the case of $K : \mathbb{F} \hookrightarrow \mathbf{Set}$, there is a density presentation for K given by *all filtered colimits in* $\mathbf{Set}$, so that by Proposition 35, an endofunctor $\mathbf{Set} \to \mathbf{Set}$ is $\mathbb{F}$-induced just when it preserves filtered colimits. Thus $\mathbb{F} \hookrightarrow \mathbf{Set}$ is a saturated class of arities.

Example 41. More generally, if Φ is a class of enriched colimit-types and $K : \mathcal{A} \to \mathcal{E}$ exhibits $\mathcal{E}$ as the free cocompletion of $\mathcal{A}$ under Φ-colimits, then there is a density presentation of K given by all Φ-colimits, and an endofunctor of $\mathcal{E}$ is K-induced just when it preserves Φ-colimits. Thus $\mathcal{A}$ is a saturated class of arities.

Example 42. Let $K : \mathcal{A} \hookrightarrow \mathbf{Set}$ be the inclusion of the one-object full subcategory $\mathcal{A}$ on the two-element set $2 = \{0, 1\}$. Since the dense generator 1 of $\mathbf{Set}$ is a retract of 2, and taking retracts does not change categories of presheaves, $\mathcal{A}$ is dense in $\mathbf{Set}$. We claim it does *not* give a saturated class of arities.

To see this, note first that $(–)^2 : \mathbf{Set} \to \mathbf{Set}$ is $\mathcal{A}$-induced, being a left Kan extension along K of the representable $\mathcal{A}(2, –) : \mathcal{A} \to \mathbf{Set}$. We claim that $(–)^2 \circ (–)^2$ is *not* $\mathcal{A}$-induced. For indeed, by the Yoneda lemma, any $X \in [\mathcal{A}, \mathbf{Set}]$ has an epimorphic cover by copies of the unique representable $\mathcal{A}(2, –)$. Since left Kan extension preserves epimorphisms, each $\mathrm{Lan}_K(X)$ admits an epimorphic cover by copies of $(–)^2$. But $(–)^2 \circ (–)^2 \cong (–)^4$ can admit no such cover, since the identity map on 4 does not factor through 2, and so cannot be $\mathcal{A}$-induced.

The proof of the following result will again be deferred to Section 8 below.

Theorem 43. *Let $\mathcal{A}$ be a saturated class of arities in $\mathcal{E}$. The following are equivalent properties of a monad $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$:*

(i) T *is $\mathcal{A}$-nervous;*
(ii) $T: \mathcal{E} \to \mathcal{E}$ *is $\mathcal{A}$-induced;*
(iii) $T: \mathcal{E} \to \mathcal{E}$ *preserves Φ-colimits for any density presentation Φ of K.*

6.3. The monad–theory equivalence in practice

We now apply our characterisation results to the examples of Section 2.1. In many cases, the explicit descriptions we obtain of the $\mathcal{A}$-nervous monads, the $\mathcal{A}$-theories, and their models will allow us to reconstruct a familiar monad–theory correspondence from the literature.

Examples 44. As before, we begin with the unenriched examples where $\mathcal{V} = \mathbf{Set}$.

(i) The case $\mathcal{E} = \mathbf{Set}$ and $\mathcal{A} = \mathbb{F}$ corresponds to the instance of the presheaf context described in Examples 34(i). Applied to the density presentations for I given there, Theorem 36 tells us that an $\mathbb{F}$-pretheory $J: \mathbb{F} \to \mathcal{T}$ is an $\mathbb{F}$-theory just when it preserves finite copowers of 1, or equally (using the larger density presentation) all finite coproducts. It thus follows that the $\mathbb{F}$-theories are the *Lawvere theories* of [24]. Moreover a functor $F: \mathcal{T}^{\mathrm{op}} \to \mathbf{Set}$ is a $\mathcal{T}$-model if and only if $FJ^{\mathrm{op}}: \mathbb{F}^{\mathrm{op}} \to \mathbf{Set}$ preserves finite products. Since, in this case, J also *reflects* finite coproducts, this happens just when $F: \mathcal{T}^{\mathrm{op}} \to \mathbf{Set}$ is itself finite-product-preserving, that is, just when F is a model of the Lawvere theory $\mathcal{T}$.

On the other hand, by Example 40, $\mathbb{F}$ is a saturated class of arities, and the $\mathbb{F}$-induced endofunctors are the finitary ones; so by Theorem 43, a monad on $\mathbf{Set}$ is $\mathbb{F}$-nervous just when it is finitary. Theorem 19 thus specialises to the classical *finitary monad–Lawvere theory* correspondence, while Theorem 26 recaptures its compatibility with semantics.

(ii) When $\mathcal{E}$ is locally finitely presentable and $\mathcal{A} = \mathcal{E}_f$, the category of K-nerves is, by [13, Kollar 7.9], the full subcategory of $[\mathcal{E}_f^{\mathrm{op}}, \mathbf{Set}]$ on the *finite-limit-preserving* functors. So an $\mathcal{E}_f$-pretheory $J: \mathcal{E}_f \to \mathcal{T}$ is an $\mathcal{E}_f$-theory just when each $\mathcal{T}(J{-}, a): {\mathcal{E}_f}^{\mathrm{op}} \to \mathbf{Set}$ preserves finite limits. By the Yoneda lemma, this happens just when J preserves finite colimits, so that the $\mathcal{E}_f$-theories are precisely [31]'s *Lawvere $\mathcal{E}$-theories*.

The concrete $\mathcal{T}$-models in this setting are exactly the models of [31, Definition 2.2]. The general $\mathcal{T}$-models are those functors $F: \mathcal{T}^{\mathrm{op}} \to \mathbf{Set}$ for which $FJ^{\mathrm{op}}: \mathcal{E}_f^{\mathrm{op}} \to \mathbf{Set}$ is a K-nerve, i.e., finite-limit-preserving; these are the more general models of [22, Definition 12], and the correspondence between the two notions in Proposition 25 recaptures Proposition 15 of [22].

On the monad side, since $K\colon \mathcal{E}_f \to \mathcal{E}$ exhibits $\mathcal{E}$ as the free filtered-colimit completion of $\mathcal{E}_f$, Example 41 and Theorem 43 imply that $\mathcal{E}_f$ is a saturated class, and that the $\mathcal{E}_f$-nervous monads are the finitary ones. So in this case, Theorem 19 and Corollary 24 reconstruct (the unenriched version of) the monad–theory correspondence given in [31, Theorem 5.2].

(iii) More generally, when $\mathcal{E}$ is locally λ-presentable and $\mathcal{A} = \mathcal{E}_\lambda$ is a skeleton of the full subcategory of λ-presentable objects, the $\mathcal{E}_\lambda$-theories are those pretheories $J\colon \mathcal{E}_\lambda \to \mathcal{T}$ which preserve λ-small colimits; the $\mathcal{T}$-models are functors $F\colon \mathcal{T}^{\mathrm{op}} \to \mathbf{Set}$ for which FJ^{op} preserves λ-small limits; and the $\mathcal{E}_\lambda$-nervous monads are those whose endofunctor preserves λ-filtered colimits.

(iv) When $\mathcal{E} = [\mathbb{G}_1^{\mathrm{op}}, \mathbf{Set}]$ and $\mathcal{A} = \Delta_0$, we are in the presheaf context of Examples 34(ii). For the density presentation for I given there, Theorem 36 tells us that a pretheory $J\colon \Delta_0 \to \mathcal{T}$ is a Δ_0-theory just when it preserves each of the wide pushouts $[n] \cong [1] +_{[0]} [1] +_{[0]} \ldots +_{[0]} [1]$. Moreover, a functor $X\colon \mathcal{T}^{\mathrm{op}} \to \mathbf{Set}$ is a $\mathcal{T}$-model just when it sends each of these wide pushouts to a limit in $\mathbf{Set}$. This is precisely the *Segal condition* of [33]; in elementary terms, it requires the invertibility of each canonical map

$$Xn \longrightarrow X1 \times_{X0} X1 \times_{X0} \cdots \times_{X0} X1\ . \tag{6.3}$$

In Corollary 49 below we will see that Δ_0 is *not* a saturated class of arities, and so we have no more direct characterisations of the Δ_0-nervous monads than is given by Corollary 21 or Theorem 38. However, Example 11 provides us with natural examples of Δ_0-nervous monads: namely, the monads T and T_g for *categories* and for *groupoids* on $[\mathbb{G}_1^{\mathrm{op}}, \mathbf{Set}]$. As was already noted in [35], the nervosity of T recaptures the classical nerve theorem relating categories and simplicial sets. Indeed, the Δ_0-theory associated to T is the first part of the (bijective-on-objects, fully faithful) factorisation

$$\Delta_0 \xrightarrow{J_\mathsf{T}} \Delta \xrightarrow{K_\mathsf{T}} \mathbf{Cat}$$

of the composite $F_\mathsf{T} K\colon \Delta_0 \to \mathbf{Cat}$. The interposing object here is the topologist's simplex category Δ, with K_T the standard inclusion into $\mathbf{Cat}$. Thus, to say that T is Δ_0-nervous is to say that:

(a) The classical nerve functor $N_{K_\mathsf{T}}\colon \mathbf{Cat} \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful;
(b) The essential image of N_{K_T} comprises those $X \in [\Delta^{\mathrm{op}}, \mathbf{Set}]$ for which XJ_T is a K-nerve.

This much is already done in [35], but our use of density presentations allows for a small improvement. To say that XJ_T is a K-nerve in (b) is equally to say that X is a $\mathcal{T}$-model, or equally that X satisfies the Segal condition expressed by the invertibility of each (6.3). This is a mild sharpening of [35], where the "Segal condition" is left in the abstract form given in (b) above.

In a similar way, the nervosity of the monad T_g for small groupoids captures the "symmetric nerve theorem". This states that the functor $\mathbf{Gpd} \to [\mathbb{F}_+^{\mathrm{op}}, \mathbf{Set}]$ sending a groupoid to its *symmetric nerve*—indexed by the category of non-empty finite sets—is fully faithful, and characterises the essential image once again as the functors satisfying the Segal condition (6.3).

(v) With $\mathcal{E} = [\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ and $\mathcal{A} = \Theta_0$, we are now in the presheaf context

$$\mathbb{G} \xrightarrow{I} \Theta_0 \xrightarrow{K} [\mathbb{G}^{\mathrm{op}}, \mathbf{Set}] \ .$$

I has a density presentation given by the *I-globular sums*

$$(n_1, \dots, n_k) \cong (n_1) +_{(n_2)} +(n_3) + \dots +_{(n_{k-1})} (n_k)$$

in Θ_0; whence by Theorem 36, a pretheory $J\colon \Theta_0 \to \mathcal{T}$ is a Θ_0-theory when it preserves these I-globular sums—that is, when it is a *globular theory* in the sense of [8].[2] A functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathbf{Set}$ is a $\mathcal{T}$-model when it sends I-globular sums to limits, thus when each map

$$X\vec{n} \longrightarrow Xn_1 \times_{Xn_2} Xn_3 \times_{Xn_4} \dots \times_{Xn_{k-1}} Xn_k$$

is invertible. Once again, Θ_0 is *not* a saturated class of arities, and so there is no direct characterisation of the Θ_0-nervous monads; however, their interaction with Θ_0-theories is important in the literature on globular approaches to higher category theory, as we now outline.

Globular theories can describe structures on globular sets such as strict or weak ω-categories and ω-groupoids. For the strict variants, we pointed out in Section 3.2 that these may be modelled by Θ_0-pretheories; and since reflecting a pretheory $\mathcal{T}$ into a theory $\Phi\Psi\mathcal{T}$ does not change the models, it is immediate that there are Θ_0-theories modelling these structures too.

The original definition of globular weak ω-category was given by Batanin in [7]; he defines them be globular sets equipped with algebraic structure controlled by a *globular operad*. Globular operads can be understood as certain cartesian monads on globular sets. Berger [8] introduced globular theories and described the passage from a globular operad T to a globular theory Θ_T just as in Section 2.4 above. In our language, his Theorem 1.17 states exactly that each globular operad T is Θ_0-nervous, so that algebras for the globular operad are the same as models of the associated theory Θ_T. In particular, Batanin's weak ω-categories are the models of a globular theory.[3] On the other hand, Grothendieck weak ω-groupoids [27] are, *by definition*, models for certain globular theories called *coherators*.

[2] The definition of globular theory in [8] has the extra condition, satisfied in most cases, that J be a faithful functor.

[3] As an aside, we note that a complete understanding of those globular theories corresponding to globular operads was obtained in Theorem 6.6.8 of [2]. See also Section 3.12 of [9].

We now proceed to our examples over a more general base for enrichment $\mathcal{V}$.

(vi) With $\mathcal{V} = \mathcal{E}$ a locally finitely presentable symmetric monoidal category and with $\mathcal{A} = \mathcal{V}_f$, we are in the presheaf context

$$\mathcal{I} \xrightarrow{I} \mathcal{V}_f \xrightarrow{K} \mathcal{V} ,$$

wherein I has a density presentation given by the class of all *finite tensors*—tensors by finitely presentable objects of $\mathcal{V}$. Thus by Theorem 36, the $\mathcal{V}_f$-theories are the pretheories $J\colon \mathcal{V}_f \to \mathcal{T}$ which preserve finite tensors, which are precisely the *Lawvere $\mathcal{V}$-theories* of [32, Definition 3.1]. Furthermore, like in (i), a functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{V}$ is a $\mathcal{T}$-model just when it preserves finite cotensors, just as in Definition 3.2 of [32]. On the other hand, $\mathcal{V}_f \to \mathcal{V}$ exhibits $\mathcal{V}$ as the free filtered-colimit completion of $\mathcal{V}_f$; whence by Example 41 it is a saturated class of arities, and by Theorem 43 the $\mathcal{V}_f$-nervous monads are again the finitary ones. So Theorems 19 and 26 specialise to Theorems 4.3, 3.4 and 4.2 of [32].

(vii) Now taking $\mathcal{E}$ to be any locally finitely presentable $\mathcal{V}$-category and $\mathcal{A} = \mathcal{E}_f$, we may argue as in (ii) to recapture the fully general enriched monad–theory correspondence of [31], and its interaction with semantics.

(viii) Now suppose we are in the situation of Examples 8(viii), provided with a class Φ of enriched colimit-types satisfying Axiom A of [23]. With $\mathcal{E} = \mathcal{V}$ and $\mathcal{A} = \mathcal{V}_\Phi$, we are now in the presheaf context

$$\mathcal{I} \xrightarrow{I} \mathcal{V}_\Phi \xrightarrow{K} \mathcal{V} .$$

By [17, Theorem 5.35], I has a density presentation given by Φ-tensors (i.e., tensors by objects in Φ) while by [23, Theorem 3.1], K exhibits $\mathcal{V}$ as the free Φ-flat cocompletion of $\mathcal{V}_\Phi$. Arguing as in the preceding parts, we see that $\mathcal{V}_\Phi$-theories are pretheories $J\colon \mathcal{V}_\Phi^{\mathrm{op}} \to \mathcal{T}$ which preserve Φ-tensors, that $\mathcal{T}$-models are Φ-tensor-preserving functors $F\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{V}$, and that a monad is $\mathcal{V}_\Phi$-nervous if its underlying endofunctor preserves Φ-flat colimits. This sharpens slightly the results obtained in [23] in the special case $\mathcal{E} = \mathcal{V}$.

(ix) Finally, in the situation of Examples 8(ix), we find that the $\mathcal{A}$-theories are the Φ-colimit preserving pretheories $J\colon \mathcal{A} \to \mathcal{T}$; that the $\mathcal{T}$-models are functors $F\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{V}$ such that FJ^{op} preserves Φ-limits; and that a monad is $\mathcal{A}$-nervous just when it preserves Φ-flat colimits. In this way, our Theorems 19 and 26 reconstruct Theorems 7.6 and 7.7 of [23].

7. Monads with arities and theories with arities

In the introduction, we mentioned the general framework for monad–theory correspondences obtained in [35,9]. Similar to this paper, the basic setting involves a category

$\mathcal{E}$ and a small, dense subcategory $K\colon \mathcal{A} \hookrightarrow \mathcal{E}$; given these data, one defines notions of *monad with arities* $\mathcal{A}$ and *theory with arities* $\mathcal{A}$, and proves an equivalence between the two that is compatible with semantics.

In this section, we compare this framework with ours by comparing the classes of monads and of theories. We will see that our setting yields *strictly* larger classes of monads and theories which are better-behaved in practically useful ways. On the other hand, in the more restrictive setting of [35,9], checking that a monad or theory is in the required class may give greater combinatorial insight into the structure which it describes.

7.1. Monads with arities versus nervous monads

In [35,9] the authors work in the *unenriched* setting; the introduction to [9] states that the results "should be applicable" also in the enriched one. To ease the comparison to our results, we take it for granted that this is true, and transcribe their framework into the enriched context without further comment.

Another difference is that we assume local presentability of $\mathcal{E}$ while [35] assumes only *cocompleteness*, and [9] not even that. Given a small dense subcategory, there is no readily discernible difference between cocompleteness and local presentability[4]; however, cocompleteness is substantively different from nothing, so that in this respect [9]'s results are more general than ours. However all known applications are in the context of a locally presentable $\mathcal{E}$, and so we do not lose much in restricting to this context. In conclusion, when we make our comparison we will work in exactly the same general setting as in Section 2.1, and now have:

Definition 45. [35, Definition 4.1] An endofunctor $T\colon \mathcal{E} \to \mathcal{E}$ is said to *have arities* $\mathcal{A}$ if the composite $\mathcal{V}$-functor $N_K T\colon \mathcal{E} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is the left Kan extension of its own restriction along K. A monad $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$ is a *monad with arities* $\mathcal{A}$ if its underlying endofunctor has arities $\mathcal{A}$.

We consider the following way of restating this to be illuminating.

Proposition 46. *An endofunctor $T\colon \mathcal{E} \to \mathcal{E}$ has arities $\mathcal{A}$ if and only if it sends K-absolute colimits to K-absolute colimits. In particular, each endofunctor with arities $\mathcal{A}$ is $\mathcal{A}$-induced.*

Proof. By Proposition 35, T has arities $\mathcal{A}$ just when $N_K T$ sends K-absolute colimits to colimits. Since N_K is fully faithful, it reflects colimits, and so T has arities $\mathcal{A}$ just when T sends K-absolute colimits to colimits which are preserved by N_K—that is, to K-absolute colimits.

[4] Indeed, if there were, then it would negate the large cardinal axiom known as *Vopěnka's principle* [1, Chapter 6].

For the second claim, recall from Definition 39 that an endofunctor $T: \mathcal{E} \to \mathcal{E}$ is $\mathcal{A}$-induced if it is the left Kan extension of its own restriction to $\mathcal{A}$, or equivalently, by Proposition 35, when it sends K-absolute colimits to colimits. $\square$

Recall also that we call a class of arities $\mathcal{A}$ *saturated* when $\mathcal{A}$-induced endofunctors are closed under composition. Example 42 shows that this condition is not always satisfied. In light of the preceding result, the endofunctors with arities $\mathcal{A}$ can be seen as a natural subclass of the $\mathcal{A}$-induced endofunctors for which composition-closure is *always* verified.

The reason that Weber introduced monads with arities was in order to prove his *nerve theorem* [35, Theorem 4.10], which in our language may be restated as:

Theorem 47. *Monads with arities $\mathcal{A}$ are $\mathcal{A}$-nervous.*

One may reasonably ask whether the classes of monads with arities and $\mathcal{A}$-nervous monads in fact coincide. In many cases, this is true; in particular, in the situation of Example 41, where $K: \mathcal{A} \to \mathcal{E}$ exhibits $\mathcal{E}$ as the free Φ-cocompletion of $\mathcal{A}$ for some class of colimit-types Φ. Indeed, this condition implies that a monad T is $\mathcal{A}$-nervous precisely when T sends Φ-colimits to Φ-colimits; since Φ-colimits are K-absolute, this in turn implies that $N_K T$ sends Φ-colimits to colimits, and so is the left Kan extension of its own restriction along K. So in this case, every $\mathcal{A}$-nervous monad has arities $\mathcal{A}$; so in particular, the two notions coincide in each of Examples 8(i), (ii), (iii), (vi), (vii), (viii) and (ix).

However, they do *not* coincide in general. That is, in some instances of our basic setting, there exist monads which are $\mathcal{A}$-nervous but do not have arities $\mathcal{A}$. We give three examples of this. The first two arise in the setting of Example 8(iv), and concern the monads for groupoids and involutive graphs respectively.

Proposition 48. *The monad T on* $\mathbf{Grph} := [\mathbb{G}_1^{\mathrm{op}}, \mathbf{Set}]$ *whose algebras are groupoids is Δ_0-nervous but does not have Δ_0-induced underlying endofunctor. It follows that T does not have arities Δ_0.*

Proof. From Example 11 we know that T is Δ_0-nervous. To see that T is not Δ_0-induced, consider the graph X with vertices and arrows as to the left in:

$$a \xrightarrow{r} b \xleftarrow{s} c \qquad\qquad \begin{array}{ccc} [0] & \xrightarrow{\tau} & [1] \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle s} \\ [1] & \xrightarrow{r} & X \end{array} \quad . \tag{7.1}$$

This X is equally the K-absolute pushout right above; so if T were Δ_0-induced then it would preserve this pushout. But $T[1] +_{T[0]} T[1]$ is the graph

$$\begin{tikzcd}
a \arrow[loop left, "1_a"] \arrow[r, shift right, "r"'] & b \arrow[l, shift right, "r^{-1}"'] \arrow[loop above, "1_b"] \arrow[r, shift left, "s^{-1}"] & c \arrow[l, shift left, "s"] \arrow[loop right, "1_c"]
\end{tikzcd}$$

wherein, in particular, there is no edge $a \to c$; while in TX we have $s^{-1} \circ r \colon a \to c$. So the pushout is not preserved. This shows that T is not Δ_0-induced and so, by Proposition 46, that T does not have arities Δ_0. $\square$

Since the above result exhibits a Δ_0-nervous monad whose underlying endofunctor is not Δ_0-induced, we can apply Theorem 43 to deduce:

Corollary 49. $K \colon \Delta_0 \hookrightarrow \mathbf{Grph}$ *is not a saturated class of arities.*

Our second example, originally due to Melliès [29, Appendix III], shows that even monads with Δ_0-induced endofunctor need not have arities Δ_0. In this example, we call a graph $s, t \colon X_1 \rightrightarrows X_0$ *involutive* if it comes endowed with an order-2 automorphism $i \colon X_1 \to X_1$ reversing source and target, i.e., with $si = t$ (and hence also $ti = s$).

Proposition 50. *The monad* T *on* $\mathbf{Grph} := [\mathbb{G}_1^{\mathrm{op}}, \mathbf{Set}]$ *whose algebras are involutive graphs is* Δ_0*-nervous and has* Δ_0*-induced underlying endofunctor, but does not have arities* Δ_0.

Proof. The value of T at $s, t \colon X_1 \rightrightarrows X_0$ is given by $\langle s, t\rangle, \langle t, s\rangle \colon X_1 + X_1 \rightrightarrows X_0$. It follows that T is *cocontinuous* and so certainly Δ_0-induced. To see it does not have arities Δ_0, consider again the graph (7.1) and its K-absolute pushout presentation. If this were preserved by $N_K T \colon \mathbf{Grph} \to [\Delta_0^{\mathrm{op}}, \mathbf{Set}]$ then, on evaluating at $[2]$, the maps $\mathbf{Grph}([2], T[1]) \rightrightarrows \mathbf{Grph}([2], TX)$ given by postcomposition with Tr and Ts would be jointly surjective. To show this is not so, consider the map $f \colon [2] \to TX$ picking out the composable pair $(r \colon a \to b, i(s) \colon b \to c)$. Since neither Tr nor Ts are surjective on objects, the bijective-on-objects f cannot factor through either of them. This shows that T does not have arities Δ_0. $\square$

Our final example shows that not even free monads on $\mathcal{A}$-signatures—which are $\mathcal{A}$-nervous by Theorem 38 above—need necessarily have arities $\mathcal{A}$.

Proposition 51. *Let* $\mathcal{V} = \mathcal{E} = \mathbf{Set}$ *and let* $\mathcal{A}$ *be the one-object full subcategory on a two-element set. The free monad on the terminal* $\mathcal{A}$*-signature does not have* $\mathcal{A}$*-induced underlying endofunctor and therefore does not have arities* $\mathcal{A}$.

Proof. The algebras for the free monad T on the terminal signature are sets equipped with a binary operation. Elements of the free T-algebra on X are binary trees with leaves labelled by elements of X, yielding the formula

$$TX = \textstyle\sum_{n \in \mathbb{N}} C_n \times X^{n+1}$$

where C_n is the nth Catalan number. In particular, T contains at least one coproduct summand $(-)^4$ and so, as in Example 42, is not $\mathcal{A}$-induced; in particular, by Proposition 46, it does not have arities $\mathcal{A}$. $\square$

7.2. Theories with arities $\mathcal{A}$ versus $\mathcal{A}$-theories

The paper [9] introduced *theories with arities* $\mathcal{A}$. These are $\mathcal{A}$-pretheories $J\colon \mathcal{A} \to \mathcal{T}$ for which the composite

$$[\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \xrightarrow{\mathrm{Lan}_J} [\mathcal{T}^{\mathrm{op}}, \mathcal{V}] \xrightarrow{[J^{\mathrm{op}},1]} [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \tag{7.2}$$

takes K-nerves to K-nerves. This functor takes the representable $\mathcal{A}(-, x)$ to $\mathcal{T}(J-, x)$, so that in this language, we may describe the $\mathcal{A}$-theories as the pretheories for which (7.2) takes each representable to a K-nerve. It follows that:

Proposition 52. *Theories with arities $\mathcal{A}$ are $\mathcal{A}$-theories.*

Proof. It suffices to observe that each representable $\mathcal{A}(-, x)$ is a K-nerve since $\mathcal{A}(-, x) \cong \mathcal{E}(K-, Kx) = N_K(Kx)$. $\square$

Theorem 3.4 of [9] establishes an equivalence between the categories of monads with arities $\mathcal{A}$ and of theories with arities $\mathcal{A}$. The functor taking a monad with arities to the corresponding theory with arities is defined in the same way as the Φ of Section 2.4, and so it follows that:

Proposition 53. *The equivalence of monads with arities $\mathcal{A}$ and theories with arities $\mathcal{A}$ is a restriction of the equivalence between $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories.*

In particular, there exist $\mathcal{A}$-theories which are not theories with arities $\mathcal{A}$; it is this statement which was verified in [29, Appendix III].

7.3. Colimits of monads with arities

In Theorem 38 we saw that the $\mathcal{A}$-nervous monads are the closure of the free monads on $\mathcal{A}$-signatures under colimits in $\mathbf{Mnd}(\mathcal{E})$. Since colimits of monads are algebraic, this allows us to give intuitive presentations for $\mathcal{A}$-nervous monads as suitable colimits of frees. The pretheory presentations of Section 3 can be understood as particularly direct descriptions of such colimits.

Since not every $\mathcal{A}$-nervous monad has arities $\mathcal{A}$, the monads with arities are *not* the colimit-closure of the frees on signatures. We already saw one explanation for this in Proposition 51: the free monads on signatures need not have arities. However, this leaves open the possibility that the monads with arities $\mathcal{A}$ are the colimit-closure of some smaller class of basic monads—which would allow for the same kind of intuitive presentation as we have for $\mathcal{A}$-nervous monads. The following result shows that even this is not the case.

Theorem 54. *Monads with arities $\mathcal{A}$ need not be closed in $\mathbf{Mnd}(\mathcal{E})$ under colimits.*

Proof. We saw in Proposition 50 that, when $\mathcal{E} = \mathbf{Grph}$ and $\mathcal{A} = \Delta_0$, the monad T for involutive graphs does not have arities Δ_0. To prove the result it will therefore suffice to exhibit T as a colimit in $\mathbf{Mnd}(\mathbf{Grph})$ of a diagram of monads with arities Δ_0. This diagram will be a coequaliser involving a pair of monads P and Q, whose respective algebras are:

- For P: graphs X endowed with a function $u\colon X_1 \to X_0$;
- For Q: graphs X endowed with an order-2 automorphism $i\colon X_1 \to X_1$.

We construct this coequaliser of monads in terms of the categories of algebras. The category $\mathbf{Grph}^{\mathsf{T}}$ of involutive graphs is an equaliser in $\mathbf{CAT}$ as to the left in:

$$\mathbf{Grph}^{\mathsf{T}} \overset{E}{\rightarrowtail} \mathbf{Grph}^{\mathsf{Q}} \underset{G}{\overset{F}{\rightrightarrows}} \mathbf{Grph}^{\mathsf{P}} \qquad\qquad \mathsf{P} \underset{\gamma}{\overset{\varphi}{\rightrightarrows}} \mathsf{Q} \overset{\varepsilon}{\twoheadrightarrow} \mathsf{T}$$

where the functors F and G send a Q-algebra (X, i) to the respective P-algebras (X, si) and (X, t). Since each of these functors commutes with the forgetful functors to $\mathbf{Grph}$, we have an equaliser of forgetful functors in $\mathbf{CAT}/\mathbf{Grph}$. Since the functor $\mathrm{Alg}\colon \mathbf{Mnd}(\mathbf{Grph})^{\mathrm{op}} \to \mathbf{CAT}/\mathbf{Grph}$ is fully faithful, this equaliser must be the image of a coequaliser diagram in $\mathbf{Mnd}(\mathbf{Grph})$ as right above.

It remains to show that in this coequaliser presentation both P and Q have arities Δ_0. By Proposition 35, this means showing that $N_K P$ and $N_K Q$ send K-absolute colimits to colimits, or equally, that each $\mathbf{Grph}([n], P\text{–})$ and $\mathbf{Grph}([n], Q\text{–})$ sends K-absolute colimits to colimits. To see this, we calculate P and Q explicitly. On the one hand, the free P-algebra on a graph X is obtained by freely adjoining an element $u(f)$ to X_0 for each $f \in X_1$. On the other hand, the free Q-algebra on X is obtained by freely adjoining an element $i(f) \in X_1$ for each $f \in X_1$. Thus we have

$$PX = X + X_1 \cdot [0] \qquad \text{and} \qquad QX = X + X_1 \cdot [1]$$

where we use $\cdot$ to denote copower. Since each $[n] \in \mathbf{Grph}$ is connected, and since each hom-set $\mathbf{Gph}([n], [m])$ has cardinality $\max(0, m - n + 1)$, we conclude that

$$\mathbf{Grph}([n], PX) = \begin{cases} \mathbf{Grph}([0], X) + \mathbf{Grph}([1], X) & \text{if } n = 0; \\ \mathbf{Grph}([n], X) & \text{if } n > 0. \end{cases}$$

$$\mathbf{Grph}([n], QX) = \begin{cases} \mathbf{Grph}([0], X) + 2 \cdot \mathbf{Grph}([1], X) & \text{if } n = 0; \\ \mathbf{Grph}([1], X) + \mathbf{Grph}([1], X) & \text{if } n = 1; \\ \mathbf{Grph}([n], X) & \text{if } n > 1. \end{cases} \tag{7.3}$$

Now by definition, N_K sends K-absolute colimits to colimits, whence also each $\mathbf{Grph}([n], –)\colon \mathbf{Grph} \to \mathbf{Set}$. The functors with this property are closed under colimits in $[\mathbf{Grph}, \mathbf{Set}]$, and so (7.3) ensures that each $\mathbf{Grph}([n], P–)$ and $\mathbf{Grph}([n], Q–)$ sends K-absolute colimits to colimits as desired. □

It is not even clear to us if the category of monads with arities $\mathcal{A}$ is always cocomplete. The argument for local presentability of $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ in Theorem 38 does not seem to adapt to the case of monads with arities, and no other obvious argument presents itself. In any case, the preceding result shows that, even if the category of monads with arities does have colimits, they do not always coincide with the usual colimits of monads, and, in particular, are not always *algebraic*. This dashes any hope we might have had of giving a sensible notion of presentation for monads with arities.

8. Deferred proofs

8.1. Identifying the monads

In this section, we complete the proofs of the results deferred from Section 6 above, beginning with Theorem 38. Recall that the category $\mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ of *signatures* is the (ordinary) category $\mathcal{V}\text{-}\mathbf{CAT}(\mathrm{ob}\,\mathcal{A}, \mathcal{E})$, and that $V\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ is the functor sending T to $(Ta)_{a \in \mathcal{A}}$.

Proposition 55. *$V\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ has a left adjoint F which takes values in $\mathcal{A}$-nervous monads.*

Proof. We can decompose V as the composite

$$\mathbf{Mnd}(\mathcal{E}) \xrightarrow{V_1} \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{E}, \mathcal{E}) \xrightarrow{V_2} \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$$

where V_1 takes the underlying endofunctor, and V_2 is given by evaluation at each $a \in \mathrm{ob}\,\mathcal{A}$. Since V_2 is equally given by restriction along $\mathrm{ob}\,\mathcal{A} \to \mathcal{A} \to \mathcal{E}$, it has a left adjoint F_2 given by pointwise left Kan extension, with the explicit formula:

$$F_2(\Sigma) = \textstyle\sum_{a \in \mathcal{A}} \mathcal{E}(Ka, –) \cdot \Sigma a \colon \mathcal{E} \to \mathcal{E} \ ,$$

where $\cdot$ denotes $\mathcal{V}$-enriched copower. So it suffices to show that the free monad on each endofunctor $F_2(\Sigma)$ exists and is $\mathcal{A}$-nervous. By [16, Theorem 23.2], such a free monad

T is characterised by the property that $\mathcal{E}^{F_2(\Sigma)} \cong \mathcal{E}^{\mathsf{T}}$ over $\mathcal{E}$, where on the left we have the $\mathcal{V}$-category of algebras for the mere endofunctor $F_2(\Sigma)$. Thus, to complete the proof, it suffices by Theorem 6 to exhibit $\mathcal{E}^{F_2(\Sigma)}$ as isomorphic to the $\mathcal{V}$-category of concrete models of some $\mathcal{A}$-pretheory.

To this end, we let $\mathcal{B}$ be the *collage* of the $\mathcal{V}$-functor $N_K\Sigma\colon \operatorname{ob}\mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$. Thus $\mathcal{B}$ is the $\mathcal{V}$-category with object set $\operatorname{ob}\mathcal{A} + \operatorname{ob}\mathcal{A}$ and the following hom-objects, where we write $\ell, r\colon \operatorname{ob}\mathcal{A} \to \operatorname{ob}\mathcal{B}$ for the two injections:

$$\begin{aligned} \mathcal{B}(\ell a', \ell a) &= \mathcal{A}(a', a) & \qquad \mathcal{B}(ra', ra) &= (\operatorname{ob}\mathcal{A})(a', a) \\ \mathcal{B}(\ell a', ra) &= \mathcal{E}(Ka', \Sigma a) & \qquad \mathcal{B}(ra', \ell a) &= 0\ . \end{aligned}$$

Let $\ell\colon \mathcal{A} \to \mathcal{B}$ and $r\colon \operatorname{ob}\mathcal{A} \to \mathcal{B}$ be the two injections into the collage, and now form the pushout $J\colon \mathcal{A} \to \mathcal{T}$ of $\langle \ell, r\rangle\colon \mathcal{A} + \operatorname{ob}\mathcal{A} \to \mathcal{B}$ along $\langle 1, \iota\rangle\colon \mathcal{A} + \operatorname{ob}\mathcal{A} \to \mathcal{A}$. Since $\langle \ell, r\rangle$ is identity-on-objects, so is $J\colon \mathcal{A} \to \mathcal{T}$, and so we have an $\mathcal{A}$-pretheory. To conclude the proof, it now suffices to show that $\mathcal{E}^{F_2(\Sigma)} \cong \mathbf{Mod}_c(\mathcal{T})$ over $\mathcal{E}$.

By the universal property of the collage and the pushout, to give a functor $H\colon \mathcal{T} \to \mathcal{X}$ is equally to give a functor $F = HJ\colon \mathcal{A} \to \mathcal{X}$ together with $\mathcal{V}$-natural transformations $\alpha_a\colon \mathcal{E}(K{-}, \Sigma a) \Rightarrow \mathcal{X}(F{-}, Fa)$ for each $a \in \operatorname{ob}\mathcal{A}$. In particular, taking $\mathcal{X} = \mathcal{V}^{\mathrm{op}}$ and $F = \mathcal{E}(K{-}, X)$, we see that a concrete $\mathcal{T}$-model structure on $X \in \mathcal{E}$ is given by an $\operatorname{ob}\mathcal{A}$-indexed family of $\mathcal{V}$-natural transformations

$$\alpha_a\colon \mathcal{E}(K{-}, \Sigma a) \Rightarrow [\mathcal{E}(Ka, X), \mathcal{E}(K{-}, X)]$$

or equally under transpose, by a family of maps

$$\mathcal{E}(Ka, X) \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](\mathcal{E}(K{-}, \Sigma a), \mathcal{E}(K{-}, X))\ .$$

By full fidelity of N_K, the right-hand side above is isomorphic to $\mathcal{E}(\Sigma a, X)$, and so concrete $\mathcal{T}$-model structure on X is equally given by a family of maps $\mathcal{E}(Ka, X) \to \mathcal{E}(\Sigma a, X)$. Finally, using the universal properties of copowers and coproducts, this is equivalent to giving a single map

$$\bar{\alpha}\colon \textstyle\sum_{a \in \mathcal{A}} \mathcal{E}(Ka, X) \cdot \Sigma a \to X$$

exhibiting X as an $F_2(\Sigma)$-algebra. We thus have a bijection over $\mathcal{E}$ between objects of $\mathcal{E}^{F_2(\Sigma)}$ and objects of $\mathbf{Mod}_c(\mathcal{T})$.

A similar analysis shows that a morphism $A \to \mathcal{E}(X, Y)$ in $\mathcal{V}$ lifts through the monomorphism $\mathbf{Mod}_c(\mathcal{T})((X, \alpha), (Y, \beta)) \to \mathcal{E}(X, Y)$ if and only if it lifts through the monomorphism $\mathcal{E}^{F_2(\Sigma)}((X, \bar{\alpha}), (Y, \bar{\beta})) \to \mathcal{E}(X, Y)$. It follows that we have an isomorphism of $\mathcal{V}$-categories $\mathcal{E}^{F_2(\Sigma)} \cong \mathbf{Mod}_c(\mathcal{T})$ over $\mathcal{E}$ as desired. $\square$

In proving the rest of Theorem 38, the following lemma will be useful.

Lemma 56. *Let $\mathcal{C}_1 \subseteq \mathcal{C}_2$ be replete, full, colimit-closed sub-$\mathcal{V}$-categories of $\mathcal{C}$; for example, they could be coreflective. If $V \colon \mathcal{C} \to \mathcal{D}$ has a left adjoint F taking values in $\mathcal{C}_1$, and the restriction $V|_{\mathcal{C}_2} : \mathcal{C}_2 \to \mathcal{D}$ is monadic, then $\mathcal{C}_1 = \mathcal{C}_2$.*

Proof. Since F takes values in $\mathcal{C}_1 \subseteq \mathcal{C}_2$, the left adjoint to $V|_{\mathcal{C}_2} : \mathcal{C}_1 \to \mathcal{D}$ is still given by F. So monadicity of $V|_{\mathcal{C}_2}$ means that each $X \in \mathcal{C}_2$ can be written as a coequaliser in $\mathcal{C}_2$, and hence also in $\mathcal{C}$, of objects in the image of F. Since $\operatorname{im} F \subseteq \mathcal{C}_1$ and since $\mathcal{C}_1$ is closed in $\mathcal{C}$ under colimits, it follows that $X \in \mathcal{C}_1$. □

Theorem 38. *$V \colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ has a left adjoint $F \colon \mathbf{Sig}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Mnd}(\mathcal{E})$ taking values in $\mathcal{A}$-nervous monads. Moreover:*

(i) The restricted functor $V \colon \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ is monadic;
(ii) A monad $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$ is $\mathcal{A}$-nervous if and only if it is a colimit in $\mathbf{Mnd}(\mathcal{E})$ of monads in the image of F;
(iii) Each of $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$, $\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$ and $\mathbf{Th}_{\mathcal{A}}(\mathcal{E})$ is locally presentable.

Proof. We begin with (i). Let $H \colon \mathbf{Preth}_{\mathcal{A}}(\mathcal{E}) \to \mathcal{V}\text{-}\mathbf{CAT}(\operatorname{ob}\mathcal{A}, [\mathcal{A}^{\mathrm{op}}, \mathcal{V}])$ be the functor sending a pretheory $J \colon \mathcal{A} \to \mathcal{T}$ to the family of presheaves $(\mathcal{T}(J-, Ja))_{a \in \mathcal{A}}$. Since an $\mathcal{A}$-pretheory is a theory just when each of these presheaves is a K-nerve, we have a pullback square as to the right in:

$$\begin{tikzcd}
\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) \arrow[r, "\Phi"] \arrow[d, "V"'] \arrow[dr, phantom, "\cong"] & \mathbf{Th}_{\mathcal{A}}(\mathcal{E}) \arrow[r, hook] \arrow[d, "P"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Preth}_{\mathcal{A}}(\mathcal{E}) \arrow[d, "H"] \\
\mathcal{V}\text{-}\mathbf{CAT}(\operatorname{ob}\mathcal{A}, \mathcal{E}) \arrow[r, "N_K \circ (-)"'] & \mathcal{V}\text{-}\mathbf{CAT}(\operatorname{ob}\mathcal{A}, K\text{-}\mathbf{Ner}) \arrow[r, hook] & \mathcal{V}\text{-}\mathbf{CAT}(\operatorname{ob}\mathcal{A}, [\mathcal{A}^{\mathrm{op}}, \mathcal{V}])\ .
\end{tikzcd} \tag{8.1}$$

Since $K\text{-}\mathbf{Ner} \hookrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is replete, this square is a pullback along a discrete isofibration, and so by [14, Corollary 1] also a bipullback. On the other hand, to the left, we have a pseudocommuting square as witnessed by the isomorphisms:

$$(PJ_{\mathsf{T}})(A) = \mathcal{A}_{\mathsf{T}}(J_{\mathsf{T}}-, J_{\mathsf{T}}A) = \mathcal{E}^{\mathsf{T}}(F^{\mathsf{T}}K-, F^{\mathsf{T}}KA) \cong \mathcal{E}(K-, TKA) = N_K(TKA)\ .$$

Since both horizontal edges of this square are equivalences, it is also a bipullback.

To show the required monadicity, we must prove that V creates V-absolute coequalisers. Since the large rectangle is a bipullback—as the pasting of two bipullbacks—it suffices to show that H creates H-absolute coequalisers. As the definition of H depends only on $\mathcal{A}$ and not $\mathcal{E}$, we lose no generality in proving this if we assume that $\mathcal{E} = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ and $K = Y$. In this case, *every* presheaf on $\mathcal{A}$ is a K-nerve, and so the horizontal composites in (8.1) are equivalences; and so, finally, it suffices to prove that V is monadic when $\mathcal{E} = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ and $K = Y$.

Note that, in this case, $\mathcal{A}$ is a saturated class of arities: for indeed, by the universal property of free cocompletion, a functor $F \colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is $\mathcal{A}$-induced if and only if it is *cocontinuous*. It thus follows from Proposition 58 below that the restriction $V_c\colon \mathbf{Mnd}_c(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ of V to cocontinuous monads is monadic; so we will be done if $\mathbf{Mnd}_c(\mathcal{E}) = \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$. In this case, $\Psi\colon \mathbf{Preth}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ sends $J\colon \mathcal{A} \to \mathcal{T}$ to a monad isomorphic to that induced by the adjunction $\mathrm{Lan}_J\colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \leftrightarrows [\mathcal{T}^{\mathrm{op}}, \mathcal{V}]\colon [J^{\mathrm{op}}, 1]$, and so $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) \subseteq \mathbf{Mnd}_c(\mathcal{E})$. To obtain equality, we apply Lemma 56. We have that:

- $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ and $\mathbf{Mnd}_c(\mathcal{E})$ are coreflective in $\mathbf{Mnd}(\mathcal{E})$ by Corollary 22 and Lemma 57 respectively;
- $V\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ has a left adjoint taking values in $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$;
- The restriction $V_c\colon \mathbf{Mnd}_c(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ is monadic;

and so $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) = \mathbf{Mnd}_c(\mathcal{E})$. This proves monadicity of V in the special case $\mathcal{E} = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, whence also, by the preceding argument, in the general case.

In order to prove (ii), we let $\mathcal{C}_1$ be the colimit-closure in $\mathbf{Mnd}(\mathcal{E})$ of the image of F. Since $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ contains this image and is colimit-closed, we have $\mathcal{C}_1 \subseteq \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) \subseteq \mathbf{Mnd}(\mathcal{E})$. Thus, applying Lemma 56 to this triple and $V\colon \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ gives $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) = \mathcal{C}_1$ as desired.

Finally we prove (iii). The monadicity of V above implies that of P and hence also of H (by taking $\mathcal{E} = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$). Since filtered colimits of $\mathcal{A}$-pretheories can be computed at the level of underlying *graphs*, the forgetful H preserves them; which is to say that $\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$ is *finitarily* monadic over the locally presentable $\mathcal{V}\text{-}\mathbf{CAT}(\mathrm{ob}\,\mathcal{A}, [\mathcal{A}^{\mathrm{op}}, \mathcal{V}])$, whence locally presentable by [13, Satz 10.3]. So in the right-hand and the large bipullback squares in (8.1), the bottom and right sides are right adjoints between locally presentable categories. Since by [10, Theorem 2.18], the 2-category of locally presentable categories and right adjoint functors is closed under bilimits in $\mathbf{CAT}$, we conclude that each $\mathbf{Th}_{\mathcal{A}}(\mathcal{E})$ and each $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ is also locally presentable. □

8.2. Saturated classes

We now turn to the deferred proof of Theorem 43. Recall the context: an endo-$\mathcal{V}$-functor $F\colon \mathcal{E} \to \mathcal{E}$ is called *$\mathcal{A}$-induced* when the pointwise left Kan extension of its restriction along K, and $\mathcal{A}$ is a *saturated class of arities* if $\mathcal{A}$-induced endofunctors of $\mathcal{E}$ are composition-closed.

We begin by recording the basic properties of this situation. We write $\mathcal{A}\text{-}\mathbf{End}(\mathcal{E})$ and $\mathcal{A}\text{-}\mathbf{Mnd}(\mathcal{E})$ for the full subcategories of $\mathbf{End}(\mathcal{E}) = \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{E}, \mathcal{E})$ and $\mathbf{Mnd}(\mathcal{E})$ on, respectively, the $\mathcal{A}$-induced endofunctors, and the monads with $\mathcal{A}$-induced underlying endofunctor.

Lemma 57. *$\mathcal{A}\text{-}\mathbf{End}(\mathcal{E})$ is coreflective in $\mathbf{End}(\mathcal{E}) = \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{E}, \mathcal{E})$ via the coreflector $R(F) = \mathrm{Lan}_K(FK)$, as on the left in:*

$$\mathcal{A}\text{-}\mathbf{End}(\mathcal{E}) \underset{I}{\overset{R}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} \mathbf{End}(\mathcal{E}) \qquad\qquad \mathcal{A}\text{-}\mathbf{Mnd}(\mathcal{E}) \underset{I}{\overset{R}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} \mathbf{Mnd}(\mathcal{E})\ . \tag{8.2}$$

*If $\mathcal{A}$ is a saturated class, then $\mathcal{A}$-**End**$(\mathcal{E})$ is right-closed monoidal, and the coreflection left above lifts to the corresponding categories of monads as on the right.*

Proof. Restriction and left Kan extension along the fully faithful K exhibits $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ as equivalent to $\mathcal{V}$-$\mathbf{CAT}(\mathcal{A}, \mathcal{E})$, whence locally presentable. Since restriction along K is a coreflector of $\mathbf{End}(\mathcal{E})$ into $\mathcal{V}$-$\mathbf{CAT}(\mathcal{A}, \mathcal{E})$, it follows that $R(F) = \mathrm{Lan}_K(FK)$ is a coreflector of $\mathbf{End}(\mathcal{E})$ into $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$.

If $\mathcal{A}$ is saturated then $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ is monoidal under composition. Since each endofunctor $(–) \circ F$ of $\mathbf{End}(\mathcal{E})$ is cocontinuous, and $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ is closed in $\mathbf{End}(\mathcal{E})$ under colimits, each endofunctor $(–) \circ F$ of $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ is cocontinuous, and so has a right adjoint by local presentability. Thus $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ is right-closed monoidal.

Furthermore, the inclusion of $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ into $\mathbf{End}(\mathcal{E})$ is strict monoidal, whence by [15, Theorem 1.5] the coreflection to the left of (8.2) lifts to a coreflection in the 2-category **MONCAT** of monoidal categories, lax monoidal functors and monoidal transformations. Applying the 2-functor $\mathbf{MONCAT}(1, –) \colon \mathbf{MONCAT} \to \mathbf{CAT}$ yields the coreflection to the right of (8.2). $\square$

The key step towards establishing Theorem 43 above is now:

Proposition 58. *The left adjoint F of $V \colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ takes values in $\mathcal{A}$-induced monads; furthermore, the restriction of V to $\mathcal{A}$-$\mathbf{Mnd}(\mathcal{E})$ is monadic.*

Proof. For any $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$, its $\mathcal{A}$-induced coreflection $\varepsilon_{\mathsf{T}} \colon IR(\mathsf{T}) \to \mathsf{T}$ has as underlying map in $\mathbf{End}(\mathcal{E})$ the component $\mathrm{Lan}_K(TK) \to T$ of the counit of the adjunction given by restriction and left Kan extension along K. Since K is fully faithful, the restriction of this map along K is invertible, whence in particular, $V\varepsilon \colon VIR \Rightarrow V \colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ is invertible. So $\eta \colon \mathrm{id} \Rightarrow VF$ factors through $V\varepsilon_F \colon VIRF \Rightarrow VF$ whence, by adjointness, $\mathrm{id} \colon F \Rightarrow F$ factors through ε_F. Therefore each $F(\Sigma)$ is a retract of $IRF(\Sigma)$; since $\mathcal{A}$-$\mathbf{Mnd}(\mathcal{E})$ is closed under colimits in $\mathbf{Mnd}(\mathcal{E})$, it is retract-closed and so each $F(\Sigma)$ belongs to $\mathcal{A}$-$\mathbf{Mnd}(\mathcal{E})$.

It remains to prove that the restriction of V to $\mathcal{A}$-$\mathbf{Mnd}(\mathcal{E})$ is monadic. To do so, we decompose this restriction as

$$\mathcal{A}\text{-}\mathbf{Mnd}(\mathcal{E}) \xrightarrow{V_1} \mathcal{A}\text{-}\mathbf{End}(\mathcal{E}) \xrightarrow{V_2} \mathbf{Sig}_{\mathcal{A}}(\mathcal{E}) \ ,$$

where V_1 forgets the monad structure and V_2 is given by precomposition with the functor $\mathrm{ob}\,\mathcal{A} \to \mathcal{A} \to \mathcal{E}$, and apply the following result, which is [21, Theorem 2]:

Theorem. *Let $\mathcal{M}$ be a right-closed monoidal category, and $V_2 \colon \mathcal{M} \to \mathcal{N}$ a monadic functor for which there exists a functor $\diamond \colon \mathcal{M} \times \mathcal{N} \to \mathcal{N}$ with natural isomorphisms $X \diamond VY \cong V(X \otimes Y)$. If the forgetful functor $V_1 \colon \mathbf{Mon}(\mathcal{M}) \to \mathcal{M}$ has a left adjoint, then the composite $V_2V_1 \colon \mathbf{Mon}(\mathcal{M}) \to \mathcal{N}$ is monadic.*

Indeed, by Lemma 57, $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ is a right-closed monoidal category, and $\mathcal{A}$-$\mathbf{Mnd}(\mathcal{E})$ the category of monoids therein. Under the equivalence $\mathcal{A}\text{-}\mathbf{End}(\mathcal{E}) \simeq \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A}, \mathcal{E})$, we may identify V_2 with precomposition along $\mathrm{ob}\,\mathcal{A} \to \mathcal{A}$. It is thus cocontinuous, and has a left adjoint given by left Kan extension; whence is monadic. Now since $V_2 V_1$ has a left adjoint and V_2 is monadic, it follows that V_1 also has a left adjoint. Finally, we have a functor

$$\diamond\colon \mathcal{A}\text{-}\mathbf{End}(\mathcal{E}) \times \mathbf{Sig}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$$

defined by $(F, G) \mapsto FG$, and this clearly has the property that $M(FG) = F \diamond M(G)$. So applying the above theorem yields the desired monadicity. □

We are now ready to prove:

Theorem 43. *Let $\mathcal{A}$ be a saturated class of arities in $\mathcal{E}$. The following are equivalent properties of a monad $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$:*

(i) T is $\mathcal{A}$-nervous;
(ii) $T\colon \mathcal{E} \to \mathcal{E}$ is $\mathcal{A}$-induced;
(iii) $T\colon \mathcal{E} \to \mathcal{E}$ preserves Φ-colimits for any density presentation Φ of K.

Proof. For (i) ⇔ (ii), the monadicity of $V\colon \mathcal{A}\text{-}\mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ verified in the previous proposition implies, as in the proof of Theorem 38(iii), that $\mathcal{A}$-$\mathbf{Mnd}(\mathcal{E})$ is the colimit-closure in $\mathbf{Mnd}(\mathcal{E})$ of the free monads on signatures. Since $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ is also this closure, we have $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) = \mathcal{A}\text{-}\mathbf{Mnd}(\mathcal{E})$ as desired. For (ii) ⇔ (iii), we apply Proposition 35. □

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