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Operadic categories and décalage [☆]Richard Garner ^{a,*}, Joachim Kock ^b, Mark Weber ^c

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ABSTRACT

Batanin and Markl's operadic categories are categories in which each map is endowed with a finite collection of "abstract fibres"—also objects of the same category—subject to suitable axioms. We give a reconstruction of the data and axioms of operadic categories in terms of the décalage comonad D on small categories. A simple case involves *unary* operadic categories—ones wherein each map has exactly one abstract fibre—which are exhibited as categories which are, first of all, coalgebras for the comonad D , and, furthermore, algebras for the monad \bar{D} induced on Cat^D by the forgetful-cofree adjunction. A similar description is found for general operadic categories arising out of a corresponding analysis that starts from a "modified décalage" comonad D_m on the arrow category Cat^2 .

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1. Introduction

Operads originated in algebraic topology, first appearing in Boardman and Vogt [6] under the name “category of operators in standard form”, with the modern name and modern definition being provided by May [15]. They quickly caught on, with applications subsequently being found not only in topology, but also in algebra, geometry, physics and beyond; see [14] for an overview. As the use of operads has grown, it has proven useful to recast the definition: rather than explicitly listing the data and axioms, one may re-express them in various more abstract ways [2,11,12], each of which points towards a range of practically useful generalisations of the original notion.

This has led to a rich profusion of operad-like structures, and various authors have proposed unifying frameworks to bring some order to this proliferation. One such framework is that of *operadic categories* [3], introduced by Batanin and Markl to specify certain kinds of generalised operad necessary for their proof of the duoidal Deligne conjecture. An operadic category is a combinatorial object which specifies a flavour of operad; an “algebra” for an operadic category is an operad of that flavour. Such an operad will, in turn, have its own algebras, but this extra layer will not concern us here.

As the name suggests, operadic categories are categories, but endowed with extra structure of a somewhat delicate nature. This structure seems to invite attempts at reconfiguration, so as better to link it to other parts of the mathematical landscape. One such reconfiguration was given by Lack [13], who drew a tight correspondence between operadic categories and the *skew-monoidal categories* of Szlachányi [17], which in recent years have figured prominently in categorical quantum algebra and work of the Australian school of category theorists.

The present paper gives another reconfiguration of the definition of operadic category, which links it to the (upper) *décalage* construction. While primarily an operation on simplicial sets, *décalage* may also—via the nerve functor—be seen as an operation on categories; namely, that which takes a category to the disjoint union of its slices:

$$D(\mathcal{C}) = \sum_{X \in \mathcal{C}} \mathcal{C}/X .$$

There are two main aspects to the tight relationship between operadic categories and *décalage*. To explain these, we must first recall the data for an operadic category. These are: a small category \mathcal{C} with a chosen terminal object in each connected component; a cardinality functor $|\cdot|: \mathcal{C} \rightarrow \mathcal{S}$ into the category of finite ordinals and arbitrary mappings; and an operation assigning to every $f: Y \rightarrow X$ in \mathcal{C} and $i \in |X|$ an “abstract fibre” $f^{-1}(i) \in \mathcal{C}$, functorially in Y .

The first connection between operadic categories and *décalage* arises from the fact that the *décalage* construction on categories underlies a *comonad* D on \mathbf{Cat} , whose coalgebras may be identified, as in Proposition 5 below, with categories endowed with a choice of terminal object in each connected component. In particular, each operadic category is a coalgebra for the *décalage* comonad.

The second connection arises through the functorial assignment of abstract fibres $f \mapsto f^{-1}(i)$ in an operadic category. Functoriality says that, for fixed $X \in \mathcal{C}$ and $i \in |X|$, this assignment is the action on objects of a functor $\varphi_{X,i}: \mathcal{C}/X \rightarrow \mathcal{C}$, so that the totality of the abstract fibres can be expressed via a single functor

$$\varphi: \sum_{X \in \mathcal{C}, i \in |X|} \mathcal{C}/X \rightarrow \mathcal{C}. \tag{1.1}$$

The domain of this functor is clearly related to the décalage of \mathcal{C} , and in due course, we will explain it in terms of a *modified décalage* construction on categories endowed with a functor to \mathcal{S} . However, there is a special case where no modification is necessary. We call an operadic category *unary* if each $|X|$ is a singleton; in this case, the domain of (1.1) is precisely the décalage $D(\mathcal{C})$, so that the fibres of a unary operadic category are encoded in a single functor $D(\mathcal{C}) \rightarrow \mathcal{C}$.

So, for a unary operadic category \mathcal{C} , we have on the one hand, that \mathcal{C} is a D -coalgebra; and on the other, that \mathcal{C} is endowed with a map $D(\mathcal{C}) \rightarrow \mathcal{C}$. To reconcile these apparently distinct facts, we apply a general observation: any comonad C on a category \mathcal{A} induces a monad \tilde{C} on the category of C -coalgebras \mathcal{A}^C , namely, the monad generated by the forgetful–cofree adjunction $\mathcal{A}^C \rightleftarrows \mathcal{A}$. In the case of the décalage comonad, we induce a décalage monad \tilde{D} on Cat^D ; and the axioms of a unary operadic category turn out to be captured *precisely* by the requirement that the map $D(\mathcal{C}) \rightarrow \mathcal{C}$ giving abstract fibres should endow the D -coalgebra \mathcal{C} with \tilde{D} -algebra structure in Cat^D . Our first main result is thus:

Theorem. *The category of algebras for the décalage monad \tilde{D} on Cat^D is isomorphic to the category of unary operadic categories.*

In order to remove the qualifier “unary” from this theorem and accommodate the “multi” aspect of the general definition, we will need, as anticipated above, to adjust the décalage construction. Rather than the décalage comonad D on Cat , we will consider a *modified décalage* comonad D_m on the arrow category Cat^2 whose action on objects is given by

$$\mathcal{E} \xrightarrow{P} \mathcal{C} \quad \mapsto \quad \sum_{Y \in \mathcal{E}} \mathcal{E}/Y \xrightarrow{\sum_{Y \in \mathcal{E}} P/Y} \sum_{Y \in \mathcal{E}} \mathcal{C}/PY. \tag{1.2}$$

To relate this to operadic categories, we consider those objects of Cat^2 which are obtained to within isomorphism as the canonical projection $P_{\mathcal{C}}: \mathcal{E}_{\mathcal{C}} \rightarrow \mathcal{C}$ from the *category of elements* of a functor $|-|: \mathcal{C} \rightarrow \mathcal{S}$. These objects span a full subcategory of Cat^2 which is equivalent to the lax slice category $\text{Cat} // \mathcal{S}$; they are moreover closed under the action of D_m , which thus restricts back to a comonad on $\text{Cat} // \mathcal{S}$. Now by following the same trajectory as the unary case, starting from this comonad on $\text{Cat} // \mathcal{S}$, we already come very close to characterising operadic categories.

The first point to make is that an object $(\mathcal{C}, |-|: \mathcal{C} \rightarrow \mathcal{S}) \in \text{Cat} // \mathcal{S}$ is a D_m -coalgebra just when \mathcal{C} has chosen terminal objects in each connected component, and $|-|$ sends

each of these to $1 \in \mathcal{S}$. Since these are among the requirements for an operadic category, every operadic category gives rise to a \mathbf{D}_m -coalgebra.

Like before, we can ask what it means to equip such a \mathbf{D}_m -coalgebra with algebra structure for the induced monad $\tilde{\mathbf{D}}_m$ on $(\mathbf{Cat}/\mathcal{S})^{\mathbf{D}_m}$. The action of this monad on $(\mathcal{C}, |-|)$ is given by the category $\sum_{X \in \mathcal{C}, i \in |X|} \mathcal{C}/X$ of (1.1), endowed with a suitable functor to \mathcal{S} ; therefore, the basic datum of $\tilde{\mathbf{D}}_m$ -algebra structure is a functor of the same form as (1.1). This seems promising, but what we find is:

Theorem. *The category of algebras for the modified décalage monad $\tilde{\mathbf{D}}_m$ on $(\mathbf{Cat}/\mathcal{S})^{\mathbf{D}_m}$ is isomorphic to the category of lax-operadic categories.*

Here, a *lax-operadic category* is a new notion, which generalises that of operadic category by replacing the assertion of equalities $|f|^{-1}(i) = |f^{-1}(i)|$ on cardinalities of abstract fibres with a collection of coherent functions $|f|^{-1}(i) \rightarrow |f^{-1}(i)|$. Since it is not yet clear that this extra generality has any practical merit, our final objective is to find a version of the above result which removes the qualifier “lax”.

The source of the laxity is easy to pinpoint. A $\tilde{\mathbf{D}}_m$ -algebra structure is given by a map $D_m(\mathcal{C}, |-|) \rightarrow (\mathcal{C}, |-|)$ in \mathbf{Cat}/\mathcal{S} , whose data involves not only a functor (1.1), but also a natural transformation relating the functors to \mathcal{S} . The components of this natural transformation are the comparison functions $|f|^{-1}(i) \rightarrow |f^{-1}(i)|$, so that the genuine operadic categories correspond to those $\tilde{\mathbf{D}}_m$ -algebras whose structure map is given by a *strictly* commuting triangle over \mathcal{S} .

However, we cannot simply restrict the modified décalage comonad \mathbf{D}_m from the lax slice category \mathbf{Cat}/\mathcal{S} back to the strict slice category \mathbf{Cat}/\mathcal{S} , and then proceed as before. The problem is that \mathbf{D}_m does not restrict, since the counit maps $\varepsilon_{\mathcal{C}}: D_m(\mathcal{C}) \rightarrow \mathcal{C}$ in \mathbf{Cat}/\mathcal{S} involve triangles which are genuinely lax-commutative. On the other hand, it turns out that we *can* restrict the lifted monad $\tilde{\mathbf{D}}_m$ on $(\mathbf{Cat}/\mathcal{S})^{\mathbf{D}_m}$ back to the subcategory $(\mathbf{Cat}^{\mathbf{D}})/\mathcal{S}$ on the strictly commuting triangles. Having done so, our final result quickly follows:

Theorem. *The category of algebras for the modified décalage monad $\tilde{\mathbf{D}}_m$ on $\mathbf{Cat}^{\mathbf{D}}/\mathcal{S}$ is isomorphic to the category of operadic categories.*

The rest of this article will fill in the details of the above sketch. The plan is quite simple. In Section 2, we recall Batanin and Markl’s definition of operadic category [3]; then in Section 3 we recall the décalage construction and establish the first of the two links with the notion of operadic category. In Section 4, we prove our first main theorem, characterising unary operadic categories in terms of décalage. Section 5 is devoted to describing the modified décalage construction required to capture general operadic categories. Finally, in Sections 6 and 7, we prove our second and third theorems, giving the characterisations of lax-operadic categories and, finally, of operadic categories themselves.

2. Operadic categories

We begin with some necessary preliminaries. We say that a category \mathcal{C} is *endowed with local terminal objects* if each connected component of \mathcal{C} is provided with a chosen terminal object; we write uX for the chosen terminal in the connected component of $X \in \mathcal{C}$ and $\tau_X: X \rightarrow uX$ for the unique map.

We write \mathcal{S} for the category whose objects are the sets $\underline{n} = \{1, \dots, n\}$ for $n \in \mathbb{N}$ and whose maps are arbitrary functions. Note that \mathcal{S} has a *unique* terminal object $\underline{1}$ which we use to endow \mathcal{S} with local terminal objects; we may also sometimes write the unique element of $\underline{1}$ as $*$ rather than 1.

Given $\varphi: \underline{m} \rightarrow \underline{n}$ in \mathcal{S} and $i \in \underline{n}$, there is a unique monotone injection

$$\varepsilon_{\varphi,i}: \varphi^{-1}(i) \rightarrow \underline{m} \tag{2.1}$$

in \mathcal{S} whose image is $\{j \in \underline{m} : \varphi(j) = i\}$; we call the object $\varphi^{-1}(i)$ the *fibre of φ at i* . Given also $\psi: \underline{\ell} \rightarrow \underline{m}$ in \mathcal{S} , we write ψ_i^φ for the unique map of \mathcal{S} rendering

$$\begin{array}{ccc} (\varphi\psi)^{-1}(i) & \xrightarrow{\psi_i^\varphi} & \varphi^{-1}(i) \\ \varepsilon_{\varphi\psi,i} \downarrow & & \downarrow \varepsilon_{\varphi,i} \\ \underline{\ell} & \xrightarrow{\psi} & \underline{m} \end{array} \tag{2.2}$$

commutative, and call it the *fibre map of ψ with respect to φ at i* .

The Batanin–Markl notion of operadic category which we now reproduce can be seen as specifying a category with *formal* notions of fibre and fibre map. The fibres of a map need not be subobjects of the domain as in the case of \mathcal{S} , but the axioms will ensure that they retain many important properties of fibres in \mathcal{S} .

Definition 1. [3] An *operadic category* is given by the following data:

- (D1) A category \mathcal{C} endowed with local terminal objects;
- (D2) A *cardinality functor* $|-|: \mathcal{C} \rightarrow \mathcal{S}$;
- (D3) For each object $X \in \mathcal{C}$ and each $i \in |X|$ a *fibre functor*

$$\varphi_{X,i}: \mathcal{C}/X \rightarrow \mathcal{C}$$

whose action on objects and morphisms we denote as follows:

$$\begin{array}{ccc} Y \xrightarrow{f} X & \mapsto & f^{-1}(i) \\ \\ \begin{array}{ccc} Z & \xrightarrow{g} & Y \\ & \searrow f & \swarrow f \\ & X & \end{array} & \mapsto & g_i^f: (fg)^{-1}(i) \rightarrow f^{-1}(i), \end{array}$$

referring to the object $f^{-1}(i)$ as the *fibre of f at i* , and the morphism $g_i^f : (fg)^{-1}(i) \rightarrow f^{-1}(i)$ as the *fibre map of g with respect to f at i* ;

all subject to the following axioms, where in (A5), we write εj for the image of $j \in |f|^{-1}(i)$ under the map $\varepsilon_{|f|,i} : |f|^{-1}(i) \rightarrow |Y|$ of (2.1):

- (A1) If X is a local terminal then $|X| = \underline{1}$;
- (A2) For all $X \in \mathcal{C}$ and $i \in |X|$, the object $(1_X)^{-1}(i)$ is chosen terminal;
- (A3) For all $f \in \mathcal{C}/X$ and $i \in |X|$, one has $|f^{-1}(i)| = |f|^{-1}(i)$, while for all $g : fg \rightarrow f$ in \mathcal{C}/X and $i \in |X|$, one has $|g_i^f| = |g|_i^{|f|}$;
- (A4) For $X \in \mathcal{C}$, one has $\tau_X^{-1}(*) = X$, and for $f : Y \rightarrow X$, one has $f_*^{TX} = f$;
- (A5) For $g : fg \rightarrow f$ in \mathcal{C}/X , $i \in |X|$ and $j \in |f|^{-1}(i)$, one has that $(g_i^f)^{-1}(j) = g^{-1}(\varepsilon j)$, and given also $h : fgh \rightarrow fg$ in \mathcal{C}/X , one has $(h_i^{fg})_j^{g_i^f} = h_{\varepsilon j}^g$.

A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ between operadic categories is called an *operadic functor* if it strictly preserves local terminal objects, strictly commutes with the cardinality functors to \mathcal{S} , and preserves fibres and fibre maps in the sense that

$$F(f^{-1}(i)) = (Ff)^{-1}(i) \quad \text{and} \quad F(g_i^f) = (Fg)_i^{Ff}$$

for all $g : fg \rightarrow f$ in \mathcal{C}/X and $i \in |X|$. We write OpCat for the category of operadic categories and operadic functors.

The preceding definitions are exactly those of [3] with only some minor notational changes for clarity. The most substantial of these is that we make explicit the use of the monotone injections (2.1) in the axiom (A5), whereas in [3] this is left implicit. In light of this, let us spend a moment doing the necessary type-checking to see that this axiom makes sense.

Intuitively, the first clause of (A5) identifies the fibres of the fibre maps of a map, with the fibres of that map. Therein we have $g_i^f : (fg)^{-1}(i) \rightarrow f^{-1}(i)$, and $j \in |f|^{-1}(i) = |f^{-1}(i)|$, so that one can consider the object $(g_i^f)^{-1}(j)$. On the other hand, we have $\varepsilon j \in |Y|$ and $g : Z \rightarrow Y$ and so can equally consider the object $g^{-1}(\varepsilon j)$; now the first part of (A5) states that these two are equal.

As for the second part of (A5), this says that the fibre maps of the fibre maps of a map, are themselves fibre maps of that map. In this case, functoriality of the fibre functor $\varphi_{X,i} : \mathcal{C}/X \rightarrow \mathcal{C}$ implies that we have an equality

$$(fgh)^{-1}(i) \xrightarrow{(gh)_i^f} f^{-1}(i) = (fgh)^{-1}(i) \xrightarrow{h_i^{fg}} (fg)^{-1}(i) \xrightarrow{g_i^f} f^{-1}(i)$$

and again we have $j \in |f|^{-1}(i) = |f^{-1}(i)|$. It follows that the fibre map of h_i^{fg} with respect to g_i^f at j is given as to the left in:

$$(h_i^{fg})_j^{g_i^f} : ((gh)_i^f)^{-1}(j) \rightarrow (g_i^f)^{-1}(j) \qquad h_{\varepsilon_j}^g : (gh)^{-1}(\varepsilon_j) \rightarrow g^{-1}(\varepsilon_j) .$$

On the other hand, one could just consider the fibre map of h with respect to g at $\varepsilon_j \in |Y|$, as to the right. The first part of (A5) ensures that the domains and codomains of these maps coincide, and now the second part asserts that the maps themselves are equal.

Example 2. The most basic example of an operadic category is \mathcal{S} itself. The choice of local terminals is the unique one, the cardinality functor is the identity, and the action of the fibre functors is defined as in (2.1) and (2.2).

Many more examples of operadic categories are discussed in [3]; we give here two new examples inspired by probability theory.

Example 3. Let \mathcal{C} be the category of finite sub-probability spaces. Its objects are lists $r = (r_1, \dots, r_n)$ where each $r_i \in [0, 1]$ and $\sum_i r_i \leq 1$; its maps $\varphi : (s_1, \dots, s_m) \rightarrow (r_1, \dots, r_n)$ are maps $\varphi : \underline{m} \rightarrow \underline{n}$ of \mathcal{S} such that $r_i = \sum_{j \in \varphi^{-1}(i)} s_j$. There is an obvious cardinality functor $|\cdot| : \mathcal{C} \rightarrow \mathcal{S}$, and a *unique* choice of local terminals: indeed, \mathcal{C} is a coproduct of categories $\mathcal{C} = \sum_{r \in [0,1]} \mathcal{C}_r$ where \mathcal{C}_r has the unique terminal object (r) . For the abstract fibres, given $\varphi : s \rightarrow r$ in \mathcal{C} and $i \in |r|$, we define $\varphi^{-1}(i)$ to be $(s_{\varepsilon_1}, \dots, s_{\varepsilon_k})$ where $k = |\varphi|^{-1}(i)$ and $\varepsilon = \varepsilon_{|\varphi|, i} : |\varphi|^{-1}(i) \rightarrow |s|$ is as in (2.1); finally, fibre maps in \mathcal{C} are as in \mathcal{S} .

Example 4. Let \mathcal{C}_1 be the category of finite probability spaces, i.e., the connected component of (1) in the category \mathcal{C} of the previous example. This subcategory of \mathcal{C} is not a sub-operadic category; however, it bears a different operadic category structure which describes *disintegration* of finite probability measures.

We begin with the cardinality functor $|\cdot|_1 : \mathcal{C}_1 \rightarrow \mathcal{S}$. For any $r = (r_1, \dots, r_n) \in \mathcal{C}_1$, we let $(r_{p_1}, \dots, r_{p_k})$ be the sublist of r obtained by deleting all zeroes, and now take $|r|_1 = \underline{k}$. Given a map $\varphi : s \rightarrow r$ in \mathcal{C}_1 , where s has sublist $(s_{q_1}, \dots, s_{q_\ell})$ of non-zero entries, we determine $|\varphi|_1 : |s|_1 \rightarrow |r|_1$ by requiring that $|\varphi|(q_j) = p_{|\varphi|_1(j)}$; i.e., $|\varphi|_1$ is the restriction of $|\varphi|$ to the indices of non-zero entries.

To define the \mathcal{C}_1 -fibres, we employ the *normalisation* of a non-zero sub-probability space $r = (r_1, \dots, r_n) \in \mathcal{C} \setminus \mathcal{C}_0$; this is the probability space $\bar{r} \in \mathcal{C}_1$ with $\bar{r} = (r_1/\sum_i r_i, \dots, r_n/\sum_i r_i)$. Now given $\varphi : s \rightarrow r$ in \mathcal{C}_1 and $i \in |r|_1$, we define the \mathcal{C}_1 -fibre $\varphi^{-1}(i)$ to be the normalisation of the i th non-zero \mathcal{C} -fibre $\varphi^{-1}(p_i)$. Note that we *cannot* normalise a \mathcal{C} -fibre $\varphi^{-1}(j)$ for which $r_j = 0$; this is why we had to remove such j in defining the cardinality functor $|\cdot|_1$. Finally, given also $\psi : t \rightarrow s$ in \mathcal{C}_1 , we define the \mathcal{C}_1 -fibre map ψ_i^φ to have underlying \mathcal{S} -map $\psi_{p_i}^\varphi$.

3. Décalage

In this section, we recall the décalage comonad on Cat , characterise its category of coalgebras, and explain how this links up with the notion of operadic category. Throughout, “décalage” will always mean *upper décalage*.

The décalage comonad on \mathcal{Cat} can be obtained as a restriction of Illusie’s décalage comonad [9] on $[\Delta^{\text{op}}, \text{Set}]$, the category of simplicial sets. This is, in turn, obtained from the monad $T = (T, \eta, \mu)$ on the category Δ of non-empty finite ordinals and monotone maps given by freely adjoining a top element. In terms of the usual presentation of Δ in terms of “coface” maps δ_i and “codegeneracy” maps σ_j , this monad is given by the data:

$$T[n] = [n + 1], \quad T\delta_i = \delta_i, \quad T\sigma_j = \sigma_j, \quad \eta_{[n]} = \delta_{n+1} \quad \text{and} \quad \mu_{[n]} = \sigma_{n+1} .$$

It follows that T^{op} is a comonad on Δ^{op} , so that precomposition with T^{op} is a comonad on $[\Delta^{\text{op}}, \text{Set}]$; this is the décalage comonad.

The classical *nerve* functor $N: \mathcal{Cat} \rightarrow [\Delta^{\text{op}}, \text{Set}]$ exhibits the category of small categories as equivalent to a full subcategory of simplicial sets. The simplicial sets in this full subcategory happen to be closed under the action of the décalage comonad, which thereby restricts to a comonad D on \mathcal{Cat} . The underlying endofunctor D of this comonad sends a category \mathcal{C} to the coproduct of its slices:

$$D(\mathcal{C}) = \sum_{X \in \mathcal{C}} \mathcal{C}/X ; \tag{3.1}$$

the counit $\varepsilon_{\mathcal{C}}: D(\mathcal{C}) \rightarrow \mathcal{C}$ is the *copairing* of the domain projections $\mathcal{C}/X \rightarrow \mathcal{C}$ from the slices (i.e., the map induced from the family of domain projections by the universal property of coproduct); while the comultiplication $\delta_{\mathcal{C}}: D(\mathcal{C}) \rightarrow DD(\mathcal{C})$, which is a functor

$$\delta_{\mathcal{C}}: \sum_{X \in \mathcal{C}} \mathcal{C}/X \rightarrow \sum_{f \in D(\mathcal{C})} D(\mathcal{C})/f ,$$

sends the X -summand to the 1_X -summand via the isomorphism $\mathcal{C}/X \rightarrow D(\mathcal{C})/1_X$.

We now characterise the category of coalgebras for the décalage comonad as the category $\mathcal{Cat}_{\ell t}$ whose objects are small categories endowed with local terminal objects, and whose morphisms are functors which preserve chosen local terminals.

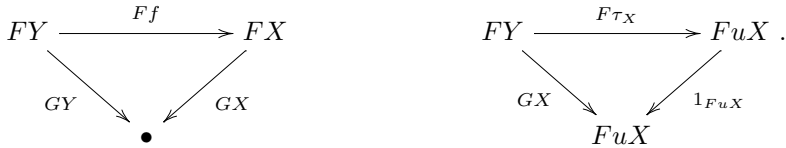
Proposition 5. *The category \mathcal{Cat}^D of D -coalgebras is isomorphic to $\mathcal{Cat}_{\ell t}$ over \mathcal{Cat} . Under this isomorphism, the D -coalgebra structure on $\mathcal{C} \in \mathcal{Cat}_{\ell t}$ is given by the functor $\tau: \mathcal{C} \rightarrow D(\mathcal{C})$ which takes $X \in \mathcal{C}$ to $\tau_X: X \rightarrow uX \in D(\mathcal{C})$.*

Proof. It suffices to show that the forgetful functor $U: \mathcal{Cat}_{\ell t} \rightarrow \mathcal{Cat}$ is strictly comonadic, and that the induced comonad is isomorphic to D . Towards the first of these, it is clear that U strictly creates limits and is faithful, and so by the Beck theorem will be strictly comonadic so long as it has a right adjoint.

We can endow the category $D(\mathcal{C})$ with the chosen terminal object 1_X in each connected component \mathcal{C}/X , so making it into an object of $\mathcal{Cat}_{\ell t}$; we claim this gives the value at \mathcal{C} of the desired right adjoint. Thus, for any $\mathcal{B} \in \mathcal{Cat}_{\ell t}$ and functor $F: \mathcal{B} \rightarrow \mathcal{C}$, we must exhibit a unique factorisation

$$F = \mathcal{B} \xrightarrow{G} D(\mathcal{C}) \xrightarrow{\varepsilon_{\mathcal{C}}} \mathcal{C} \tag{3.2}$$

where G strictly preserves chosen local terminals. Such a G must send each object $X \in \mathcal{C}$ to an object of $D(\mathcal{C})$ with domain projection FX . In particular, each chosen terminal uX of \mathcal{B} must be sent to a chosen terminal of $D(\mathcal{C})$ with domain FuX , and so we must have $G(uX) = 1_{FuX}$. Furthermore, such a G , if it exists, must send each map $f: Y \rightarrow X$ of \mathcal{B} to a map in $D(\mathcal{C})$ as to the left in:



In particular, taking $f = \tau_X$ yields the commuting triangle to the right, so that on objects we must have $GX = (F\tau_X: FX \rightarrow FuX)$. So G is unique if it exists; but it is easy to see that defining G in this way does indeed yield a map $G: \mathcal{B} \rightarrow D(\mathcal{C})$ in $\text{Cat}_{\ell t}$ preserving chosen terminals and factorising (3.2) as required.

So $U: \text{Cat}_{\ell t} \rightarrow \text{Cat}$ has a right adjoint R , and by strict comonadicity, $\text{Cat}_{\ell t}$ is isomorphic to the category of UR -coalgebras. By construction, the underlying functor and counit of UR are equal to D and ε , while the comultiplication at \mathcal{C} is the unique factorisation (3.2) of $F = 1_{D(\mathcal{C})}: D(\mathcal{C}) \rightarrow D(\mathcal{C})$ through a map in $\text{Cat}_{\ell t}$. As $\delta_{\mathcal{C}}: D(\mathcal{C}) \rightarrow DD(\mathcal{C})$ is easily seen to be such a factorisation, we conclude that $D = UR$ and so $\text{Cat}_{\ell t} \cong \text{Cat}^D$ as required. \square

To motivate the developments which will follow, we now establish a first link between operadic categories and décalage, by showing how the data and axioms for an operadic category can be partially re-expressed in terms of structure in $\text{Cat}_{\ell t} \cong \text{Cat}^D$. Of course, (D1) asserts that \mathcal{C} is an object in $\text{Cat}_{\ell t}$, whereupon axiom (A1) asserts that the cardinality functor $|\cdot|: \mathcal{C} \rightarrow \mathcal{S}$ is a map therein. Similarly, axiom (A2) states that each functor $\varphi_{X,i}: \mathcal{C}/X \rightarrow \mathcal{C}$ is a map of $\text{Cat}_{\ell t}$, where we take the chosen (local) terminal object in \mathcal{C}/X to be the identity 1_X .

To express (A3), we define for each $X \in \mathcal{C}$ and $i \in |X|$ a cardinality functor $|\cdot|_{X,i}: \mathcal{C}/X \rightarrow \mathcal{S}$ as the composite of $|\cdot|/X: \mathcal{C}/X \rightarrow \mathcal{S}/|X|$ with the fibre functor $\varphi_{|X|,i}: \mathcal{S}/|X| \rightarrow \mathcal{S}$ of the operadic category \mathcal{S} ; thus, on objects, $|f|_{X,i} = |f|^{-1}(i)$. Now (A3) asserts that the following diagram commutes for all $X \in \mathcal{C}$, $i \in |X|$:

$$\begin{array}{ccc}
 \mathcal{C}/X & \xrightarrow{\varphi_{X,i}} & \mathcal{C} \\
 \searrow & & \swarrow \\
 & & \mathcal{S}
 \end{array}
 \tag{3.3}$$

We may express all of the above more compactly as follows. For any object $|\cdot|_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{S}$ of $\text{Cat}_{\ell t}/\mathcal{S}$, we write $D_m(\mathcal{C})$ for the category $\Sigma_{X \in \mathcal{C}, i \in |X|} \mathcal{C}/X$, seen as an object of $\text{Cat}_{\ell t}$

by choosing each identity map as a local terminal, and write $|-|_{D_m(\mathcal{C})} : D_m(\mathcal{C}) \rightarrow \mathcal{S}$ for the copairing of the maps $|-|_{X,i} : \mathcal{C}/X \rightarrow \mathcal{S}$. Now to give the data (D1)–(D3) and axioms (A1)–(A3) for an operadic category is to give an object $(\mathcal{C}, |-|_{\mathcal{C}})$ of $\text{Cat}_{\ell t}/\mathcal{S}$ and a map $\varphi : (D_m(\mathcal{C}), |-|_{D_m(\mathcal{C})}) \rightarrow (\mathcal{C}, |-|_{\mathcal{C}})$.

It remains to account for axioms (A4) and (A5). In fact, it turns out that the assignment $(\mathcal{C}, |-|_{\mathcal{C}}) \mapsto (D_m(\mathcal{C}), |-|_{D_m(\mathcal{C})})$ is the action on objects of a monad \tilde{D}_m on the category $\text{Cat}_{\ell t}/\mathcal{S}$, and that the remaining axioms are just those needed for $\varphi : (D_m(\mathcal{C}), |-|_{D_m(\mathcal{C})}) \rightarrow (\mathcal{C}, |-|_{\mathcal{C}})$ to endow $(\mathcal{C}, |-|)$ with \tilde{D}_m -algebra structure. While we could verify this straight away in a hands-on fashion, we prefer to give an argument which justifies the constructions in terms of a deeper link to the décalage construction. In the end, the claimed monad structure on \tilde{D}_m will be exhibited in Definition 26 below, and the characterisation of its algebras as operadic categories given in Theorem 27.

4. Characterising unary operadic categories

The characterisation of general operadic categories in terms of décalage will require a modification of the décalage construction, to be introduced in Section 5 below. As a warm-up for this, we consider the case of *unary* operadic categories, for which the usual décalage will suffice.

Definition 6. An operadic category is *unary* if $|X| = \underline{1}$ for all $X \in \mathcal{C}$. We write OpCat_1 for the category of unary operadic categories and operadic functors.

Example 7. For any category \mathcal{C} , the category $D(\mathcal{C}) = \sum_{X \in \mathcal{C}} \mathcal{C}/X$ is a unary operadic category. The chosen local terminals are the identity maps, and the unique fibre of a map $g : fg \rightarrow f$ is the object g . Given another map $h : fgh \rightarrow fg$, the fibre map of h with respect to g at $*$ is taken to be $h : gh \rightarrow g$.

Example 8. If \mathcal{C} is a pointed category with a chosen zero object and chosen kernels, we can attempt to impose a unary operadic structure as follows: the chosen (local) terminal is the zero object; the unique fibre of a map $f : Y \rightarrow X$ is its kernel; and the fibre map of $g : Z \rightarrow Y$ with respect to f is the restriction $g|_{\ker fg} : \ker fg \rightarrow \ker f$. However, whether these data satisfy the required axioms is sensitive to the choice of kernels. For instance, if $g : Z \rightarrow Y$ and $f : Y \rightarrow X$, then the chosen kernel of g , though always *isomorphic* to the chosen kernel of $g|_{\ker fg} : \ker fg \rightarrow \ker f$, need not be *equal* to it as required by axiom (A5).

Often, there *is* an appropriate choice of kernels; for example if \mathcal{C} is Set_* or Ab or $k\text{-Vect}$ or $\text{Ch}(R\text{-Mod})$, then we can take the kernel of any identity map to be the chosen zero object, and the kernel of any other map to be given by the usual subset formula; this yields the necessary axioms for a unary operadic category.

Yet even for a \mathcal{C} where we cannot choose kernels appropriately, we can always consider the *equivalent* category $\mathcal{Pt}(\mathcal{C}^{\text{op}}, \text{Set}_*)_{\text{rep}}$ of representable zero-preserving functors to Set_* ,

and endow this with unary operadic structure given pointwise as in Set_* . Note that this structure need *not* transport back to an operadic structure on \mathcal{C} , since the notion of operadic category is not invariant under equivalence (in the terminology of [5] it is not *flexible*).

In the unary case, we can effectively ignore the cardinality functor down to \mathcal{S} ; so on repeating the analysis at the end of the preceding section, we find that the data and first three axioms for a unary operadic category \mathcal{C} are encoded precisely by a map $D(\mathcal{C}) \rightarrow \mathcal{C}$ in $\text{Cat}_{\ell t}$. To complete this analysis, we will show that the assignment $\mathcal{C} \mapsto D(\mathcal{C})$ underlies a monad on $\text{Cat}_{\ell t}$ whose category of algebras is isomorphic to OpCat_1 . The monad structure arises as follows.

Definition 9. The *décalage monad* $\tilde{D} = (\tilde{D}, \eta, \mu)$ on $\text{Cat}_{\ell t} \cong \text{Cat}^{\mathcal{D}}$ is the monad induced by the forgetful–cofree adjunction $\text{Cat}^{\mathcal{D}} \rightleftarrows \text{Cat}$.

Since the proof of Proposition 5 furnishes us with an explicit description of the forgetful–cofree adjunction $\text{Cat}^{\mathcal{D}} \rightleftarrows \text{Cat}$, we can read off from it the following description of the décalage monad:

- (i) The underlying functor \tilde{D} on objects sends \mathcal{C} to $\sum_{X \in \mathcal{C}} \mathcal{C}/X$ endowed with the local terminal objects $1_X \in \mathcal{C}/X$; while on morphisms, it sends $F: \mathcal{C} \rightarrow \mathcal{C}'$ to the functor which maps the X -summand of $\sum_{X \in \mathcal{C}} \mathcal{C}/X$ to the FX -summand of $\sum_{Y \in \mathcal{C}'} \mathcal{C}'/Y$ via $F/X: \mathcal{C}/X \rightarrow \mathcal{C}'/FX$;
- (ii) The unit map $\eta_{\mathcal{C}}: \mathcal{C} \rightarrow \tilde{D}(\mathcal{C})$ is defined on objects by $\eta_{\mathcal{C}}(X) = \tau_X: X \rightarrow uX$ and on morphisms by $\eta_{\mathcal{C}}(f: Y \rightarrow X) = f: \tau_Y \rightarrow \tau_X$;
- (iii) The multiplication map $\mu_{\mathcal{C}}: \tilde{D}\tilde{D}(\mathcal{C}) \rightarrow \tilde{D}(\mathcal{C})$, which is given by a functor $\sum_{f \in D(\mathcal{C})} D(\mathcal{C})/f \rightarrow \sum_{X \in \mathcal{C}} \mathcal{C}/X$, sends the summand indexed by $f: Y \rightarrow X$ to the summand indexed by Y via the isomorphism $D(\mathcal{C})/f \rightarrow \mathcal{C}/Y$.

Using this description, we can now prove our first main theorem.

Theorem 10. *The category of algebras for the décalage monad \tilde{D} on $\text{Cat}_{\ell t} \cong \text{Cat}^{\mathcal{D}}$ is isomorphic to the category OpCat_1 of unary operadic categories.*

Proof. We have already argued that the data and first three axioms for a unary operadic category \mathcal{C} are encapsulated by giving the object $\mathcal{C} \in \text{Cat}_{\ell t}$ together with the map $\varphi: \tilde{D}(\mathcal{C}) \rightarrow \mathcal{C}$ in $\text{Cat}_{\ell t}$ obtained as the copairing of the fibre functors $\varphi_{X,*}: \mathcal{C}/X \rightarrow \mathcal{C}$. Given this, we can read off from Definition 9 that (A4) asserts precisely the unit axiom $\varphi \circ \eta_{\mathcal{C}} = 1_{\mathcal{C}}$, while (A5) asserts the multiplication axiom $\varphi \circ \mu_{\mathcal{C}} = \varphi \circ \tilde{D}(\varphi): \tilde{D}\tilde{D}(\mathcal{C}) \rightarrow \mathcal{C}$. So \tilde{D} -algebras in $\text{Cat}_{\ell t}$ are in bijection with unary operadic categories; the corresponding bijection on maps is direct. \square

Using this result, we may obtain a further description of unary operadic categories which, though not necessary for the subsequent results of this paper, is nonetheless enlightening. We observed above that the décalage comonad on \mathbf{Cat} is the restriction along the full inclusion $N: \mathbf{Cat} \rightarrow [\Delta^{\text{op}}, \mathbf{Set}]$ of the décalage comonad on simplicial sets. It follows that we have a full inclusion

$$\mathbf{OpCat} \xrightarrow{\cong} (\mathbf{Cat}^D)^{\tilde{D}} \xrightarrow{(N^D)^{\tilde{D}}} ([\Delta^{\text{op}}, \mathbf{Set}]^D)^{\tilde{D}} \tag{4.1}$$

(where we re-use the notation D and \tilde{D} for the décalage comonad on $[\Delta^{\text{op}}, \mathbf{Set}]$ and the induced monad on $[\Delta^{\text{op}}, \mathbf{Set}]^D$) whose essential image comprises just those \tilde{D} -algebras in $[\Delta^{\text{op}}, \mathbf{Set}]^D$ whose underlying simplicial set satisfies the Segal condition. On the other hand, we have a straightforward characterisation of the category $([\Delta^{\text{op}}, \mathbf{Set}]^D)^{\tilde{D}}$:

Lemma 11. *The comparison functor*

$$[\Delta^{\text{op}}, \mathbf{Set}] \rightarrow ([\Delta^{\text{op}}, \mathbf{Set}]^D)^{\tilde{D}}$$

sending a simplicial set X to $D(X)$ with its canonical \tilde{D} -algebra structure, is an equivalence of categories.

Proof. The functor part of the comonad D on $[\Delta^{\text{op}}, \mathbf{Set}]$ is given by precomposition with $T^{\text{op}}: \Delta^{\text{op}} \rightarrow \Delta^{\text{op}}$, and so is cocontinuous. Thus, for the forgetful–cofree adjunction

$$[\Delta^{\text{op}}, \mathbf{Set}]^D \begin{array}{c} \xleftarrow{G^D} \\ \xrightarrow{U^D} \end{array} [\Delta^{\text{op}}, \mathbf{Set}]$$

the functor G^D is again cocontinuous. Moreover, U^D is conservative, and it is easy to see that $U^D G^D = D$ is conservative—since the set of 0-simplices of a simplicial set is the splitting of an idempotent on the set of 1-simplices—so that G^D is also conservative. Thus by the Beck monadicity theorem G^D is monadic, and so the comparison functor $[\Delta^{\text{op}}, \mathbf{Set}] \rightarrow ([\Delta^{\text{op}}, \mathbf{Set}]^D)^{\tilde{D}}$ is an equivalence. \square

Combining this with the characterisation of the essential image of (4.1) yields:

Corollary 12. *The category \mathbf{OpCat}_1 of unary operadic categories is isomorphic to the full (reflective) subcategory of $[\Delta^{\text{op}}, \mathbf{Set}]$ on those simplicial sets C for which $D(C)$ satisfies the Segal condition.*

Explicitly, the simplicial set C giving the “undecked” of a unary operadic category \mathcal{C} has as 0-simplices, the chosen terminal objects of \mathcal{C} , and as $(n + 1)$ -simplices the n -simplices of the nerve of \mathcal{C} . The faces of a 1-simplex X are

$$\varphi(1_X) \xrightarrow{X} uX$$

where we write $\varphi(f)$ for the unique fibre $f^{-1}(*)$ of a map $f: Y \rightarrow X$ of \mathcal{C} . The faces of a 2-simplex $f \in \mathcal{C}(Y, X)$ are given by

$$\begin{array}{ccc} & \varphi(1_X) & \\ \varphi(f) \nearrow & & \searrow X \\ \varphi(1_Y) & \xrightarrow{Y} & uX \end{array};$$

while the faces of a 3-simplex $(g, f) \in \mathcal{C}(Z, Y) \times \mathcal{C}(Y, X)$ are given by

$$\begin{array}{ccc} \varphi(1_Y) & \xrightarrow{\varphi(f)} & \varphi(1_X) \\ \varphi(g) \nearrow & & \searrow X \\ \varphi(1_Z) & \xrightarrow{Z} & uX \end{array} \quad \begin{array}{ccc} \varphi(1_Y) & \xrightarrow{\varphi(f)} & \varphi(1_X) \\ \varphi(g) \nearrow & g_*^f \nearrow & \searrow X \\ \varphi(1_Z) & \xrightarrow{Z} & uX \end{array} .$$

The degeneracies are easily written down, and the remaining data is determined by coskeletality. Note that $D(\mathcal{C})$ is the nerve of \mathcal{C} , which satisfies the Segal condition. Conversely, if C is a simplicial set for which $D(C)$ satisfies the Segal condition, then $D(C) \cong N(\mathcal{C})$ for a category \mathcal{C} , and by working backwards through the above description we may read off the operadic structure on \mathcal{C} .

Remark 13. The condition on a simplicial set X that $D(X)$ should satisfy the Segal condition gives half of the axioms for a discrete *decomposition space* [8]. (Decomposition spaces are also known as *2-Segal spaces* [7].) In particular, for any discrete decomposition space $X: \Delta^{op} \rightarrow \text{Set}$, its décalage is a unary operadic category, generalising Example 7. For example, there is a discrete decomposition space X of (combinatorialists’) graphs, wherein X_n is the set of graphs with a map from the set of vertices to \underline{n} . The corresponding unary operadic category has graphs as objects; a map is the opposite of a full inclusion of graphs, and the fibre of such a map is the induced graph on the complementary set of vertices.

In fact, the remaining axioms for a discrete decomposition space X can be expressed in terms of the associated unary operadic category \mathcal{C} : they say precisely that the fibre functor $\varphi: D(\mathcal{C}) \rightarrow \mathcal{C}$ is a discrete opfibration. This establishes a link with Lack’s [13], which characterises operadic categories with object set O in terms of certain left-normal *skew monoidal* [17] structures on Set/O , and provides conditions for these skew structures to be genuinely monoidal; in the unary case, the necessary condition is, again, that φ be a discrete opfibration.

In the following result, the equivalence between (i) and (ii) is thus due to Lack; we omit the proof, since the result is not needed elsewhere in this paper.

Theorem. *Let \mathcal{C} be a unary operadic category. The following are equivalent:*

- (i) *The fibre functor $\varphi: D(\mathcal{C}) \rightarrow \mathcal{C}$ is a discrete opfibration;*
- (ii) *The associated skew monoidal structure on $\text{Set}/\text{ob } \mathcal{C}$ is genuinely monoidal;*
- (iii) *The “undecking” C is a discrete decomposition space.*

In fact (cf. [13, Remark 7.2]) the left-normal skew monoidal structures induced by unary operadic categories are precisely those whose tensor preserves colimits in each variable; these can be identified with *skew monoidales* in the monoidal bicategory Span , and in this case Lack’s characterisation reduces to one given by Andrianopoulos [1]. Under this identification, the unary operadic categories satisfying the equivalent conditions of the above theorem correspond to genuine monoidales in Span : in the language of [8], this monoidale is the *incidence algebra* of the corresponding discrete decomposition space.

Remark 14. The equivalence of Corollary 12 is also interesting in the other direction. If \mathcal{C} is a unary operadic category derived from a category with a zero object and kernels, as in Example 8, then the associated simplicial set is a discrete version of Waldhausen’s S_\bullet construction.

5. Modified décalage

We now wish to expand on Theorem 10 to give a characterisation of general operadic categories in terms of décalage. As explained in the introduction, the key to this will be a comonad D_m on the arrow category Cat^2 given on objects by

$$\mathcal{E} \xrightarrow{P} \mathcal{C} \quad \mapsto \quad \sum_{Y \in \mathcal{E}} \mathcal{E}/Y \xrightarrow{\sum_{Y \in \mathcal{E}} P/Y} \sum_{Y \in \mathcal{E}} \mathcal{C}/PY, \tag{5.1}$$

which we call *modified décalage*. In this section, we describe this comonad, and show that it restricts back to the lax slice category Cat/\mathcal{S} , identified with the full subcategory of Cat^2 on the discrete opfibrations with finite fibres.

While we could describe the comonad D_m and its coalgebras by hand, we prefer in the spirit of the rest of the paper to obtain it by way of more general considerations. The key is the following construction on a functor $P: \mathcal{E} \rightarrow \mathcal{C}$. It begins by decomposing \mathcal{E} and \mathcal{C} into their connected components:

$$\mathcal{E} = \sum_{y \in Y} \mathcal{E}_y \quad \text{and} \quad \mathcal{C} = \sum_{x \in X} \mathcal{C}_x .$$

Now for each $y \in Y$, the restriction of P to \mathcal{E}_y must factor through a single connected component \mathcal{C}_{f_y} of \mathcal{C} . If we write $P_y: \mathcal{E}_y \rightarrow \mathcal{C}_{f_y}$ for this factorisation, then summing the P_y ’s over all $y \in Y$ yields the first map L_P in a factorisation:

$$\begin{array}{ccc}
 \sum_{y \in Y} \mathcal{C}_y & \xrightarrow{P} & \sum_{x \in X} \mathcal{C}_x \\
 \searrow^{L_P} & & \nearrow^{R_P} \\
 & \sum_{y \in Y} \mathcal{C}_{fy} &
 \end{array}
 \tag{5.2}$$

whose second map R_P maps the y -summand to the fy -summand via $1_{\mathcal{C}_{fy}}$. Let us call a functor π_0 -*bijection* if, like L_P , the induced function on connected components is invertible, and π_0 -*cartesian* if, like R_P , it maps each connected component of its domain bijectively onto a connected component of its codomain. As these two classes of functors are easily seen to be orthogonal, we have a factorisation system (π_0 -bijection, π_0 -cartesian) on \mathcal{Cat} ; and so by [10, Theorem 5.10] we have:

Lemma 15. *The full subcategory $\pi_0\text{-Bij}$ of \mathcal{Cat}^2 whose objects are the π_0 -bijection functors is a coreflective subcategory. The counit of the coreflection at P is given by the morphism $(1, R_P): L_P \rightarrow P$ in \mathcal{Cat}^2 .*

Remark 16. Whenever $H: \mathcal{T} \rightarrow \mathcal{B}$ is a Grothendieck fibration, there is a factorisation system on \mathcal{T} whose left and right classes are, respectively, the maps inverted by H , and the cartesian maps with respect to H . The above factorisation system arises in this way from the connected components functor $\pi_0: \mathcal{Cat} \rightarrow \mathcal{Set}$.

Now, if the $P: \mathcal{E} \rightarrow \mathcal{C}$ of (5.2) is a strictly local-terminal-preserving functor between categories endowed with local terminal objects, then there is a *unique* way of endowing the interposing $\sum_y \mathcal{C}_{fy}$ with local terminal objects such that both L_P and R_P preserve them strictly. It follows that the (π_0 -bijection, π_0 -cartesian) factorisation system on \mathcal{Cat} lifts to $\mathcal{Cat}_{\ell t}$, and so again by [10, Theorem 5.10]:

Lemma 17. *The full subcategory $\pi_0\text{-Bij}_{\ell t}$ of $(\mathcal{Cat}_{\ell t})^2$ whose objects are the π_0 -bijection functors is a coreflective subcategory. The counit of the coreflection at P is given by the morphism $(1, R_P): L_P \rightarrow P$ in $(\mathcal{Cat}_{\ell t})^2$.*

Remark 18. The lifting of the (π_0 -bijection, π_0 -cartesian) factorisation system from \mathcal{Cat} to $\mathcal{Cat}_{\ell t}$ is in fact also the lifting of the *comprehensive* factorisation system [16], whose classes are the final functors and the discrete fibrations. So the category $\pi_0\text{-Bij}_{\ell t}$ is equally the full subcategory of $(\mathcal{Cat}_{\ell t})^2$ on the final functors.

Now, if we let L and $L_{\ell t}$ denote the idempotent comonads on \mathcal{Cat}^2 and $(\mathcal{Cat}_{\ell t})^2$ corresponding to the coreflective subcategories of the last two lemmas, then it is evident from their explicit descriptions that $L_{\ell t}$ is a *lifting*—in the sense of [4]—of L along the strictly comonadic $(\mathcal{Cat}_{\ell t})^2 \rightarrow \mathcal{Cat}^2$. It follows by the proposition in [4, §2] that the composite adjunction

$$\pi_0\text{-Bij}_{\ell t} \begin{array}{c} \longleftarrow \top \\ \longrightarrow \end{array} (\text{Cat}_{\ell t})^2 \begin{array}{c} \longleftarrow \top \\ \longrightarrow \end{array} \text{Cat}^2 \tag{5.3}$$

is also strictly comonadic. Thus, if we define the *modified décalage comonad* D_m to be the comonad generated by this adjunction, then we have:

Proposition 19. *The category $(\text{Cat}^2)^{D_m}$ of D_m -coalgebras is isomorphic over Cat^2 to the full subcategory $\pi_0\text{-Bij}_{\ell t}$ of $(\text{Cat}_{\ell t})^2$ on the π_0 -bijective functors.*

By combining Proposition 5 and Lemma 17, we see that the cofree functor $\text{Cat}^2 \rightarrow (\text{Cat}^2)^{D_m}$ sends the object $P: \mathcal{E} \rightarrow \mathcal{C}$ of Cat^2 to the object

$$D_m(P) = \sum_{Y \in \mathcal{E}} \mathcal{E}/Y \xrightarrow{\sum_{Y \in \mathcal{E}} P/Y} \sum_{Y \in \mathcal{E}} \mathcal{C}/PY \tag{5.4}$$

endowed in domain and codomain with the respective local terminals 1_Y and 1_{PY} for each $Y \in \mathcal{E}$. Furthermore, the counit at P of the adjunction (5.3) is the map $D_m(P) \rightarrow P$ of Cat^2 whose two components $\sum_Y \mathcal{E}/Y \rightarrow \mathcal{E}$ and $\sum_Y \mathcal{C}/PY \rightarrow \mathcal{C}$ are given by the appropriate copairings of slice projections.

We now show that the comonad D_m on Cat^2 restricts to the *lax slice category* Cat/\mathcal{S} . The objects of this category are pairs of a small category \mathcal{C} and a functor $|-|_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{S}$, while morphisms $(\mathcal{C}, |-|_{\mathcal{C}}) \rightarrow (\mathcal{C}', |-|_{\mathcal{C}'})$ are pairs of a functor F and natural transformation ν fitting into a diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\ & \searrow & \swarrow \\ & \mathcal{S} & \end{array} \begin{array}{c} \nu \\ \Rightarrow \end{array} \tag{5.5}$$

To embed Cat/\mathcal{S} into Cat^2 , we use the *category of elements* construction. For a functor $Q: \mathcal{C} \rightarrow \text{Set}$, its category of elements $\text{el}(Q)$ has objects given by pairs $(X \in \mathcal{C}, i \in QX)$, and maps $(Y, j) \rightarrow (X, i)$ given by maps $f \in \mathcal{C}(Y, X)$ with $(Qf)(j) = i$. Associated to the category of elements we have a discrete opfibration $\pi_Q: \text{el}(Q) \rightarrow \mathcal{C}$ sending (X, i) to X ; recall that a functor $P: \mathcal{E} \rightarrow \mathcal{C}$ is a *discrete opfibration* if, for every $Y \in \mathcal{E}$ and $f: PY \rightarrow X$ in \mathcal{C} , there is a unique map $\bar{f}: Y \rightarrow \bar{X}$ with $P\bar{f} = f$. In particular, to each $(\mathcal{C}, |-|_{\mathcal{C}}) \in \text{Cat}/\mathcal{S}$ we can associate the discrete opfibration $P_{\mathcal{C}}: \mathcal{E}_{\mathcal{C}} \rightarrow \mathcal{C}$ obtained as the projection from the category of elements of $|-|_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{S} \hookrightarrow \text{Set}$.

Proposition 20. *The assignment $(\mathcal{C}, |-|_{\mathcal{C}}) \mapsto (P_{\mathcal{C}}: \mathcal{E}_{\mathcal{C}} \rightarrow \mathcal{C})$ is the action on objects of a fully faithful functor $\Upsilon: \text{Cat}/\mathcal{S} \rightarrow \text{Cat}^2$. Its essential image comprises the discrete opfibrations with finite fibres, and choosing an isomorphism with an object in the image amounts to endowing each of these fibres with a linear order.*

While this result is well known, we prove it for the sake of self-containedness.

Proof. If $(\mathcal{C}, |-|_{\mathcal{C}})$ and $(\mathcal{C}', |-|_{\mathcal{C}'})$ are objects of Cat/\mathcal{S} , then a map $P_{\mathcal{C}} \rightarrow P_{\mathcal{C}'}$ of Cat^2 is a commutative square

$$\begin{array}{ccc} \mathcal{E}_{\mathcal{C}} & \xrightarrow{G} & \mathcal{E}_{\mathcal{C}'} \\ P_{\mathcal{C}} \downarrow & & \downarrow P_{\mathcal{C}'} \\ \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \end{array} .$$

Commutativity forces $G(X, i) = (FX, \nu_X(i))$ for suitable $\nu_X(i) \in |FX|_{\mathcal{C}'}$, so yielding functions $\nu_X: |X|_{\mathcal{C}} \rightarrow |FX|_{\mathcal{C}'}$, which by applying G to morphisms we see are natural in X . So every map $P_{\mathcal{C}} \rightarrow P_{\mathcal{C}'}$ arises from a lax triangle (5.5), and it is easy to see that any such triangle induces a map $P_{\mathcal{C}} \rightarrow P_{\mathcal{C}'}$ in this manner.

So Υ is well-defined and fully faithful. As for its essential image, it is well known (and easily proved) that $H: \mathcal{E} \rightarrow \mathcal{C}$ is a discrete opfibration just when it is isomorphic over \mathcal{C} to $\pi_Q: \text{el}(Q) \rightarrow \mathcal{C}$ for some functor $Q: \mathcal{C} \rightarrow \text{Set}$. In this case, H will have finite fibres just when π_Q does so, which happens just when each $Q(B)$ is finite. But such a Q may always be replaced by an isomorphic one which factors through $\mathcal{S} \subseteq \text{Set}$, and so the discrete opfibration H has finite fibres just when it is in the essential image of Υ .

Finally, the fibre of $P_{\mathcal{C}}: \mathcal{E}_{\mathcal{C}} \rightarrow \mathcal{C}$ over $X \in \mathcal{C}$ is the set $\{(X, i) : i \in |X|_{\mathcal{C}}\}$ which inherits a linear order from $|X|_{\mathcal{C}}$. So any specified isomorphism $H \cong P_{\mathcal{C}}$ induces by transport of structure a linear order on each fibre of H . Conversely, given a linear order on the fibres of H , we may reconstruct an isomorphism with $P_{\mathcal{C}}$ by requiring each map on fibres to be a *monotone* isomorphism. \square

We now show that the modified décalage comonad D_m on Cat^2 restricts back to a comonad on Cat/\mathcal{S} .

Proposition 21. *The essential image of $\Upsilon: \text{Cat}/\mathcal{S} \rightarrow \text{Cat}^2$ is closed under the action of modified décalage, which thus restricts to a comonad D_m on Cat/\mathcal{S} . The category of coalgebras $(\text{Cat}/\mathcal{S})^{D_m}$ is isomorphic to the lax slice $\text{Cat}_{\text{lt}}/\mathcal{S}$.*

Proof. Given $(\mathcal{C}, |-|)$ in Cat/\mathcal{S} , applying D_m to the corresponding $P_{\mathcal{C}}: \mathcal{E}_{\mathcal{C}} \rightarrow \mathcal{C}$ in Cat^2 yields by (5.4) the functor

$$\sum_{X \in \mathcal{C}, i \in |X|} \mathcal{E}_{\mathcal{C}}/(X, i) \xrightarrow{\Sigma_{X, i} P_{\mathcal{C}}/(X, i)} \sum_{X \in \mathcal{C}, i \in |X|} \mathcal{C}/X . \tag{5.6}$$

We must show this is a discrete opfibration with finite fibres. Since functors of this kind are closed under coproducts, it suffices to show that each $P_{\mathcal{C}}/(X, i): \mathcal{E}_{\mathcal{C}}/(X, i) \rightarrow \mathcal{C}/X$ is a discrete opfibration with finite fibres. It is a discrete opfibration since it is a slice of the discrete opfibration $P_{\mathcal{C}}$; as for the fibres, given $f: Y \rightarrow X$ in \mathcal{C}/X , the objects over it in $\mathcal{E}_{\mathcal{C}}/(X, i)$ are maps of $\mathcal{E}_{\mathcal{C}}$ of the form $f: (Y, j) \rightarrow (X, i)$, which are indexed by the *finite* set $\{j \in |Y| : |f|(j) = i\}$.

It follows that D_m restricts back to a comonad on $\mathcal{C}at//\mathcal{S}$, and the corresponding category of coalgebras fits into a pullback

$$\begin{array}{ccc}
 (\mathcal{C}at//\mathcal{S})^{D_m} & \longrightarrow & (\mathcal{C}at^2)^{D_m} \\
 U \downarrow \lrcorner & & \downarrow U \\
 \mathcal{C}at//\mathcal{S} & \xrightarrow{\Upsilon} & \mathcal{C}at^2 .
 \end{array}$$

Now given $(\mathcal{C}, |-|) \in \mathcal{C}at//\mathcal{S}$, endowing its image $P_{\mathcal{C}}: \mathcal{E}_{\mathcal{C}} \rightarrow \mathcal{C}$ under Υ with D_m -coalgebra structure means, first of all, endowing \mathcal{C} with local terminal objects. Having done this, we must endow $\mathcal{E}_{\mathcal{C}}$ with local terminals such that $P_{\mathcal{C}}$ preserves them, and it is easy to see that the unique way of doing this is by choosing the set of objects $\{(X, i) : X \text{ is local terminal in } \mathcal{C}, i \in |X|\}$. Finally, to assert that $P_{\mathcal{C}}$ is π_0 -bijective, there must be a *unique* (X, i) over each chosen local terminal of \mathcal{C} , which is to say that $|X| = \underline{1}$ for each local terminal of \mathcal{C} . So objects of $(\mathcal{C}at//\mathcal{S})^{D_m}$ are in bijection with those of $\mathcal{C}at_{\text{lt}}//\mathcal{S}$. The argument on maps is similar and left to the reader. \square

6. Characterising lax-operadic categories

In this section, we take the procedure employed in Section 4 for the décalage comonad on $\mathcal{C}at$ —considering its category of coalgebras, then the monad induced on the category of coalgebras, and then the algebras for *that* monad—and apply it to the modified décalage comonad on $\mathcal{C}at//\mathcal{S}$. By doing so, we come very close to obtaining a characterisation of operadic categories. What we in fact characterise are instances of the more general notion of *lax-operadic* category. These generalise operadic categories by replacing the fact of the commutativity of the triangles (3.3) by the data of coherent 2-cells filling these triangles.

Definition 22. A *lax-operadic category* is given by the following data, which augment those of an operadic category by the addition of (D4):

- (D1) A category \mathcal{C} endowed with local terminal objects;
- (D2) A cardinality functor $|-|: \mathcal{C} \rightarrow \mathcal{S}$;
- (D3) For all $X \in \mathcal{C}$ and $i \in |X|$ a fibre functor $\varphi_{X,i}: \mathcal{C}/X \rightarrow \mathcal{C}$ notated as before;
- (D4) For each $f: Y \rightarrow X$ in \mathcal{C} and $i \in |X|$, a *relabelling function*

$$\gamma_{f,i}: |f|^{-1}(i) \rightarrow |f^{-1}(i)| .$$

These data are subject to the following axioms, which are as for an operadic category, except that (A3) and (A5) are suitably modified to take account of the relabelling functions of (D4). In stating (A5-lax), we write γj and εj for the images of $j \in |f|^{-1}(i)$ under $\gamma_{f,i}: |f|^{-1}(i) \rightarrow |f^{-1}(i)|$ and $\varepsilon_{|f|,i}: |f|^{-1}(i) \rightarrow |Y|$.

- (A1) If X is a local terminal then $|X| = \mathbf{1}$;
- (A2) For all $X \in \mathcal{C}$ and $i \in |X|$, the object $(1_X)^{-1}(i)$ is chosen terminal;
- (A3-lax) For all $g: fg \rightarrow f$ in \mathcal{C}/X and $i \in |X|$, the fibre map is compatible with relabelling, in the sense that $|g_i^f| \circ \gamma_{fg,i} = \gamma_{f,i} \circ |g_i^f|$;
- (A4) For $X \in \mathcal{C}$, one has $\tau_X^{-1}(*) = X$, and for $f: Y \rightarrow X$, one has $f_*^{\tau_X} = f$;
- (A5-lax) For $g: fg \rightarrow f$ in \mathcal{C}/X , $i \in |X|$ and $j \in |f|^{-1}(i)$ one has that $(g_i^f)^{-1}(\gamma j) = g^{-1}(\varepsilon j)$ and that the square left below commutes:

$$\begin{array}{ccc}
 (|g_i^f|^{-1}(j)) \xrightarrow{\tilde{\gamma}_{fg,i}} |g_i^f|^{-1}(\gamma j) & & (|g_i^f|^{-1}(j)) \xrightarrow{\tilde{\gamma}_{fg,i}} |g_i^f|^{-1}(\gamma j) \\
 \parallel & \downarrow \gamma_{g_i^f, \gamma j} & \varepsilon_{|g_i^f|, j} \downarrow \\
 |g|^{-1}(\varepsilon j) \xrightarrow{\gamma_{g, \varepsilon j}} |g^{-1}(\varepsilon j)| & & |fg|^{-1}(i) \xrightarrow{\gamma_{fg,i}} |(fg)^{-1}(i)|
 \end{array} \tag{6.1}$$

where $\tilde{\gamma}_{fg,i}$ is the unique map making the square right above commute. Given moreover $h: fgh \rightarrow fg$ in \mathcal{C}/X , one has $(h_i^{fg})_{\gamma j}^g = h_{\varepsilon j}^g$.

A strictly local-terminal-preserving functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ between lax-operadic categories is called a *lax-operadic functor* if it comes endowed with a natural family of relabelling functions $\nu_X: |X| \rightarrow |FX|$, which are compatible with fibre functors in the sense of rendering commutative each diagram of the form:

$$\begin{array}{ccc}
 \mathcal{C}/X & \xrightarrow{\varphi_{X,i}} & \mathcal{C} \\
 F/X \downarrow & & \downarrow F \\
 \mathcal{C}'/FX & \xrightarrow{\varphi_{FX, \nu_X(i)}} & \mathcal{C}' ;
 \end{array}$$

in other words, we have $F(f^{-1}(i)) = (Ff)^{-1}(\nu_X(i))$ and $F(g_i^f) = (Fg)_{\nu_X(i)}^{Ff}$ for all $g: fg \rightarrow f$ in \mathcal{C}/X and $i \in |X|$. We write $\mathcal{LaxOpCat}$ for the category of lax-operadic categories and lax-operadic functors.

It is perhaps worth type-checking the display in (A5-lax) to see that it makes sense. In the left square, the left edge is well-defined simply by computing cardinalities of fibres; while the right edge is well-defined by the equality $(g_i^f)^{-1}(\gamma j) = g^{-1}(\varepsilon j)$ asserted directly beforehand. In the right square, for the factorisation $\tilde{\gamma}_{fg,i}$ to exist, we must know that $\gamma_{fg,i}$ maps each $k \in |fg|^{-1}(i)$ with $|g_i^f|(k) = j$ to an element $k' \in |(fg)^{-1}(i)|$ with $|g_i^f|(k') = \gamma_{f,i}(j)$; but this follows from the equality $|g_i^f| \circ \gamma_{fg,i} = \gamma_{f,i} \circ |g_i^f|$ asserted in (A3-lax).

We now begin our abstract rederivation of lax-operadic categories in terms of modified décalage. Recall that in Proposition 21, we exhibited the category of coalgebras for the

modified décalage comonad on $\mathcal{C}at//\mathcal{S}$ as isomorphic to the lax slice category $\mathcal{C}at_{\ell t}//\mathcal{S}$. Thus we are justified in giving:

Definition 23. The *modified décalage monad* \tilde{D}_m on $\mathcal{C}at_{\ell t}//\mathcal{S} \cong (\mathcal{C}at//\mathcal{S})^{D_m}$ is the monad induced by the forgetful–cofree adjunction $(\mathcal{C}at//\mathcal{S})^{D_m} \rightleftarrows \mathcal{C}at//\mathcal{S}$.

Towards a concrete description of the modified décalage monad, we note that the sets $\{j \in |Y| : |f|(j) = i\}$ giving the fibres of (5.6) inherit linear orders from $|Y|$, so that we may use the last clause of Proposition 20 to obtain a particular instantiation of the forgetful–cofree adjunction for the modified décalage comonad on $\mathcal{C}at//\mathcal{S}$. The cofree functor $\mathcal{C}at//\mathcal{S} \rightarrow (\mathcal{C}at//\mathcal{S})^{D_m}$ sends an object $(\mathcal{C}, |-|)$ to the object $(D_m(\mathcal{C}), |-|_{D_m(\mathcal{C})})$, where $D_m(\mathcal{C}) = \Sigma_{X \in \mathcal{C}, i \in |X|} \mathcal{C}/X$ is the codomain of (5.6), with the chosen terminal 1_X in the connected component indexed by (X, i) , and where $|-|_{D_m(\mathcal{C})} : D_m(\mathcal{C}) \rightarrow \mathcal{S}$ is defined on objects and morphisms by

$$\begin{aligned} (X \in \mathcal{C}, i \in |X|, f : Y \rightarrow X) &\mapsto |f|^{-1}(i) \\ (X, i, fg) \xrightarrow{g} (X, i, f) &\mapsto |fg|^{-1}(i) \xrightarrow{|g|_i^{f|}} |f|^{-1}(i) . \end{aligned} \tag{6.2}$$

The counit at $(\mathcal{C}, |-|)$ of the forgetful–cofree adjunction is given by a lax triangle

$$\begin{array}{ccc} D_m(\mathcal{C}) & \xrightarrow{E_e} & \mathcal{C} \\ & \searrow & \swarrow \\ & \mathcal{S} & \end{array} \begin{array}{c} \\ \varepsilon_e \\ \Rightarrow \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \end{array} \tag{6.3}$$

wherein the functor $E_e : \Sigma_{X \in \mathcal{C}, i \in |X|} \mathcal{C}/X \rightarrow \mathcal{C}$ is the copairing of the slice projections, and the natural transformation ε_e has component at (X, i, f) given by the map $\varepsilon_{|f|, i} : |f|^{-1}(i) \rightarrow |Y|$ of (2.1). We now use this to read off a description of the modified décalage monad on $(\mathcal{C}at//\mathcal{S})^{D_m} \cong \mathcal{C}at_{\ell t}//\mathcal{S}$.

- (i) The underlying functor $\tilde{D}_m : \mathcal{C}at_{\ell t}//\mathcal{S} \rightarrow \mathcal{C}at_{\ell t}//\mathcal{S}$ is given on objects by $(\mathcal{C}, |-|) \mapsto (D_m(\mathcal{C}), |-|_{D_m(\mathcal{C})})$ as above, and on morphisms by:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\ & \searrow & \swarrow \\ & \mathcal{S} & \end{array} \begin{array}{c} \\ \nu \\ \Rightarrow \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \end{array} \mapsto \begin{array}{ccc} D_m(\mathcal{C}) & \xrightarrow{D_m(F)} & D_m(\mathcal{C}') \\ & \searrow & \swarrow \\ & \mathcal{S} & \end{array} \begin{array}{c} \\ D_m(\nu) \\ \Rightarrow \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \end{array}$$

Here $D_m(F)$ has action on objects $(X, i, f) \mapsto (FX, \nu_X(i), Ff)$ and action on maps inherited from F ; while the component of $D_m(\nu)$ at an object (X, i, f) is the unique map rendering commutative the square

$$\begin{array}{ccc}
 |f|_{\mathcal{C}}^{-1}(i) & \xrightarrow{D_m(\nu)_{(X,i,f)}} & |Ff|_{\mathcal{C}'}^{-1}(\nu_X(i)) \\
 \varepsilon_{|f|,i} \downarrow & & \downarrow \varepsilon_{|Ff|,\nu_X(i)} \\
 |X|_{\mathcal{C}} & \xrightarrow{\nu_Y} & |FX|_{\mathcal{C}'} .
 \end{array}$$

- (ii) The unit $\eta_{\mathcal{C}}: (\mathcal{C}, |-|) \rightarrow \tilde{D}_m(\mathcal{C}, |-|)$ is a *strictly* commuting triangle, whose upper edge is the functor $\mathcal{C} \rightarrow D_m(\mathcal{C})$ sending X to $(uX, 1, \tau_X)$ and sending $f: Y \rightarrow X$ to $f: (uX, 1, \tau_X) \rightarrow (uY, 1, \tau_Y)$.
- (iii) The multiplication $\mu_{\mathcal{C}}: \tilde{D}_m \tilde{D}_m(\mathcal{C}, |-|) \rightarrow \tilde{D}_m(\mathcal{C}, |-|)$ is also a *strictly* commuting triangle, whose upper edge is the functor

$$\sum_{(X,i,f) \in D_m \mathcal{C}, j \in |f|^{-1}(i)} D_m \mathcal{C} / (X, i, f) \rightarrow \sum_{X \in \mathcal{C}, i \in |X|} \mathcal{C} / X \tag{6.4}$$

defined as follows. Since a typical map in $D_m \mathcal{C}$ is of the form $g: (X, i, fg) \rightarrow (X, i, f)$, an object of the domain of (6.4) comprises the data of

$$X \in \mathcal{C}, i \in |X|, f: Y \rightarrow X, j \in |f|^{-1}(i), g: Z \rightarrow Y \tag{6.5}$$

while each morphism is of the form $h: (X, i, f, j, gh) \rightarrow (X, i, f, j, g)$. In these terms, we can define the functor (6.4) on objects and morphisms by

$$\begin{array}{ccc}
 (X, i, f, j, g) & \mapsto & (Y, \varepsilon j, g) \\
 (X, i, f, j, gh) \xrightarrow{h} (X, i, f, j, g) & \mapsto & (Y, \varepsilon j, gh) \xrightarrow{h} (Y, \varepsilon j, g) ,
 \end{array} \tag{6.6}$$

where, like before, we write εj for $\varepsilon_{|f|,i}(j)$.

Using this description, we can now give our second main result.

Theorem 24. *The category of algebras for the modified décalage monad \tilde{D}_m on $\text{Cat}_{\ell t} // \mathcal{S} \cong (\text{Cat} // \mathcal{S})^{D_m}$ is isomorphic to the category $\mathcal{LaxOpCat}$ of lax-operadic categories.*

Proof. The data (D1)–(D2) and axiom (A1) specify exactly an object $(\mathcal{C}, |-|)$ in $\text{Cat}_{\ell t} // \mathcal{S}$. Giving the fibre functors (D3) is equivalent to giving a single functor $\varphi: D_m(\mathcal{C}) \rightarrow \mathcal{C}$, and the relabelling maps of (D4) give the components of a natural transformation

$$\begin{array}{ccc}
 D_m(\mathcal{C}) & \xrightarrow{\varphi} & \mathcal{C} \\
 \searrow \downarrow \downarrow_{|-|_{D_m(\mathcal{C})}} & \Downarrow \gamma & \swarrow \downarrow \downarrow_{|-|} \\
 & \mathcal{S} &
 \end{array} \tag{6.7}$$

whose naturality is then asserted by (A3-lax). Since axiom (A2) asserts that φ in (6.7) is a map of $\text{Cat}_{\ell t}$, we conclude that giving the data for a lax-operadic category plus the

first three axioms is the same as giving an object $(\mathcal{C}, |-|)$ of $(\text{Cat} // \mathcal{S})^{\mathcal{D}_m}$ endowed with a morphism $(\varphi, \gamma): \tilde{\mathcal{D}}_m(\mathcal{C}, |-|) \rightarrow (\mathcal{C}, |-|)$.

It is not hard to see that (A4) is equivalent to (φ, γ) satisfying the $\tilde{\mathcal{D}}_m$ -algebra unit axiom $(\varphi, \gamma) \circ (\eta_{(\mathcal{C}, |-|)}, 1) = (1_{\mathcal{C}}, 1_{|-|})$; we claim, finally, that (A5-lax) asserts the multiplication axiom given by the equality of pastings:

$$\begin{array}{ccc}
 D_m D_m \mathcal{C} & \xrightarrow{D_m \varphi} & D_m \mathcal{C} & \xrightarrow{\varphi} & \mathcal{C} \\
 & \searrow^{D_m(\gamma)} & \downarrow & \searrow^{\gamma} & \\
 & & |-|_{D_m \mathcal{C}} & \xrightarrow{\gamma} & \\
 & \swarrow_{|-|_{D_m D_m(\mathcal{C})}} & \downarrow & \swarrow_{|-|} & \\
 & & \mathcal{S} & &
 \end{array}
 =
 \begin{array}{ccc}
 D_m D_m \mathcal{C} & \xrightarrow{\mu_e} & D_m \mathcal{C} & \xrightarrow{\varphi} & \mathcal{C} \\
 & \searrow & \downarrow & \searrow^{\gamma} & \\
 & & |-|_{D_m \mathcal{C}} & \xrightarrow{\gamma} & \\
 & \swarrow_{|-|_{D_m D_m(\mathcal{C})}} & \downarrow & \swarrow_{|-|} & \\
 & & \mathcal{S} & &
 \end{array}
 \tag{6.8}$$

Now, the functors across the top of (6.8) act on a typical object (6.5) of $D_m D_m \mathcal{C}$ by the respective assignments:

$$\begin{aligned}
 (X, i, f, j, g) &\mapsto (f^{-1}(i), \gamma j, g_i^f) \mapsto (g_i^f)^{-1}(\gamma j) \\
 \text{and } (X, i, f, j, g) &\mapsto (Y, \varepsilon j, g) \mapsto g^{-1}(\varepsilon j),
 \end{aligned}$$

whose equality is precisely the first clause of (A5-lax). On the other hand, at this same object (6.5), the components of the two composite natural transformations in (6.8) are given by the two sides of the left square of (6.1)—whose equality is the second clause of (A5-lax). Finally, the actions on a map $h: (X, i, f, j, gh) \rightarrow (X, i, f, j, g)$ of $D_m D_m \mathcal{C}$ of the functors across the top of (6.8) are given by

$$h \mapsto h_i^{f g} \mapsto (h_i^{f g})_{\gamma_j}^{g_i^f} \quad \text{and} \quad h \mapsto h \mapsto h_{\varepsilon_j}^g,$$

whose equality is precisely the final clause of (A5-lax). This proves that $\tilde{\mathcal{D}}_m$ -algebras in $(\text{Cat} // \mathcal{S})^{\mathcal{D}_m}$ correspond bijectively with lax-operadic categories. A similar argument verifies the same for the maps between them, and we leave this to the reader. \square

7. Characterising operadic categories

There is not much left to do to get from the preceding result to our main result, characterising genuine operadic categories in terms of décalage. If we define a morphism of $\text{Cat}_{\ell t} // \mathcal{S}$ as in (5.5) to be *strict* whenever the natural transformation ν therein is an identity, then it is immediate from the preceding proof that:

Proposition 25. *Under the isomorphism of Theorem 24, a $\tilde{\mathcal{D}}_m$ -algebra corresponds to an operadic category just when its algebra structure map in $\text{Cat}_{\ell t} // \mathcal{S}$ is strict; while a $\tilde{\mathcal{D}}_m$ -algebra morphism corresponds to an operadic functor just when its underlying map in $\text{Cat}_{\ell t} // \mathcal{S}$ is strict.*

At this point, it is *not* possible to restrict the modified décalage comonad D_m on $\text{Cat} // \mathcal{S}$ back to the strict slice category Cat / \mathcal{S} , and obtain operadic categories as algebras for

the induced monad \tilde{D}_m on $(\mathcal{Cat}/\mathcal{S})^{\mathcal{D}_m}$. The reason for this, as noted in the introduction, is simply that modified décalage \mathcal{D}_m does not restrict from $\mathcal{Cat}//\mathcal{S}$ to $\mathcal{Cat}/\mathcal{S}$, since its counit maps (6.3) are only lax triangles.

However, the modified décalage monad \tilde{D}_m on $(\mathcal{Cat}//\mathcal{S})^{\mathcal{D}_m} \cong \mathcal{Cat}_{\ell t}//\mathcal{S}$ does interact well with strictness: inspection of the description following Definition 23 shows that the functor \tilde{D}_m preserves strictness of triangles, and that each unit and multiplication component is a strict triangle. We are thus justified in giving:

Definition 26. The *modified décalage monad* \tilde{D}_m on $\mathcal{Cat}_{\ell t}/\mathcal{S}$ is the restriction to $\mathcal{Cat}_{\ell t}/\mathcal{S}$ of the modified décalage monad $\mathcal{Cat}_{\ell t}//\mathcal{S}$.

And so, from Theorem 24 and Proposition 25, our main result immediately follows:

Theorem 27. *The category of algebras for the modified décalage monad \tilde{D}_m on $\mathcal{Cat}_{\ell t}/\mathcal{S}$ is isomorphic to the category OpCat of operadic categories.*

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