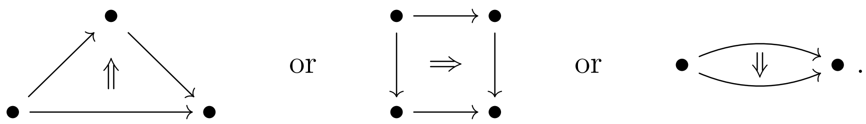
ORIENTALS AND CUBES, INDUCTIVELY

MITCHELL BUCKLEY AND RICHARD GARNER

Abstract. We provide direct inductive constructions of the orientals and the cubes, exhibiting them as the iterated cones, respectively, the iterated cylinders, of the terminal strict globular ω-category.

1. Introduction

A notorious aspect of the theory of weak higher dimensional categories is the proliferation of models that have been proposed for the notion [4]; a major outstanding problem is showing these different models to be suitably equivalent. Among the technical challenges facing anyone looking to do so is one of *geometry*, since in every kind of model, one has a notion of "n-cell", but between models the shapes of these n-cells may differ. There is a general agreement that "0-cell" and "1-cell" should mean "point" and "arrow"; but beyond this, the n-cells could be, among other things, simplicial, cubical or globular in shape. In dimension two, for example, this means that cells could take any of the following forms:



In comparing two notions of model, then, a first step must always be to describe a construction by which the basic cell-shapes of the one kind of model may be built out of the cell-shapes of the other.

In the literature there are certain equivalences of models which have been fully realised; one is the equivalence of strict globular ω-categories and strict cubical ω-categories with connections [1]; another is the equivalence of strict globular ω-categories with *complicial sets* [9], whose geometry is simplicial in nature. In particular, this means that the basic n-cell shapes of these cubical and simplicial models can be realised as strict globular ω-categories, known respectively as the *cubes* and the *orientals*. The orientals were constructed by Street in [6]; his later *parity complexes* [7] generalised the construction to

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permit the realisation by strict globular ω-categories of a wide range of oriented polyhedra, including not only the orientals but also the cubes.

Now, in undertaking the as-yet-unrealised task of relating simplicial, cubical and globular models of *weak* ω-categories, it is clear from the discussion above that a reasonable first step would be the construction of suitably weakened analogues of the orientals or cubes—that is, realisations of each n-simplex or n-cube as a weak globular ω-category. In this context, the theory of parity complexes is of no use, since it makes free and implicit use of the *middle-four interchange* axioms present in a strict higher category, but absent from a truly weak model; and so it is of interest to find alternate constructions of the (strict) orientals and cubes that may be more liable to adapt to the weak context.

In this paper, we describe one such alternate construction, which builds the orientals and cubes inductively: the $(n+1)$st oriental will be obtained as the *cone* of the nth oriental, and the $(n+1)$st cube as the *cylinder* of the nth one. Here, "cone" and "cylinder" are certain operations on ω-categories to be introduced below; the nomenclature comes, of course, from topology, where the *cylinder* of a topological space is its product with the interval, and the *cone* the result of collapsing one end of the cylinder to a point.

We have not yet attempted to adapt our inductive constructions from strict to weak ω-categories, but even without having done so, we may still justify the worth of our inductive constructions from another perspective: simplicity. The theory of parity complexes is challenging, and the proof that any parity complex can be realised by a strict ω-category is both substantial and combinatorially intricate. Our construction, by contrast, is relatively elementary, and the proof of the equivalence with the original approach is straightforward.

We have obtained further results concerning the ω-categorical cone and cylinder constructions; for reasons of space, the details of these results are reserved for a future paper, but let us at least outline them here. The first makes precise the analogy between our cones and cylinders and the topological ones, by exhibiting the cylinder of a strict globular ω-category X as its *lax Gray tensor product* [2] $X \otimes \mathbf{2}$ with the arrow category, and exhibiting the cone of X as the pushout of the codomain inclusion $X \to X \otimes \mathbf{2}$ along the unique map $X \to 1$. The second additional result has to do with the *freeness* of the orientals and cubes. In [7], a strict globular ω-category is called *free* (also *cofibrant* [5]) when it admits a presentation by iteratively adjoining new n-cells into existing n-cell boundaries. An important result of [7] (the "excision of extremals" algorithm) shows that the strict globular ω-category on any parity complex—so in particular, any oriental or cube—is free. Our second additional result allows us to recover the freeness of the orientals and cubes inductively, by showing that that both cone and cylinder *preserve freeness of ω-categories*.

Beyond this introduction, this paper comprises the following parts. Section 2 describes some necessary background on ω-categories; Section 3 introduces our cone and cylinder constructions; Section 4 proves that the iterated cones of the terminal ω-category are the orientals, while Section 5 proves that the iterated cylinders of the terminal ω-category are the cubes. Appendix A gives proofs of well-definedness deferred from Section 3.

2. Background

In the rest of the paper, *ω-category* will mean *strict globular ω-category*; in this section, we recall those aspects of their theory necessary for our development. A *globular set* X is a diagram of sets

$$\cdots \underset{t}{\overset{s}{\rightrightarrows}} X_{n+1} \underset{t}{\overset{s}{\rightrightarrows}} X_n \underset{t}{\overset{s}{\rightrightarrows}} \cdots \underset{t}{\overset{s}{\rightrightarrows}} X_0$$

satisfying the globularity equations $ss = st$ and $ts = tt$. If X is a globular set, then its *n-cells* are the elements of X_n; a pair of n-cells x, y are *parallel* if $n = 0$ or if $n > 0$ and $(sx, tx) = (sy, ty)$. Given $0 \leqslant n < k$, we write s_n, t_n for the maps $s^{k-n}, t^{k-n} \colon X_k \to X_n$, and for $x \in X_k$ we call the parallel pair $(s_n x, t_n x)$ the *n-boundary* of x. We write $x \colon y \rightsquigarrow z$ to indicate that (y, z) is the n-boundary of x; when $k = n + 1$, we may write $x \colon y \to z$ instead.

A *small ω-category* is a globular set X with *identity* and *composition* functions

$$i \colon X_n \to X_{n+1} \qquad \text{and} \qquad \circ_n \colon X_k \, {}_{t_n}\!\times_{s_n} X_k \to X_k$$

for all $0 \leqslant n < k$, satisfying the following three kinds of axioms. First, the *source–target* axioms that $s(ix) = t(ix) = x$ for all cells x and that:

$$s_k(x \circ_n y) = \begin{cases} s_k(x) \circ_n s_k(y) & \text{if } k > n; \\ s_k(y) & \text{if } k \leqslant n, \end{cases} \qquad t_k(x \circ_n y) = \begin{cases} t_k(x) \circ_n t_k(y) & \text{if } k > n; \\ t_k(x) & \text{if } k \leqslant n, \end{cases}$$

for all suitable cells x and y. Second, the *category* axioms that $x \circ_n i(sx) = x = i(tx) \circ_n x$ and $x \circ_n (y \circ_n z) = (x \circ_n y) \circ_n z$ for all suitable cells x, y, z. Finally, the *interchange* axiom that $(x \circ_n y) \circ_k (z \circ_n w) = (x \circ_k z) \circ_n (y \circ_k w)$ for all $n < k$ and suitable cells x, y, z, w.

The *dual* X^{op} of a globular set X is the globular set obtained by interchanging s and t at each stage; the *dual* X^{op} of a small ω-category is given by the dual of the underlying globular set of X equipped with the same identities and the reversed compositions at each dimension.

A map $f \colon X \to Y$ between globular sets comprises functions $f_n \colon X_n \to Y_n$ satisfying $sf_{n+1} = f_n s$ and $tf_{n+1} = f_n t$. An *ω-functor* $f \colon X \to Y$ between ω-categories is a map of underlying globular sets which preserve composition and identities, in the sense that $f(ix) = i(fx)$ and $f(x \circ_n y) = fx \circ_n fy$ for all suitable cells x and y. Of course, ω-functors compose, and so we have the category ω-**Cat** of small ω-categories and ω-functors.

The category ω-**Cat** has finite products, computed at the level of underlying globular sets, and so we can consider the category (ω-**Cat**)-**Cat** of small ω-**Cat**-*enriched* [3] categories; this is in fact equivalent to ω-**Cat**. Indeed, given an ω-category X, we obtain an ω-**Cat**-category with object set X_0, with hom

$X(x,y)$ the ω-category whose n-cells are the $(n+1)$-cells $x \rightsquigarrow y$ in X, and with composition ω-functors $X(y,z) \times X(x,y) \to X(x,z)$ given by $\circ_0$ in X. Conversely, if X is an ω-**Cat**-category, then there is an ω-category whose 0-cells are the objects of X and whose $(n+1)$-cells are the disjoint union of the n-cells of each $X(x,y)$, with composition $\circ_0$ given by the composition maps of X, and composition $\circ_{n+1}$ given by $\circ_n$ in the appropriate hom-ω-category.

Using this identification, we obtain the standard enriched-categorical notion of module (= profunctor) for ω-categories. A *right module* over an ω-category X comprises ω-categories $M(x)$ for each $x \in X_0$ together with ω-functors $m \colon M(y) \times X(x,y) \to M(x)$ for each $x, y \in X_0$ making each diagram:

$$\begin{array}{ccc} M(z) \times X(y,z) \times X(x,y) & \xrightarrow{m \times 1} & M(y) \times X(x,y) \\ {\scriptstyle 1\times m}\downarrow & & \downarrow{\scriptstyle m} \\ M(z) \times X(x,z) & \xrightarrow[m]{} & M(x) \end{array} \qquad \begin{array}{ccc} & M(x) \times 1 & \\ {\scriptstyle 1\times i}\swarrow & & \searrow{\scriptstyle \cong} \\ M(x) \times X(x,x) & \xrightarrow[m]{} & M(x) \end{array}$$

commute in ω-**Cat**. A *left module* over X is defined dually, while if X and Y are ω-categories, then a Y-X-*bimodule* comprises ω-categories $M(x,y)$ for $x, y \in X_0 \times Y_0$ such that each $M(x,–)$ is a left Y-module, each $M(–,y)$ is a right X-module, and each diagram of the following form commutes:

$$\begin{array}{ccc} Y(y,y') \times M(x,y) \times X(x',x) & \xrightarrow{m \times 1} & M(x,y') \times X(x',x) \\ {\scriptstyle 1\times m}\downarrow & & \downarrow{\scriptstyle m} \\ Y(y,y') \times M(x',y) & \xrightarrow[m]{} & M(x',y') \end{array} .$$

General enriched-categorical principles allow us to assign to any right X-module M a new ω-category $\mathrm{coll}(M)$, called the *collage* [8] of M. As an ω-**Cat**-category, $\mathrm{coll}(M)$ has object-set $X_0 + \{\star\}$ and hom-ω-categories:

$$\mathrm{coll}(M)(x,y) = \begin{cases} X(x,y) & \text{if } x, y \in X_0; \\ M(x) & \text{if } x \in X_0 \text{ and } y = \star; \\ \emptyset & \text{if } y \in X_0 \text{ and } x = \star; \\ 1 & \text{if } x = y = \star. \end{cases}$$

The non-trivial compositions in $\mathrm{coll}(M)$ are obtained from composition in X augmented by the action morphisms $M(y) \times X(x,y) \to M(x)$. Dually, each right X-module also has a collage, while if M is a Y-X-bimodule, then its collage $\mathrm{coll}(M)$ has object set $X_0 + Y_0$, hom-categories

$$\mathrm{coll}(M)(u,v) = \begin{cases} X(u,v) & \text{if } x, y \in X_0; \\ \emptyset & \text{if } u \in Y_0 \text{ and } v \in X_0; \\ M(u,v) & \text{if } u \in X_0 \text{ and } v \in Y_0; \\ Y(u,v) & \text{if } x, y \in Y_0, \end{cases}$$

and non-trivial compositions given by the composition morphisms of X together with the left and right M-action morphisms $M(x,y) \times X(x',x) \to M(x',y)$ and $Y(y,y') \times M(x,y) \to M(x,y')$.

3. Cones and cylinders

We now introduce the lax *coslices* and *slices* of an ω-category, and use them to define the basic *cone* and *cylinder* constructions, whose iterated application will yield the orientals and cubes. To simplify notation, it will be convenient henceforth to adopt the following conventions. First, we assume that $\circ_n$ binds more tightly than $\circ_k$ whenever $n < k$. In other words, we take it that:

$$x \circ_n y \circ_k z := (x \circ_n y) \circ_k z \qquad \text{and} \qquad x \circ_k y \circ_n z := x \circ_k (y \circ_n z) \ ,$$

and similarly for longer unbracketed composites. Second, we implicitly identify any k-cell with the identity $(k+\ell)$-cell thereon where necessary to make binary composition type-check. In other words, we take it that

$$x \circ_n y := x \circ_n i^\ell(y) \qquad \text{and} \qquad w \circ_n z := i^\ell(w) \circ_n z$$

for all suitable $x \in X_{k+\ell}$ and $y \in X_k$ or $w \in X_{k+\ell}$ and $z \in X_k$. We refer to the resultant composite as the *whiskering* of the $(k+\ell)$-cell by the k-cell.

Definition 1. If X is an ω-category and $a \in X_0$, then the *lax coslice* ω-category a/X is defined as follows.

- 0-cells $\boldsymbol{x} = (x, \bar{x})$ are pairs $x \in X_0$ and $\bar{x} \colon a \to x$.
- $(n+1)$-cells $\boldsymbol{x} = (x, \bar{x})$ with i-boundary $(\boldsymbol{m}_i, \boldsymbol{p}_i)$ for $i \leqslant n$ are given by pairs of the following form when n is even:

  $$(x \colon m_n \to p_n,\ \bar{x} \colon \bar{p}_{n-1} \circ_{n-1} \cdots \bar{p}_3 \circ_3 \bar{p}_1 \circ_1 x \circ_0 \bar{m}_0 \circ_2 \bar{m}_2 \cdots \circ_n \bar{m}_n \to \bar{p}_n) \ ,$$

  and by pairs of the following form when n is odd:

  $$(x \colon m_n \to p_n,\ \bar{x} \colon \bar{m}_n \to \bar{p}_n \circ_n \cdots \bar{p}_3 \circ_3 \bar{p}_1 \circ_1 x \circ_0 \bar{m}_0 \circ_2 \bar{m}_2 \cdots \circ_{n-1} \bar{m}_{n-1}) \ .$$

- If $\boldsymbol{x}$ and $\boldsymbol{y}$ satisfy $t_n(\boldsymbol{x}) = s_n(\boldsymbol{y})$, with common i-boundary $(\boldsymbol{m}_i, \boldsymbol{p}_i)$ for each $i < n$, then $\boldsymbol{y} \circ_n \boldsymbol{x}$ is given by the following pair when n is even:

  $$(y \circ_n x,\ \bar{y} \circ_{n+1} \bar{p}_{n-1} \circ_{n-1} \cdots \bar{p}_3 \circ_3 \bar{p}_1 \circ_1 s_{n+1} y \circ_0 \bar{m}_0 \circ_2 \bar{m}_2 \cdots \circ_{n-2} \bar{m}_{n-2} \circ_n \bar{x})$$

  and by the following pair when n is odd:

  $$(y \circ_n x,\ \bar{y} \circ_n \bar{p}_{n-2} \circ_{n-2} \cdots \bar{p}_3 \circ_3 \bar{p}_1 \circ_1 t_{n+1} x \circ_0 \bar{m}_0 \circ_2 \bar{m}_2 \cdots \circ_{n-1} \bar{m}_{n-1} \circ_{n+1} \bar{x}) \ .$$

- The identity $(n+1)$-cell on an n-cell $(x, \bar{x})$ is $(ix, i\bar{x})$.

We write $\pi \colon a/X \to X$ for the ω-functor defined by $\pi(x, \bar{x}) = x$.

Dually, for any $b \in X_0$, we define the *lax slice* ω-category X/b to be $(b/X^{\mathrm{op}})^{\mathrm{op}}$; explicitly, this means that:

- 0-cells $\boldsymbol{x} = (x, \hat{x})$ are pairs $x \in X_0$ and $\hat{x} \colon x \to b$.

- $(n+1)$-cells $\boldsymbol{x} = (x, \hat{x})$ with i-boundary $(\boldsymbol{m}_i, \boldsymbol{p}_i)$ for $i \leqslant n$ are given by pairs of the following form when n is even:

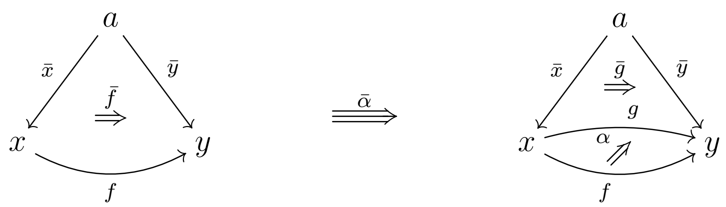
  $$(x\colon m_n \to p_n,\ \hat{x}\colon \hat{m}_n \to \hat{p}_n \circ_n \cdots \hat{p}_2 \circ_2 \hat{p}_0 \circ_0 x \circ_1 \hat{m}_1 \circ_3 \hat{m}_3 \cdots \circ_{n-1} \hat{m}_{n-1})\ ,$$

  and by pairs of the following form when n is odd:

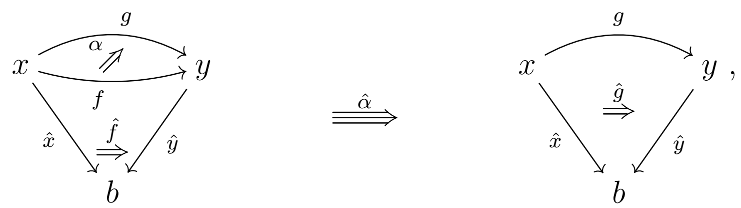
  $$(x\colon m_n \to p_n,\ \hat{x}\colon \hat{p}_{n-1} \circ_{n-1} \cdots \hat{p}_2 \circ_2 \hat{p}_0 \circ_0 x \circ_1 \hat{m}_1 \circ_3 \hat{m}_3 \cdots \circ_n \hat{m}_n \to \hat{p}_n)\ ,$$

with composition being given dually to that in a/X. Combining the preceding two constructions, if X is an ω-category and $a, b \in X_0$, then the *lax bislice* $a/X/b$ is the pullback of ω-categories $a/X \times_X X/b$. Explicitly, this means that objects in $a/X/b$ are triples $(x, \bar{x}, \hat{x})$ where $x \in X_0$ and $\bar{x}\colon a \to x$ and $\hat{x}\colon x \to b$, and similarly for higher cells.

In pasting notation, a 1-cell $(f, \bar{f})\colon (x, \bar{x}) \to (y, \bar{y})$ in a/X is given by a lax-commutative triangle as on the left of the following diagram; the entirety of this same diagram depicts a 2-cell $(\alpha, \bar{\alpha})\colon (f, \bar{f}) \to (g, \bar{g})$ in a/X.



Correspondingly, a 1-cell in the lax slice X/b is as on the right in:



while the whole diagram depicts a typical 2-cell.

It is not immediate that the lax coslice (and hence slice) ω-categories are well-defined. One way to show this is to view their construction as a particular case of the *Grothendieck construction* for ω-categories of [10, Section 4.1]. This construction assigns to each left module M over an ω-category X, an ω-functor (an "opfibration") $\int M \to X$; applying it to the representable left module $X(a, –)$ yields $\pi\colon a/X \to X$, so that the well-definedness of a/X is a consequence of [10, Proposition 4.6]. However, one may also prove well-definedness directly; so as to have a self-contained presentation, we give this proof as Proposition 11 below.

Definition 2. Let X be an ω-category and $a, b \in X_0$. We define the ω-functor $\bullet\colon a/X \times X(b, a) \to b/X$ to have action on cells given inductively as follows:

- On 0-cells, $\boldsymbol{x} \bullet h = (x, \bar{x} \circ_0 h)$;
- On $(n+1)$-cells, $\boldsymbol{x} \bullet h = (x, \bar{x} \circ_0 h)\colon s\boldsymbol{x} \bullet sh \to t\boldsymbol{x} \bullet th$.

Once again, it is not immediate that this gives a well-defined ω-functor; we verify this in Proposition 12 below. Thereafter, it is immediate that these actions satisfy the necessary associativity and unit axioms to give a right X-module $(\text{–})/X$. By duality, we obtain from the lax slice categories X/b a left module $X/(\text{–})$, and from the lax bislice categories an X-X-bimodule $(\text{–})/X/(\text{–})$ whose value at a, b is $a/X/b$ with right X-action inherited from a/X and left X-action inherited from X/b. Using these modules, we may now give:

Definition 3. If X is an ω-category, then $s(X)$, the *cone under* X is the collage of the right X-module $(\text{–})/X$. The *cone over* X, $\bar{s}(X)$, is the collage of the left X-module $(\text{–})/X$, while the *cylinder on* X, $c(X)$, is the collage of the X-X-bimodule $(\text{–})/X/(\text{–})$.

4. Orientals

In this section, we prove our first main result, identifying the iterated cones of the terminal ω-category with the *orientals* of [6]. We begin by recalling the definition of the orientals, following the presentation of [7]. For any natural numbers n and j, we write $[n]$ for the set $\{0, \dots, n\}$, and write $[n]_j$ for the set of order-preserving injections $[j] \rightarrowtail [n]$. If $j > 0$ and $a \in [n]_j$, then we define the sets of *even* and *odd* faces $a^+, a^- \subset [n]_{j-1}$ by

$$a^+ = \{a\delta_{2i} : 0 \leqslant 2i \leqslant j\} \quad \text{and} \quad a^- = \{a\delta_{2i+1} : 0 \leqslant 2i+1 \leqslant j\} \ ,$$

where here $\delta_k \colon [j-1] \rightarrowtail [j]$ is the unique order-preserving injection for which $k \notin \operatorname{im} \delta_k$. If $\xi \subset [n]_j$, then we write $\xi^- = \bigcup_{a\in\xi} a^-$ and $\xi^+ = \bigcup_{a\in\xi} a^+$. We may write elements $a \in [n]_j$ as increasing lists $(a_0 \cdots a_j)$ of elements in $[n]$; with this notation, we have for example:

$$\{(135), (125)\}^- = \{(15)\} \quad \text{and} \quad \{(135), (125)\}^+ = \{(35), (13), (12), (25)\} \ .$$

Definition 4. The nth oriental $\mathcal{O}(n)$ is the strict ω-category defined as follows.

- 0-cells are natural numbers $i \in \{0, \dots, n\}$; we identify i with the singleton subset $\{(i)\}$ of $[n]_0$.
- $(j+1)$-cells ξ with successive k-boundaries (μ_k, π_k) for each $k \leqslant j$ are finite subsets $\xi \subset [n]_{j+1}$ such that:
  - (i) If $a \neq b \in \xi$ then $a^+ \cap b^+ = a^- \cap b^- = \emptyset$;
  - (ii) $\pi_j = (\mu_j \cup \xi^+) \setminus \xi^-$ and $\mu_j = (\pi_j \cup \xi^-) \setminus \xi^+$ in $[n]_j$.

Given $(j+1)$-cells $\xi \colon x \to y$ and $\zeta \colon y \to z$, their $\circ_j$-composite is $\zeta \cup \xi \colon x \to z$; while if the $(j+1)$-cells $\xi \colon x \to y$ and $\zeta \colon w \to z$ satisfy $t_i(\xi) = s_i(\zeta)$ for a fixed $i < j$ then their $\circ_i$-composite is $\zeta \cup \xi \colon w \circ_i x \to z \circ_i y$. The identity $(j+1)$-cell on the j-cell x is given by $\emptyset \colon x \to x$. (In particular, *whiskering* a k-cell of $\mathcal{O}(n)$ by a lower-dimensional cell does not change its k-dimensional part).

As with the coslices of Section 3, it is by no means immediate that the orientals are well-defined ω-categories; the problem is showing that the well-formedness and movement conditions are stable under composition, and it is one of the main theorems of [6] that this is so. Following [7], when a set $\xi \subset [n]_j$ satisfies the condition in (i) above, we say that ξ is *well-formed*, and when it satisfies the two conditions in (ii), we say that ξ *moves* μ_j *to* π_j, and write $\xi \colon \mu_j \twoheadrightarrow \pi_j$; we refer to the two conditions involved as the *first* and *second movement conditions*.

Before giving our first main result, we recall from [6, §2] a useful characterisation of the 1-cells of the orientals:

Lemma 5. *Any* 1*-cell* $\xi \colon i \to j$ *in* $\mathcal{O}(n)$ *is either* $\emptyset \colon i \to i$ *or takes the form* $\xi = \{(k_0\, k_1), (k_1\, k_2), \dots, (k_{r-1}\, k_r)\}$ *for* $i = k_0 < \cdots < k_r = j$ *in* $[n]$.

Proof. If $i \neq j$, then by movement we have $i \in \xi^- \setminus \xi^+$ and $j \in \xi^+ \setminus \xi^-$ and for all $k \neq i, j$ that $k \in \xi^-$ iff $k \in \xi^+$. By well-formedness, it follows that the values i and j appear *exactly* once in ξ, as odd and even faces respectively, and all other values appear exactly twice, as an even and odd face respectively. This gives the required form; a similar argument shows that, when $i = j$, the only possibility is $\xi = \emptyset$. □

Theorem 6. *The nth cone* $s^n(1)$ *under the terminal* ω*-category is isomorphic to the nth oriental* $\mathcal{O}(n)$.

Proof. We prove this by induction on n, simultaneously with the result that:

(4.1) if $\xi \colon x \to y$ and $\zeta \colon y \to z$ are $(j+1)$-cells in $\mathcal{O}(n)$, then $\xi \cap \zeta = \emptyset$.

The case $n = 0$ is clear. For the inductive step, we assume the result for n, and begin by showing $s(\mathcal{O}(n)) \cong \mathcal{O}(n+1)$. Removing $n+1$ from $\mathcal{O}(n+1)$ or $\star$ from $s(\mathcal{O}(n))$ yields in both cases $\mathcal{O}(n)$, and in both cases, the only maps from this removed object are identities. It thus suffices to find ω-isomorphisms $\varphi_i \colon s(\mathcal{O}(n))(i, \star) = i/\mathcal{O}(n) \to \mathcal{O}(n+1)(i, n+1)$ which are compatible with composition, in the sense that for each $(\boldsymbol{x}, h) \in j/\mathcal{O}(n) \times \mathcal{O}(n)(i,j)$, we have $\varphi_i(\boldsymbol{x} \bullet h) = \varphi_j(\boldsymbol{x}) \circ_0 h$.

First we introduce some notation. Given $a = (a_0 \cdots a_j) \in [n]_j$, we write $a^\vee$ for $(a_0 \cdots a_j\; n{+}1) \in [n{+}1]_{j+1}$, and given $\xi \subset [n]_j$, we write $\xi^\vee$ for $\{a^\vee : a \in \xi\}$. Note that for any $\xi \subset [n]_j$ and $j > 0$, we have:

$$(\xi^\vee)^+ = \begin{cases} (\xi^+)^\vee & \text{if } j \text{ even;} \\ (\xi^+)^\vee \cup \xi & \text{if } j \text{ odd,} \end{cases} \qquad (\xi^\vee)^- = \begin{cases} (\xi^-)^\vee \cup \xi & \text{if } j \text{ even;} \\ (\xi^-)^\vee & \text{if } j \text{ odd.} \end{cases} \tag{4.2}$$

We now define $\varphi_i \colon i/\mathcal{O}(n) \to \mathcal{O}(n+1)(i, n+1)$ on cells of all dimension by

$$\varphi_i(\boldsymbol{x}) = \begin{cases} \bar{x} \cup x^\vee \colon i \to n+1 & \text{for } \boldsymbol{x} \text{ a 0-cell;} \\ \bar{x} \cup x^\vee \colon \varphi_i(\boldsymbol{m}) \to \varphi_i(\boldsymbol{p}) & \text{for } \boldsymbol{x} \colon \boldsymbol{m} \to \boldsymbol{p} \text{ a } (j+1)\text{-cell.} \end{cases}$$

We will show by induction on dimension that this assignation is well-defined and bijective. For the base case, we use Lemma 5. Any 0-cell $(x, \bar{x} \colon i \to x)$ of $i/\mathcal{O}(n)$ has either $i = x$ and $\bar{x} = \emptyset$—in which case $\varphi_i(\boldsymbol{x}) = \{(i\ n{+}1)\} \colon i \to n{+}1$ is well-defined—or has $i < x$ and $\bar{x} = \{(i\, k_1), \dots, (k_{r-1}\, x)\}$—in which case $\varphi_i(\boldsymbol{x}) = \{(i\, k_1), \dots, (k_{r-1}\, x), (x\ n+1)\} \colon i \to n+1$ is again well-defined. In fact, by Lemma 5, *any* $\xi \colon i \to n+1$ in $\mathcal{O}(n+1)$ is uniquely of one of the two forms just listed, so that φ_i is a bijection on 0-cells.

Suppose now that we have shown that φ_i is well-defined and bijective on all cells up to dimension j; then for any parallel pair of j-cells $(\boldsymbol{m}_j, \boldsymbol{p}_j)$ in $i/\mathcal{O}(n)$ with successive boundaries $(\boldsymbol{m}_k, \boldsymbol{p}_k)$ for $k < j$, we will show that φ_i gives a well-defined bijection between cells $\boldsymbol{x} \colon \boldsymbol{m}_j \to \boldsymbol{p}_j$ and ones $\xi \colon \varphi_i(\boldsymbol{m}_j) \to \varphi_i(\boldsymbol{p}_j)$. We consider only the case where j is odd; the even case is identical in form, and so omitted. Observe first that the operation on subsets

$$\begin{aligned} \mathcal{P}[n]_{j+1} \times \mathcal{P}[n]_{j+2} &\to \mathcal{P}[n+1]_{j+2} \\ (x, \bar{x}) &\mapsto \bar{x} \cup x^{\vee} \end{aligned} \tag{4.3}$$

underlying φ_i's action on $(j+1)$-cells is bijective. A cell $\boldsymbol{x} \colon \boldsymbol{m}_j \to \boldsymbol{p}_j$ is an element $(x, \bar{x})$ of the domain of (4.3) satisfying the three conditions that:

(i) x and $\bar{x}$ are well-formed; (ii) $x \colon m_j \twoheadrightarrow p_j$; (iii) $\bar{x} \colon \bar{m}_j \twoheadrightarrow \bar{p}_j \cup x$,

while a cell $\varphi_i(\boldsymbol{m}_j) \to \varphi_i(\boldsymbol{p}_j)$ is an element $\bar{x} \cup x^{\vee}$ of the codomain such that:

(iv) $\bar{x} \cup x^{\vee}$ is well-formed; (v) $\bar{x} \cup x^{\vee} \colon \bar{m}_j \cup m_j^{\vee} \twoheadrightarrow \bar{p}_j \cup p_j^{\vee}$.

Thus to check well-definedness and bijectivity of φ_i on $(j+1)$-cells, it suffices to show that (i)–(iii) are equivalent to (iv) & (v).

Now, if $\bar{x} \cup x^{\vee}$ is well-formed then clearly so is $\bar{x}$, but in fact also x, as if $a \neq b \in x$ shared an even or odd face, then so would $a^{\vee} \neq b^{\vee}$ in $x^{\vee} \subset \xi$. Thus (iv) implies (i). Conversely, if x and $\bar{x}$ are well-formed, then both components of $\bar{x} \cup x^{\vee}$ are individually well-formed, while if $a \in \bar{x}$ and $b^{\vee} \in x^{\vee}$, then clearly a and $b^{\vee}$ share no even faces since $n+1 \notin a$, and could only share an odd face if $b \in x$ were an odd face of $a \in \bar{x}$; and this is impossible if $\bar{x} \colon \bar{m}_j \twoheadrightarrow \bar{p}_j \cup x$ since then $(\bar{p}_j \cup x) \cap \bar{x}^- = \emptyset$. So (i) and (iii) imply (iv).

Turning now to the movement conditions, we have $(\bar{x} \cup x^{\vee})^+ = \bar{x}^+ \cup (x^+)^{\vee}$ and $(\bar{x} \cup x^{\vee})^- = \bar{x}^- \cup x \cup (x^-)^{\vee}$ by (4.2); so (v) is equivalent to:

$$\begin{aligned} (\bar{p}_j \cup p_j^{\vee}) &= (\bar{m}_j \cup m_j^{\vee}) \cup (\bar{x}^+ \cup (x^+)^{\vee}) \setminus (\bar{x}^- \cup x \cup (x^-)^{\vee}) \\ (\bar{m}_j \cup m_j^{\vee}) &= (\bar{p}_j \cup p_j^{\vee}) \cup (\bar{x}^- \cup x \cup (x^-)^{\vee}) \setminus (\bar{x}^+ \cup (x^+)^{\vee}) \ ; \end{aligned}$$

now as the terms which are under $(–)^{\vee}$ are disjoint from those which are not, and $(–)^{\vee}$ is a bijection, this is equivalent to the four conditions:

$$\begin{aligned} p_j &= m_j \cup x^+ \setminus x^- & \qquad \bar{p}_j &= \bar{m}_j \cup \bar{x}^+ \setminus (\bar{x}^- \cup x) \\ m_j &= p_j \cup x^- \setminus x^+ & \qquad \bar{m}_j &= \bar{p}_j \cup \bar{x}^- \cup x \setminus \bar{x}^+ \ . \end{aligned} \tag{4.4}$$

The left two are precisely (ii), and the lower right is the second movement condition for (iii). The upper right will imply the first movement condition $\bar{p}_j \cup x = \bar{m}_j \cup \bar{x}^+ \setminus \bar{x}^-$ for (iii) so long as $\bar{x}^- \cap x = \emptyset$; but this is certainly the case if $\bar{x} \cup x^\vee$ is well-formed, as if $a \in \bar{x}$ had an odd face b in x, then $a \neq b^\vee$ would share an odd face in $\bar{x} \cup x^\vee$. So (iv) and (v) imply (ii) and (iii). Finally, since (v) is equivalent to the conditions in (4.4), it will follow from (ii) and (iii) so long as we know that $x \cap \bar{p}_j = \emptyset$. Now, observe that $\bar{x}$ is a cell

$$\bar{m}_j \to \bar{p}_j \circ_j \bar{p}_{j-2} \circ_{j-2} \cdots \bar{p}_1 \circ_1 x \circ_0 \bar{m}_0 \cdots \circ_{j-1} \bar{m}_{j-1}$$

so that in particular, the cells $\bar{p}_j$ and $\bar{p}_{j-2} \circ_{j-2} \cdots \bar{p}_1 \circ_1 x \circ_0 \bar{m}_0 \cdots \circ_{j-1} \bar{m}_{j-1}$ of $\mathcal{O}(n)$ are j-composable; applying the inductive instance of (4.1) for $\mathcal{O}(n)$ we conclude that $x \cap \bar{p}_j = \emptyset$ as required.

This show that each φ_i is a bijective map on cells of all dimension; it remains only to show ω-functoriality and compatibility with the actions by $\bullet$ and $\circ_0$. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are $(j+1)$-cells of $i/\mathcal{O}(n)$ with $t_\ell(\boldsymbol{x}) = s_\ell(\boldsymbol{y})$ and with common boundary $(\boldsymbol{m}_k, \boldsymbol{p}_k)$ for each $k < \ell$, then:

$$\begin{aligned}
\varphi_i(\boldsymbol{y} \circ_\ell \boldsymbol{x}) &= \varphi_i(y \circ_\ell x,\ \bar{y} \circ_{\ell+1} \bar{p}_{\ell-1} \circ_{\ell-1} \cdots \bar{p}_1 \circ_1 s_{\ell+1} y \circ_0 \bar{m}_0 \cdots \circ_{\ell-2} \bar{m}_{\ell-2} \circ_\ell \bar{x}) \\
&= \varphi_i(y \cup x, \bar{y} \cup \bar{x}) = (\bar{y} \cup \bar{x}) \cup (y \cup x)^\vee \\
&= (x^\vee \cup \bar{x}) \cup (y^\vee \cup \bar{y}) = \varphi_i(\boldsymbol{y}) \circ_n \varphi_i(\boldsymbol{x})
\end{aligned}$$

when ℓ is even, and correspondingly when ℓ is odd. As for identity morphisms, we have $\varphi_i(i\boldsymbol{x}) = \varphi_i(ix, i\bar{x}) = \emptyset \cup \emptyset^\vee = \emptyset = i\varphi_i(\boldsymbol{x})$; so φ_i is ω-functorial as required. To show compatibility of the φ_i's with composition, we argue similarly that $\varphi_i(\boldsymbol{x} \bullet h) = \varphi_i(x, \bar{x} \circ_0 h) = (\bar{x} \cup h) \cup x^\vee = (\bar{x} \cup x^\vee) \cup h = \varphi_j(\boldsymbol{x}) \circ_0 h$.

This proves that $s(\mathcal{O}(n)) \cong \mathcal{O}(n+1)$, and it remains only to derive (4.1) for $\mathcal{O}(n+1)$. The case of 0-composable 1-cells is easy from Lemma 5, while any pair of $(j+1)$-composable $(j+2)$-cells must live in some hom-ω-category of $\mathcal{O}(n+1)$; the only new case to consider is that of $\mathcal{O}(n+1)(i, n+1) \cong i/\mathcal{O}(n)$. For this, let $\boldsymbol{x}\colon \boldsymbol{a} \to \boldsymbol{b}$ and $\boldsymbol{y}\colon \boldsymbol{b} \to \boldsymbol{c}$ be $(j+1)$-cells in $i/\mathcal{O}(n)$ with common boundary $(\boldsymbol{m}_k, \boldsymbol{p}_k)$ for all $k < j$. We have $x\colon a \to b$ and $y\colon b \to c$ in $\mathcal{O}(n)$, whence $x \cap y = \emptyset$ by (4.1) for $\mathcal{O}(n)$; moreover, assuming j is even, we have that $\bar{y}$ and $\bar{p}_{j-1} \circ_{j-1} \cdots \bar{p}_1 \circ_1 y \circ_0 \bar{m}_0 \cdots \circ_{j-2} \bar{m}_{j-2} \circ_j \bar{x}$ are $(j+1)$-composable $(j+2)$-cells, whence $\bar{x} \cap \bar{y} = \emptyset$ again by (4.1) for $\mathcal{O}(n)$; a similar argument shows $\bar{x} \cap \bar{y} = \emptyset$ when j is odd. We conclude that the composable pair $\varphi_i(\boldsymbol{x}) = \bar{x} \cup x^\vee$ and $\varphi_i(\boldsymbol{y}) = \bar{y} \cup y^\vee$ satisify $(\bar{x} \cup x^\vee) \cap (\bar{y} \cup y^\vee) = \emptyset$, as required. □

Remark 7. The condition (4.1) on composition of cells in orientals is proved by Street in [6, Theorem 3.12]; by not simply quoting his result, we have avoided using any aspect of the theory of orientals beyond the basic definitions, and this allows our main theorem to provide an alternative and simpler proof that the orientals do indeed have a well-defined composition. Arguing inductively, once we know that $\mathcal{O}(n)$ is well-defined, then so too is $s(\mathcal{O}(n))$; now transporting across the isomorphism of globular sets $s(\mathcal{O}(n)) \cong \mathcal{O}(n+1)$ shows that composition in $\mathcal{O}(n+1)$ is also well-defined.

5. Cubes

We now turn to our second main result, which will identify the iterated cylinders on the terminal ω-category with the *cubes*. We begin by recalling their definition, following again the presentation of [7]. Given natural numbers n and j, we write $[\![n]\!]$ for the set of strings of length n in the symbols $\ominus$, $\odot$, and $\oplus$, and write $[\![n]\!]_j$ for the subset of such strings in which the symbol $\odot$ appears exactly j times. If $j > 0$ and $a \in [\![n]\!]_j$, then we define the sets $a^-, a^+ \subset [\![n]\!]_{j-1}$ of *odd* and *even* faces by:

$$a^- = \{a\delta_i^- : 1 \leqslant i \leqslant j\} \qquad \text{and} \qquad a^+ = \{a\delta_i^+ : 1 \leqslant i \leqslant j\}$$

where $a\delta_i^-$ denotes the string obtained from a by replacing the ith occurence of $\odot$ therein by either $\ominus$ or $\oplus$ according as i is odd or even, and where $a\delta_i^+$ denotes similarly the string obtained by replacing the ith $\odot$ by either $\oplus$ or $\ominus$ according as i is odd or even. Like before, for any $\xi \subset [\![n]\!]_j$ we define $\xi^- = \bigcup_{a\in\xi} a^-$ and $\xi^+ = \bigcup_{a\in\xi} a^+$; with this notation, we have, for example, that:

$$\{\ominus\odot\odot\}^+ = \{\ominus\oplus\odot, \ominus\odot\ominus\} \qquad \text{and} \qquad \{\ominus\odot, \odot\ominus\}^- = \{\ominus\ominus\}\ .$$

Definition 8. The nth cube $\mathcal{Q}(n)$ is is the strict ω-category defined as follows.

- 0-cells are elements of $[\![n]\!]_0$: strings of length n of $\ominus$'s and $\oplus$'s. We identify each such string with the corresponding singleton subset of $[\![n]\!]_0$.
- $(j+1)$-cells ξ with successive k-boundaries (μ_k, π_k) for each $k \leqslant j$ are finite subsets $\xi \subset [\![n]\!]_{j+1}$ such that:
  - (i) If $a \neq b \in \xi$ then $a^+ \cap b^+ = a^- \cap b^- = \emptyset$;
  - (ii) $\pi_j = (\mu_j \cup \xi^+) \setminus \xi^-$ and $\mu_j = (\pi_j \cup \xi^-) \setminus \xi^+$ in $[\![n]\!]_j$.

Given $(j+1)$-cells $\xi\colon x \to y$ and $\zeta\colon y \to z$, their $\circ_j$-composite is $\zeta \cup \xi\colon x \to z$; while if the $(j+1)$-cells $\xi\colon x \to y$ and $\zeta\colon w \to z$ satisfy $t_i(\xi) = s_i(\zeta)$ for a fixed $i < j$ then their $\circ_i$-composite is $\zeta \cup \xi\colon w \circ_i x \to z \circ_i y$. The identity $(j+1)$-cell on the j-cell x is given by $\emptyset\colon x \to x$.

Note that this definition is identical to Definition 4 except that $[n]_j$ is replaced by $[\![n]\!]_j$ and the meaning of $(–)^+$ and $(–)^-$ adapted accordingly. This is because both are instances of the general definition in [7] of the *free ω-category on a parity complex*; the basic data of a parity complex are sets like $[n]_j$ or $[\![n]\!]_j$ equipped with functions $(–)^+$ and $(–)^-$ satisfying axioms. As before, it is quite non-trivial that the cubes are well-defined ω-categories, and as before, we will be able to deduce this well-definedness from the inductive argument we give.

As before, we refer to the conditions in (i) and (ii) above as *well-formedness* and *movement*, and with the same notational conventions. Exactly the same argument as in Lemma 5 now shows that:

Lemma 9. *Any* 1*-cell* $\xi\colon a \to b$ *in* $\mathcal{Q}(n)$ *is of the form* $\xi = \{f_1, \dots, f_r\}$*, where either* $r = 0$ *and* $a = b$*, or* $r > 0$*,* $f_1^- = a$*,* $f_i^+ = f_{i+1}^-$ *for all* $1 < i < r$ *and* $f_r^+ = b$*.*

With this in place, we are ready to give the proof of our second main result, which follows a very similar pattern to the first.

Theorem 10. *The nth cylinder* $c^n(1)$ *on the terminal* ω*-category is isomorphic to the nth cube* $\mathcal{Q}(n)$*.*

Proof. First we introduce some notation. Given $a = a_1 \cdots a_n \in [\![n]\!]$, we write $a\eta$ for $a_1 \cdots a_n\eta \in [\![n+1]\!]$ where $\eta \in \{\ominus, \odot, \oplus\}$, and given $\xi \subset [\![n]\!]$, we write $\xi\eta$ for $\{a\eta : a \in \xi\}$. Note that for any $\xi \subset [\![n]\!]_j$ and $j > 0$, we have that:

$$(5.1) \quad (\xi\odot)^+ = \begin{cases} (\xi^+)\odot \cup \xi\oplus & \text{if } j \text{ even;} \\ (\xi^+)\odot \cup \xi\ominus & \text{if } j \text{ odd,} \end{cases} \qquad (\xi\odot)^- = \begin{cases} (\xi^-)\odot \cup \xi\ominus & \text{if } j \text{ even;} \\ (\xi^-)\odot \cup \xi\oplus & \text{if } j \text{ odd,} \end{cases}$$

and that $(\xi\eta)^\epsilon = (\xi^\epsilon)\eta$ for any $\eta \in \{\oplus, \ominus\}$ and $\epsilon \in \{+, -\}$. We now prove the result by induction on n, simultaneously with the result that:

$$(5.2) \qquad \text{if } \xi\colon x \to y \text{ and } \zeta\colon y \to z \text{ are } (j+1)\text{-cells in } \mathcal{Q}(n)\text{, then } \xi \cap \zeta = \emptyset.$$

The case $n = 0$ is clear. For the inductive step, we assume the result for n, and begin by showing $c(\mathcal{Q}(n)) \cong \mathcal{Q}(n+1)$. Recall that $c(\mathcal{Q}(n))$ is the collage of the bimodule $(\text{–})/\mathcal{Q}(n)/(\text{–})$ determined by bislice and thus contains two copies of $\mathcal{Q}(n)$ embedded on the left and right which we call $\mathcal{Q}(n)_l$ and $\mathcal{Q}(n)_r$. These can be mapped into $\mathcal{Q}(n+1)$ via ω-functors $(\text{–})\ominus\colon \mathcal{Q}(n)_l \to \mathcal{Q}(n+1)$ and $(\text{–})\oplus\colon \mathcal{Q}(n)_r \to \mathcal{Q}(n+1)$ which are easily shown to be bijective on hom-ω-categories and jointly bijective on 0-cells. In this way, we determine all of the data for an ω-isomorphism $\varphi\colon c(\mathcal{Q}(n)) \to \mathcal{Q}(n+1)$ except for the action on hom-ω-categories $c(\mathcal{Q}(n))(a,b) = a/\mathcal{Q}(n)/b$ when $a \in \mathcal{Q}(n)_l$ and $b \in \mathcal{Q}(n)_r$. To give this action is equally to give ω-isomorphisms

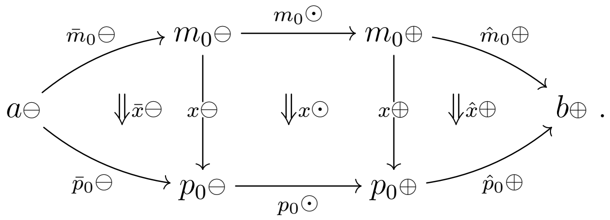
$$\varphi_{a,b}\colon c(\mathcal{Q}(n))(a,b) = a/\mathcal{Q}(n)/b \to \mathcal{Q}(n+1)(a\ominus, b\oplus)$$

which are compatible with composition, in the sense that we have $\varphi_{a,d}(k \bullet \boldsymbol{x} \bullet h) = \varphi_{c,d}(k) \circ_0 \varphi_{b,c}(\boldsymbol{x}) \circ_0 \varphi_{a,b}(h)$ for each $(k, \boldsymbol{x}, h) \in \mathcal{Q}(n)(c,d) \times b/\mathcal{O}(n)/c \times \mathcal{Q}(n)(a,b)$.

We will define $\varphi_{a,b}$ on cells of all dimension by:

$$\varphi_{a,b}(\boldsymbol{x}) = \begin{cases} \bar{x}\ominus \cup x\odot \cup \hat{x}\oplus\colon a\ominus \to b\oplus & \text{for } \boldsymbol{x} \text{ a 0-cell;} \\ \bar{x}\ominus \cup x\odot \cup \hat{x}\oplus\colon \varphi_{a,b}(\boldsymbol{m}) \to \varphi_{a,b}(\boldsymbol{p}) & \text{for } \boldsymbol{x}\colon \boldsymbol{m} \to \boldsymbol{p} \text{ a } (j+1)\text{-cell;} \end{cases}$$

for example, the action on 0- and 1-cells is as in the following diagram:



We will show by induction on dimension that this assignation is well-defined and bijective. For the base case, given a 0-cell $(x, \bar{x}, \hat{x})$ of $a/\mathcal{Q}(n)/b$ we may write $\bar{x} = \{f_1, \ldots, f_r\}\colon a \to x$ and $\hat{x} = \{g_1, \ldots, g_s\}\colon x \to b$ with the f_i's and g_k's satisfying the conditions of Lemma 9; since $(\xi\eta)^\epsilon = (\xi^\epsilon)\eta$ for any $\eta \in \{\oplus, \ominus\}$ and $\epsilon \in \{+, -\}$, it follows that

$$\{f_1\ominus, \ldots, f_r\ominus, x\odot, g_1\oplus, \ldots, g_s\oplus\}\colon a\ominus \to b\oplus$$

is a well-defined 1-cell of $\mathcal{Q}(n+1)$; in fact, it is easy to see from Lemma 9 that *any* $\xi\colon a\ominus \to b\oplus$ in $\mathcal{Q}(n+1)$ is of this form for a unique $(x, \bar{x}, \hat{x})$, and so $\varphi_{a,b}$ is not only well-defined but also bijective on 0-cells.

Suppose now that we have shown $\varphi_{a,b}$ is well-defined and bijective on all cells up to dimension j; then for any parallel pair of j-cells $(\boldsymbol{m}_j, \boldsymbol{p}_j)$ in $a/\mathcal{O}(n)/b$ with successive boundaries $(\boldsymbol{m}_k, \boldsymbol{p}_k)$ for $k < j$, we will show that $\varphi_{a,b}$ gives a well-defined bijection between cells $\boldsymbol{x}\colon \boldsymbol{m}_j \to \boldsymbol{p}_j$ and ones $\xi\colon \varphi_{a,b}(\boldsymbol{m}_j) \to \varphi_{a,b}(\boldsymbol{p}_j)$. We consider only the case where j is odd; the even case is identical in form, and so omitted. Observe first that the operation on subsets

$$\begin{aligned} \mathcal{P}\,[\![n]\!]_{j+1} \times \mathcal{P}\,[\![n]\!]_{j+2} \times \mathcal{P}\,[\![n]\!]_{j+2} &\to \mathcal{P}\,[\![n+1]\!]_{j+2} \\ (x, \bar{x}, \hat{x}) &\mapsto \bar{x}\ominus \cup x\odot \cup \hat{x}\oplus \end{aligned} \tag{5.3}$$

underlying $\varphi_{a,b}$'s action on $(j+1)$-cells is bijective. A cell $\boldsymbol{x}\colon \boldsymbol{m}_j \to \boldsymbol{p}_j$ is an element $(x, \bar{x}, \hat{x})$ of the domain of (5.3) satisfying the four conditions that:

(i) x, $\bar{x}$ and $\hat{x}$ are well-formed;
(ii) $x\colon m_j \twoheadrightarrow p_j$;
(iii) $\bar{x}\colon \bar{m}_j \twoheadrightarrow \bar{p}_j \cup x$;
(iv) $\hat{x}\colon \hat{m}_j \cup x \twoheadrightarrow \hat{p}_j$,

while a cell $\varphi_{a,b}(\boldsymbol{m}_j) \to \varphi_{a,b}(\boldsymbol{p}_j)$ is an element $\bar{x}\ominus \cup x\odot \cup \hat{x}\oplus$ of the codomain satisfying the two conditions that:

(v) $\bar{x}\ominus \cup x\odot \cup \hat{x}\oplus$ is well-formed;
(vi) $\bar{x}\ominus \cup x\odot \cup \hat{x}\oplus\colon \bar{m}_j\ominus \cup m_j\odot \cup \hat{m}_j\oplus \twoheadrightarrow \bar{p}_j\ominus \cup p_j\odot \cup \hat{p}_j\oplus$.

Thus to check well-definedness and bijectivity of $\varphi_{a,b}$ on $(j+1)$-cells, it suffices to show that (i)–(iv) are equivalent to (v) & (vi).

Now, if $\bar{x}\ominus \cup x\odot \cup \hat{x}\oplus$ is well-formed then so are $\bar{x}$, x and $\hat{x}$, since if $a \neq b \in x$ shared a positive or negative face, then so would $a\odot \neq b\odot$ in $x\odot \subset \xi$, and correspondingly for $\bar{x}$ and $\hat{x}$; so (v) implies (i). Conversely, if x, $\bar{x}$ and $\hat{x}$ are well-formed, then each component of $\bar{x}\ominus \cup x\odot \cup \hat{x}\oplus$ is individually well-formed, and so it remains to check the cross-terms. First, $a \in \bar{x}\ominus$ and $b \in \hat{x}\oplus$ cannot share *any* face, since its final symbol would be $\ominus$ and $\oplus$ simultaneously. Next, if $a \in \bar{x}\ominus$ and $b \in x\odot$ then $a^- \subset \bar{x}^-\ominus$ and $b^- \subset x\ominus \cup x^-\odot$; but (iii) ensures that $\bar{x}^-$ and x are disjoint so $a^- \cap b^- = \emptyset$. Likewise $a^+ \subset \bar{x}^+\ominus$ and $b^+ \subset x\oplus \cup x^+\odot$ and so $a^+ \cap b^+ = \emptyset$. A similar argument shows that $a \in \hat{x}\oplus$ and $b \in x\odot$ cannot share an odd face or an even face, and so (i) and (iii) imply (v).

Turning now to the movement conditions, we have $(x\odot)^+ = x^+\odot \cup x\oplus$ and $(x\odot)^- = x^-\odot \cup x\ominus$ by (5.1); so (vi) is equivalent to:

$$\begin{aligned}
\bar{p}_j\ominus \cup p_j\odot \cup \hat{p}_j\oplus &= (\bar{m}_j\ominus \cup m_j\odot \cup \hat{m}_j\oplus) \cup (\bar{x}^+\ominus \cup x^+\odot \cup x\oplus \cup \hat{x}^+\oplus) \\
&\qquad\qquad \setminus (\bar{x}^-\ominus \cup x^-\odot \cup x\ominus \cup \hat{x}^-\oplus) \\
\bar{m}_j\ominus \cup m_j\odot \cup \hat{m}_j\oplus &= (\bar{p}_j\ominus \cup p_j\odot \cup \hat{p}_j\oplus) \cup (\bar{x}^-\ominus \cup x^-\odot \cup x\ominus \cup \hat{x}^-\oplus) \\
&\qquad\qquad \setminus (\bar{x}^+\ominus \cup x^+\odot \cup x\oplus \cup \hat{x}^+\oplus) \ ;
\end{aligned}$$

now as terms ending with the three possible symbols are disjoint, and each $(-)\eta$ for $\eta \in \{\odot, \ominus, \oplus\}$ is a bijection, this is equivalent to the six conditions:

$$\begin{aligned}
p_j &= m_j \cup x^+ \setminus x^- & m_j &= p_j \cup x^- \setminus x^+ \\
\bar{p}_j &= \bar{m}_j \cup \bar{x}^+ \setminus (\bar{x}^- \cup x) & \bar{m}_j &= \bar{p}_j \cup \bar{x}^- \cup x \setminus \bar{x}^+ \\
\hat{p}_j &= \hat{m}_j \cup \hat{x}^+ \cup x \setminus \hat{x}^- & \hat{m}_j &= \hat{p}_j \cup \hat{x}^- \setminus (\hat{x}^+ \cup x)\ .
\end{aligned} \tag{5.4}$$

The top row is precisely (ii), the middle right is the second movement condition for (iii), and the bottom left is the first movement condition for (iv). The middle left will imply the first movement condition $\bar{p}_j \cup x = \bar{m}_j \cup \bar{x}^+ \setminus \bar{x}^-$ for (iii) so long as $\bar{x}^- \cap x = \emptyset$; but this is certainly the case if $\bar{x}\ominus \cup x\odot \cup \hat{x}\oplus$ is well-formed, as $(\bar{x}^- \cap x)\ominus = \bar{x}^-\ominus \cap x\ominus \subset (\bar{x}\ominus)^- \cap (x\odot)^- = \emptyset$. The bottom right will imply the first second condition $\hat{m}_j \cup x = \hat{p}_j \cup \hat{x}^- \setminus \hat{x}^+$ for (iv) so long as $\hat{x}^+ \cap x = \emptyset$; again, this is the case if $\bar{x}\ominus \cup x\odot \cup \hat{x}\oplus$ is well-formed, as $(\hat{x}^+ \cap x)\oplus = \bar{x}^+\oplus \cap x\oplus \subset (\hat{x}\oplus)^+ \cap (x\odot)^+ = \emptyset$. So (v) and (vi) imply (ii)–(iv).

Finally, since (vi) is equivalent to the conditions in (5.4), it will follow from (ii)–(iv) so long as we know that $x \cap \bar{p}_j = \emptyset$ and $\hat{m}_j \cap x = \emptyset$. Now, observe that $\bar{x}$ and $\hat{x}$ are cells

$$\begin{aligned}
&\bar{m}_j \to \bar{p}_j \circ_j \bar{p}_{j-2} \circ_{j-2} \cdots \bar{p}_1 \circ_1 x \circ_0 \bar{m}_0 \cdots \circ_{j-1} \bar{m}_{j-1} \\
\text{and}\quad &\hat{p}_{j-1} \circ_{j-1} \cdots \hat{p}_0 \circ_0 x \circ_1 \hat{m}_1 \cdots \circ_{j-2} \hat{m}_{j-2} \circ_j \hat{m}_j \to \hat{p}_j
\end{aligned}$$

so that in particular, the cells $\bar{p}_j$ and $\bar{p}_{j-2} \circ_{j-2} \cdots \bar{p}_1 \circ_1 x \circ_0 \bar{m}_0 \cdots \circ_{j-1} \bar{m}_{j-1}$ of $\mathcal{Q}(n)$ are j-composable; applying the inductive instance of (5.2) for $\mathcal{Q}(n)$ we conclude that $x \cap \bar{p}_j = \emptyset$. The same argument, applied to the domain of $\hat{x}$, shows that also $\hat{m}_j \cap x = \emptyset$ as required.

This shows that each $\varphi_{a,b}$ is a bijective map on cells of all dimension; we next show ω-functoriality and compatibility with composition. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are $(j+1)$-cells of $a/\mathcal{O}(n)/b$ with $t_\ell(\boldsymbol{x}) = s_\ell(\boldsymbol{y})$ and with common boundary

$(\boldsymbol{m}_k, \boldsymbol{p}_k)$ for each $k < \ell$, then:

$$\begin{aligned}\varphi_{a,b}(\boldsymbol{y} \circ_\ell \boldsymbol{x}) &= \varphi_i(y \circ_\ell x,\ \bar{y} \circ_{\ell+1} \bar{p}_{\ell-1} \circ_{\ell-1} \cdots \bar{p}_1 \circ_1 s_{\ell+1} y \circ_0 \bar{m}_0 \cdots \circ_{\ell-2} \bar{m}_{\ell-2} \circ_\ell \bar{x},\\ &\qquad\qquad \hat{y} \circ_\ell \hat{p}_{\ell-2} \circ_{\ell-2} \cdots \hat{p}_0 \circ_0 t_{\ell+1} x \circ_1 \hat{m}_1 \cdots \circ_{\ell-1} \hat{m}_{\ell-1} \circ_{\ell+1} \hat{x})\\ &= \varphi_{a,b}(y \cup x, \bar{y} \cup \bar{x}, \hat{y} \cup \hat{x}) = (y \cup x)\odot \cup (\bar{y} \cup \bar{x})\ominus \cup (\hat{y} \cup \hat{x})\oplus\\ &= (y\odot \cup \bar{y}\ominus \cup \hat{y}\oplus) \cup (x\odot \cup \bar{x}\ominus \cup \hat{x}\oplus) = \varphi_{a,b}(\boldsymbol{y}) \circ_\ell \varphi_{a,b}(\boldsymbol{x})\end{aligned}$$

when ℓ is even, and correspondingly when ℓ is odd. As for identity morphisms, we have $\varphi_{a,b}(i\boldsymbol{x}) = \varphi_{a,b}(ix, i\bar{x}, i\hat{x}) = \emptyset\ominus \cup \emptyset\odot \cup \emptyset\oplus = \emptyset = i\varphi_{a,b}(\boldsymbol{x})$; so $\varphi_{a,b}$ is ω-functorial as required. Compatibility of the $\varphi_{a,b}$'s with composition is similar: we have that $\varphi_{a,d}(k \bullet \boldsymbol{x} \bullet h) = \varphi_{a,d}(x, \bar{x} \circ_0 h, k \circ_0 \hat{x}) = (\bar{x} \cup h)\ominus \cup x\odot \cup (k \cup \hat{x})\oplus = k\oplus \cup (\bar{x}\ominus \cup x\odot \cup \hat{x}\oplus) \cup h\ominus = \varphi_{c,d}(k) \circ_0 \varphi_{b,c}(\boldsymbol{x}) \circ_0 \varphi_{a,b}(h)$.

This proves that $c(\mathcal{Q}(n)) \cong \mathcal{Q}(n+1)$, and it remains only to derive (5.2) for $\mathcal{Q}(n+1)$. In the case of 0-composable 1-cells, we see from Lemma 9 that if $\{f_1, \dots, f_r\}\colon a \to b$ is a 1-cell of $\mathcal{Q}(n+1)$, then each f_i will contain at least as many $\oplus$'s as a and strictly fewer $\oplus$'s than b; so if $\{g_1, \dots, g_s\}\colon b \to c$ is another 1-cell, then each g_k must contain strictly more $\oplus$'s than each f_i, thus proving disjointness. As for $(j+1)$-composable $(j+2)$-cells in $\mathcal{Q}(n+1)$, any pair of such must live in some hom-ω-category; the only new case to consider is that of $\mathcal{Q}(n+1)(a\ominus, b\oplus) \cong a/\mathcal{Q}(n)/b$. So let $\boldsymbol{x}\colon \boldsymbol{a} \to \boldsymbol{b}$ and $\boldsymbol{y}\colon \boldsymbol{b} \to \boldsymbol{c}$ be $(j+1)$-cells in $a/\mathcal{Q}(n)/b$ with common boundary $(\boldsymbol{m}_k, \boldsymbol{p}_k)$ for all $k < j$. We have $x\colon a \to b$ and $y\colon b \to c$ in $\mathcal{Q}(n)$, whence $x \cap y = \emptyset$ by (5.2) for $\mathcal{Q}(n)$; moreover, assuming j is even, we have that $\bar{y}$ and $\bar{p}_{j-1} \circ_{j-1} \cdots \bar{p}_1 \circ_1 y \circ_0 \bar{m}_0 \cdots \circ_{j-2} \bar{m}_{j-2} \circ_j \bar{x}$ are $(j+1)$-composable $(j+2)$-cells, whence $\bar{x} \cap \bar{y} = \emptyset$ again by (5.2) for $\mathcal{Q}(n)$; similarly, $\hat{x}$ and $\hat{y} \circ_n \hat{p}_{j-2} \circ_{j-2} \cdots \hat{p}_0 \circ_0 x \circ_1 \hat{m}_1 \cdots \circ_{j-1} \hat{m}_{j-1}$ are $(j+1)$-composable $(j+2)$-cells, whence $\hat{x} \cap \hat{y} = \emptyset$. A dual argument applies when j is odd, and in both cases we conclude that the composable pair $\varphi_i(\boldsymbol{x}) = \bar{x}\ominus \cup x\odot \cup \hat{x}\oplus$ and $\varphi_i(\boldsymbol{y}) = \bar{y}\ominus \cup y\odot \cup \hat{y}\oplus$ satisfy $(\bar{x}\ominus \cup x\odot \cup \hat{x}\oplus) \cap (\bar{y}\ominus \cup y\odot \cup \hat{y}\oplus) = \emptyset$, as required. ☐

As before, we have avoided using any aspect of the theory of parity complexes beyond the basic definitions, and so in an identical manner to Remark 7 we may exploit the preceding theorem to give a simpler proof that the cubes are indeed well-defined ω-categories.

Appendix A. Proofs of well-definedness

Proposition 11. *For any ω-category C and $a \in C_0$, the lax coslice a/C is a well-defined ω-category.*

Proof. We first show by induction on n that (a) the cells of a/C of dimension $\leqslant n$ are well-defined; and (b) for any parallel pair of n-cells $(\boldsymbol{m}_n, \boldsymbol{p}_n)$ with i-boundary $(\boldsymbol{m}_i, \boldsymbol{p}_i)$ for all $i < n$, there is, for n even, a well-defined ω-functor

$$M_n\colon C(m_0, p_0) \cdots (m_n, p_n) \to C(s\bar{p}_0, t\bar{p}_0)(s\bar{m}_1, t\bar{m}_1) \cdots (s\bar{p}_n, t\bar{p}_n) \tag{A.1}$$

sending x to $\bar{p}_{n-1} \circ_{n-1} \cdots \bar{p}_3 \circ_3 \bar{p}_1 \circ_1 x \circ_0 \bar{m}_0 \circ_2 \bar{m}_2 \cdots \circ_n \bar{m}_n$, and, for n odd, a well-defined ω-functor

$$P_n \colon C(m_0,p_0)\cdots(m_n,p_n) \to C(s\bar{p}_0,t\bar{p}_0)(s\bar{m}_1,t\bar{m}_1)\cdots(s\bar{m}_n,t\bar{m}_n) \tag{A.2}$$

sending x to $\bar{p}_n \circ_n \cdots \bar{p}_3 \circ_3 \bar{p}_1 \circ_1 x \circ_0 \bar{m}_0 \circ_2 \bar{m}_2 \cdots \circ_{n-1} \bar{m}_{n-1}$.

For the base case $n = 0$, it is clear for (a) that the notion of 0-cell is well-defined. As for (b), if $(\boldsymbol{m}_0, \boldsymbol{p}_0)$ are a (necessarily parallel) pair of 0-cells, then, since $\bar{m}_0 \colon a \to m_0$, the assignation $x \mapsto x \circ_0 \bar{m}_0$ defines an ω-functor $M_0 \colon C(m_0,p_0) \to C(s\bar{p}_0,t\bar{p}_0) = C(a,p_0)$ as required for (A.1).

We now assume the result for n, and verify it for $(n+1)$. First let n be even. For (a), let $(\boldsymbol{m}_n, \boldsymbol{p}_n)$ be a parallel pair of n-cells, and M_n the associated ω-functor (A.1); then an $(n+1)$-cell $\boldsymbol{x} \colon \boldsymbol{m}_n \to \boldsymbol{p}_n$ of a/C is a pair

$$(x \in C(m_0,p_0)\cdots(m_n,p_n),\ \bar{x} \colon M_n x \to \bar{p}_n)\ , \tag{A.3}$$

and so well-defined. For (b), if $\boldsymbol{m}_{n+1}, \boldsymbol{p}_{n+1}$ are both $(n+1)$-cells $\boldsymbol{m}_n \to \boldsymbol{p}_n$, then $\bar{m}_{n+1} \colon M_n m_{n+1} \to \bar{p}_n$ and $\bar{p}_{n+1} \colon M_n p_{n+1} \to \bar{p}_n$; whence the assignation $x \mapsto \bar{p}_{n+1} \circ_{n+1} M_n x$ yields an ω-functor

$$C(m_0,p_0)\cdots(m_{n+1},p_{n+1}) \to C(s\bar{p}_0,t\bar{p}_0)\cdots(s\bar{p}_n,t\bar{p}_{n+1})(M_n m_{n+1},\bar{p}_n)\ ,$$

which is of the correct form to be the P_{n+1} of (A.2). Suppose now that n is odd. For (a), if $(\boldsymbol{m}_n, \boldsymbol{p}_n)$ are parallel n-cells, and now P_n is the associated ω-functor of (A.2), then an $(n+1)$-cell $\boldsymbol{x} \colon \boldsymbol{m}_n \to \boldsymbol{p}_n$ of a/C is a pair

$$(x \in C(m_0,p_0)\cdots(m_n,p_n),\ \bar{x} \colon \bar{m}_n \to P_n x)\ , \tag{A.4}$$

and so, again, well-defined. For (b), if $\boldsymbol{m}_{n+1}, \boldsymbol{p}_{n+1} \colon \boldsymbol{m}_n \to \boldsymbol{p}_n$, then the operation $x \mapsto P_n x \circ_{n+1} \bar{m}_{n+1}$ defines an ω-functor of the right form to be the M_{n+1} of (A.1). This completes the inductive step.

So a/C is well-defined as a globular set; given $\boldsymbol{x} \in (a/C)_k$ and $n < k$, we will denote the ω-functor (A.1) or (A.2) associated to the n-boundary $(\boldsymbol{m}_n, \boldsymbol{p}_n)$ of $\boldsymbol{x}$ as $M_n^{\boldsymbol{x}}$ (for n even) or $P_n^{\boldsymbol{x}}$ (for n odd). Note that, for each $n < k$, we have by (A.3), (A.4) and induction that:

$$s_n(\bar{x}) = \begin{cases} \bar{m}_{n-1} & n \text{ even;} \\ M_{n-1}^{\boldsymbol{x}}(m_n) & n \text{ odd,} \end{cases} \quad \text{and} \quad t_n(\bar{x}) = \begin{cases} P_{n-1}^{\boldsymbol{x}}(p_n) & n \text{ even;} \\ \bar{p}_{n-1} & n \text{ odd.} \end{cases} \tag{A.5}$$

We now show that a/C is a well-defined ω-category. The identity operations are clearly well-defined; for composition, let $\boldsymbol{x} \colon \boldsymbol{a} \rightsquigarrow \boldsymbol{b}$ and $\boldsymbol{y} \colon \boldsymbol{b} \rightsquigarrow \boldsymbol{c}$ be n-composable k-cells whose common i-boundary for $i < n$ is $(\boldsymbol{m}_i, \boldsymbol{p}_i)$. First let n be odd. Writing $M = M_{n-1}^{\boldsymbol{x}} = M_{n-1}^{\boldsymbol{y}}$, the composite cell in a/C is the pair

$$\boldsymbol{y} \circ_n \boldsymbol{x} := (y \circ_n x,\ \bar{y} \circ_n M(t_{n+1}x) \circ_{n+1} \bar{x})\ .$$

The first component is clearly well-defined; writing $\bar{y} * \bar{x}$ for the second, note that the ω-functors $P_n^{\boldsymbol{x}}$ and $P_n^{\boldsymbol{y}}$ satisfy $P_n^{\boldsymbol{x}}(u) = \bar{b} \circ_n M(u)$ and $P_n^{\boldsymbol{y}}(u) = \bar{c} \circ_n M(u)$; from this and (A.5) we conclude that

$$\bar{x} \colon \bar{a} \rightsquigarrow \bar{b} \circ_n M(t_{n+1}x) \qquad \text{and} \qquad \bar{y} \colon \bar{b} \rightsquigarrow \bar{c} \circ_n M(t_{n+1}y)\ ,$$

so that $\bar{y} * \bar{x}$ is indeed a well-defined cell of C. We next check it has the correct source and target. If $k = n+1$, then we should have $\boldsymbol{y} \circ_n \boldsymbol{x} \colon \boldsymbol{a} \to \boldsymbol{c}$; so by (A.5), $\bar{y} * \bar{x}$ should be a map $\bar{a} \to \bar{c} \circ_n M(y \circ_n x)$. But this is so since it is the composite

$$\bar{a} \xrightarrow{\bar{x}} \bar{b} \circ_n Mx \xrightarrow{\bar{y} \circ_n Mx} \bar{c} \circ_n My \circ_n Mx = \bar{c} \circ_n M(y \circ_n x) \ .$$

Now let $k > n+1$; if $\boldsymbol{x} \colon \boldsymbol{u} \to \boldsymbol{v}$ and $\boldsymbol{y} \colon \boldsymbol{w} \to \boldsymbol{z}$, then we should have $\boldsymbol{y} \circ_n \boldsymbol{x} \colon \boldsymbol{w} \circ_n \boldsymbol{u} \to \boldsymbol{z} \circ_n \boldsymbol{v}$. We show by induction on k that (a) $\boldsymbol{y} \circ_n \boldsymbol{x}$ is a k-cell of this form; and (b) for all cells $f \colon u \rightsquigarrow v$ and $g \colon w \rightsquigarrow z$ we have:

$$\text{(A.6)} \qquad \begin{aligned} P^{\boldsymbol{y} \circ_n \boldsymbol{x}}_{k-1}(g \circ_n f) &= P^{\boldsymbol{y}}_{k-1}(g) \circ_n M(t_{n+1}x) \circ_{n+1} P^{\boldsymbol{x}}_{k-1}(f) && \text{if } k \text{ even;} \\ M^{\boldsymbol{y} \circ_n \boldsymbol{x}}_{k-1}(g \circ_n f) &= M^{\boldsymbol{y}}_{k-1}(g) \circ_n M(t_{n+1}x) \circ_{n+1} M^{\boldsymbol{x}}_{k-1}(f) && \text{if } k \text{ odd.} \end{aligned}$$

Assuming (a) and (b) for all $j < k$, we prove it for k. If k is odd, then by (A.5) we have $\bar{x} \colon M^{\boldsymbol{x}}_{k-1}(x) \to \bar{v}$ and $\bar{y} \colon M^{\boldsymbol{y}}_{k-1}(y) \to \bar{z}$ and require for (a) that $\bar{y} * \bar{x} \colon M'(y \circ_n x) \to \bar{z} * \bar{v}$, where M' is the ω-functor associated to the parallel pair $(\boldsymbol{w} \circ_n \boldsymbol{u}, \boldsymbol{z} \circ_n \boldsymbol{v})$. Note first that we have $t(\bar{y} * \bar{x}) = t(\bar{y} \circ_n M(t_{n+1}x) \circ_{n+1} \bar{x}) = t\bar{y} \circ_n Mv \circ_{n+1} t\bar{x} = \bar{z} \circ_n M(t_{n+1}v) \circ_{n+1} \bar{v} = \bar{z} * \bar{v}$ as required; on the other hand, we have $s(\bar{y} * \bar{x}) = s\bar{y} \circ_n M(t_{n+1}x) \circ_{n+1} s\bar{x} = M^{\boldsymbol{y}}_{k-1}(y) \circ_n M(t_{n+1}x) \circ_{n+1} M^{\boldsymbol{x}}_{n+1}(x)$ so that for $\boldsymbol{y} \circ_n \boldsymbol{x}$ to be a cell of the form required for (a), it will suffice to prove

$$\text{(A.7)} \qquad M'(g \circ_n f) = M^{\boldsymbol{y}}_{k-1}(g) \circ_n M(t_{n+1}x) \circ_{n+1} M^{\boldsymbol{x}}_{k-1}(f)$$

for all cells $f \colon u \rightsquigarrow v$ and $g \colon w \rightsquigarrow z$. Once we know $\boldsymbol{y} \circ_n \boldsymbol{x}$ is a cell, we will have $M' = M^{\boldsymbol{y} \circ_n \boldsymbol{x}}_{k-1}$, so that (A.7) gives (b) as required. We verify (A.7) first when $k = n+2$; here, (A.1), functoriality of M and interchange gives

$$\begin{aligned} M'(g \circ_n f) &= \bar{c} \circ_n M(g \circ_n f) \circ_{n+1} (\bar{w} * \bar{u}) = \big(\bar{c} \circ_n M(g \circ_n f)\big) \circ_{n+1} (\bar{w} \circ_n Mu) \circ_{n+1} \bar{u} \\ &= \big(\bar{c} \circ_n M(g \circ_n v)\big) \circ_{n+1} \big(\bar{c} \circ_n M(w \circ_n f)\big) \circ_{n+1} (\bar{w} \circ_n Mu) \circ_{n+1} \bar{u} \\ &= \big(\bar{c} \circ_n M(g \circ_n v)\big) \circ_{n+1} \big(\bar{w} \circ_n M(w \circ_n f)\big) \circ_{n+1} \bar{u} \\ &= \big(\bar{c} \circ_n M(g \circ_n v)\big) \circ_{n+1} (\bar{w} \circ_n Mv) \circ_{n+1} (\bar{b} \circ_n Mf) \circ_{n+1} \bar{u} \\ &= (\bar{c} \circ_n Mg \circ_{n+1} \bar{w}) \circ_n Mv \circ_{n+1} (\bar{b} \circ_n Mf \circ_{n+1} \bar{u}) \\ &= M^{\boldsymbol{y}}_{k-1}(g) \circ_n M(t_{n+1}x) \circ_{n+1} M^{\boldsymbol{x}}_{k-1}(f) \end{aligned}$$

as required. In the case $k > n+2$, we have that

$$\begin{aligned} M'(g \circ_n f) &= P^{\boldsymbol{w} \circ_n \boldsymbol{u}}_{k-2}(g \circ_n f) \circ_{k-1} (\bar{w} * \bar{u}) \\ &= \big(P^{\boldsymbol{w}}_{k-2}(g) \circ_n M(t_{n+1}x) \circ_{n+1} P^{\boldsymbol{u}}_{k-2}(f)\big) \circ_{k-1} (\bar{w} \circ_n Mu \circ_{n+1} \bar{u}) \\ &= (P^{\boldsymbol{w}}_{k-2}(g) \circ_{k-1} \bar{w}) \circ_n M(t_{n+1}x) \circ_{n+1} (P^{\boldsymbol{u}}_{k-2}(f) \circ_{k-1} \bar{u}) \\ &= M^{\boldsymbol{y}}_{k-1}(g) \circ_n M(t_{n+1}x) \circ_{n+1} M^{\boldsymbol{x}}_{k-1}(f) \end{aligned}$$

by (A.2), the case $(k-1)$ of (A.6), and interchange. This completes the inductive step for odd k; we omit the analogous argument for k even.

We have thus proved for odd n that composition $\circ_n$ in a/C is well-defined and satisfies the source–target axioms; the case where n is even is analogous, and so omitted. The identity axioms for a/C are easy; next, for associativity, we must show that $\boldsymbol{x} \circ_n (\boldsymbol{y} \circ_n \boldsymbol{z}) = (\boldsymbol{x} \circ_n \boldsymbol{y}) \circ_n \boldsymbol{z}$ in a/C. Suppose that n is odd, and let $M = M^{\boldsymbol{x}}_{n-1} = M^{\boldsymbol{y}}_{n-1} = M^{\boldsymbol{z}}_{n-1}$. Then the two iterated composites are

$$\begin{aligned} &\big((x \circ_n y) \circ_n z,\ (\bar{x} \circ_n M(t_{n+1}y) \circ_{n+1} \bar{y}) \circ_n M(t_{n+1}z) \circ_{n+1} \bar{z}\big) \\ \text{and } &\big(x \circ_n (y \circ_n z),\ \bar{x} \circ_n M(t_{n+1}(y \circ_n z)) \circ_{n+1} (\bar{y} \circ_n M(t_{n+1}z) \circ_{n+1} \bar{z})\big) \end{aligned}$$

which are easily equal by functoriality of M and interchange. The case of n even is dual, and so omitted; and it remains only to verify the interchange axiom $(\boldsymbol{z} \circ_k \boldsymbol{w}) \circ_n (\boldsymbol{y} \circ_k \boldsymbol{x}) = (\boldsymbol{z} \circ_n \boldsymbol{y}) \circ_k (\boldsymbol{w} \circ_n \boldsymbol{x})$ for all suitable cells $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{z}$ in a/C. Of course, the equality is clear on first components; on second components, there are four cases to consider depending on the parities of the dimensions $n < k$; we give only the case where both n and k are odd, as the others are similar. So let $M = M^{\boldsymbol{a}}_{n-1} = M^{\boldsymbol{b}}_{n-1} = M^{\boldsymbol{c}}_{n-1}$, let $M' = M^{\boldsymbol{w}}_{k-1} = M^{\boldsymbol{z}}_{k-1}$ and let $M'' = M^{\boldsymbol{x}}_{k-1} = M^{\boldsymbol{y}}_{k-1}$. The second component of $(\boldsymbol{z} \circ_k \boldsymbol{w}) \circ_n (\boldsymbol{y} \circ_k \boldsymbol{x})$ is:

$$\begin{aligned} &(\bar{z} \circ_k M't_{k+1}w \circ_{k+1} \bar{w}) \circ_n Mt_{n+1}(y \circ_k x) \circ_{n+1} (\bar{y} \circ_k M''t_{k+1}x \circ_{k+1} \bar{x}) \\ = &\Big(\big[(\bar{z} \circ_k M't_{k+1}w) \circ_n Mt_{n+1}x\big] \circ_{k+1} \big[\bar{w} \circ_n Mt_{n+1}x\big]\Big) \circ_{n+1} \Big(\big[\bar{y} \circ_k M''t_{k+1}x\big] \circ_{k+1} \bar{x}\Big) \\ = &\Big(\big[(\bar{z} \circ_k M't_{k+1}w) \circ_n Mt_{n+1}x\big] \circ_{n+1} \big[\bar{y} \circ_k M''t_{k+1}x\big]\Big) \circ_{k+1} \Big(\big[\bar{w} \circ_n Mt_{n+1}x\big] \circ_{n+1} \bar{x}\Big) \end{aligned}$$

using interchange. The left-hand bracketed term is in turn equal to

$$\begin{aligned} &([\bar{z} \circ_n Mt_{n+1}x] \circ_k [M't_{k+1}w \circ_n Mt_{n+1}x]) \circ_{n+1} (\bar{y} \circ_k M''t_{k+1}x) \\ = &([\bar{z} \circ_n Mt_{n+1}x] \circ_{n+1} \bar{y}) \circ_k ([M't_{k+1}w \circ_n Mt_{n+1}x] \circ_{n+1} M''t_{k+1}x) \\ = &([\bar{z} \circ_n Mt_{n+1}x] \circ_{n+1} \bar{y}) \circ_k M^{\boldsymbol{w} \circ_n \boldsymbol{x}}_{k-1}(t_{k+1}(w \circ_n x)) \end{aligned}$$

using interchange and (A.6), which on recomposing with the right-hand bracketed term above yields the second component of $(\boldsymbol{z} \circ_n \boldsymbol{y}) \circ_k (\boldsymbol{w} \circ_n \boldsymbol{x})$, as required. □

Proposition 12. *For any ω-category C and objects $a, b \in C_0$, the ω-functor $\bullet\colon a/C \times C(b,a) \to b/C$ is well-defined.*

Proof. Recall that $\bullet$ is defined on 0-cells by $\boldsymbol{x} \bullet h = (x, \bar{x} \circ_0 h)$ and on $(n+1)$-cells by $\boldsymbol{x} \bullet h = (x, \bar{x} \circ_0 h)\colon s\boldsymbol{x} \bullet sh \to t\boldsymbol{x} \bullet th$. Well-definedness is clear on 0-cells. At higher dimensions, we show by induction on n that for each pair $(\boldsymbol{x}, h)$ of dimension $(n+1)$, the cell $\boldsymbol{x} \bullet h$ is well-defined and satisfies

(A.8) $$M^{\boldsymbol{x} \bullet \boldsymbol{h}}_n(-) = M^{\boldsymbol{x}}_n(-) \circ_0 sh \qquad \text{or} \qquad P^{\boldsymbol{x} \bullet \boldsymbol{h}}_n(-) = P^{\boldsymbol{x}}_n(-) \circ_0 th$$

according as n is even or odd, where, as before, $M^{\boldsymbol{x}}_n$ and $P^{\boldsymbol{x}}_n$ denote the auxiliary functors (A.1) and (A.2) associated to the n-boundary $(\boldsymbol{m}_n, \boldsymbol{p}_n)$ of $\boldsymbol{x}$.

So let $\boldsymbol{x}\colon \boldsymbol{m}_n \to \boldsymbol{p}_n$ and $h\colon u \to v$ be $(n+1)$-cells of a/C and $C(b,a)$; by induction $\boldsymbol{m}_n \bullet u$ and $\boldsymbol{p}_n \bullet v$ are well-defined, and we must show that $\boldsymbol{x} \bullet h\colon \boldsymbol{m}_n \bullet u \to \boldsymbol{p}_n \bullet v$ is too. Even without knowing this, we may still verify (A.8) since $M^{\boldsymbol{x} \bullet \boldsymbol{h}}_n$ or $P^{\boldsymbol{x} \bullet \boldsymbol{h}}_n$ (as the case may be) depend only on the well-defined boundary pair $(\boldsymbol{m}_n \bullet u, \boldsymbol{p}_n \bullet v)$. But when n is even we have

$$\begin{aligned} M_n^{\boldsymbol{x}}(-) \circ_0 u &= (P_{n-1}^{\boldsymbol{m}_n}(-) \circ_n \bar{m}_n) \circ_0 u = (P_{n-1}^{\boldsymbol{m}_n}(-) \circ_n \bar{m}_n) \circ_0 (tu \circ_n u) \\ &= (P_{n-1}^{\boldsymbol{m}_n}(-) \circ_0 tu) \circ_n (\bar{m}_n \circ_0 u) = P_{n-1}^{\boldsymbol{m}_n \bullet u}(-) \circ_n (\bar{m}_n \circ_0 u) \\ &= M_n^{\boldsymbol{x} \bullet h}(-) \end{aligned}$$

as required, and correspondingly for n odd. We now use this to show that $\boldsymbol{x} \bullet h = (x, \bar{x} \circ_0 h)$ is a well-defined cell $\boldsymbol{m}_n \bullet u \to \boldsymbol{p}_n \bullet v$. Clearly the first component is a map $x \colon m_n \to p_n$ as required. For the second component, suppose first that n is even; then by (A.3), $\bar{x}$ is a cell $M_n^{\boldsymbol{x}}(x) \to \bar{p}_n$, whose 0-source is by (A.5) equal to a. Thus $\bar{x} \circ_0 h$ is a well-defined cell $M_n^{\boldsymbol{x}}(x) \circ_0 u \to \bar{p}_n \circ_0 v$ and by the above calculation $M_n^{\boldsymbol{x}}(x) \circ_0 u = M_n^{\boldsymbol{x} \bullet h}(x)$ as required. The case where n is odd is similar. □

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Department of Computing, Macquarie University, NSW 2109, Australia
E-mail address: `mitchell.buckley@mq.edu.au`

Department of Mathematics, Macquarie University, NSW 2109, Australia
E-mail address: `richard.garner@mq.edu.au`