# Polycategories via pseudo-distributive laws

## Richard Garner [1]

*University of Cambridge, Department of Pure Mathematics, Wilberforce Road, Cambridge CB3 0WB, UK*

## Abstract

In this paper, we give a novel abstract description of Szabo's *polycategories*. We use the theory of *double clubs* – a generalisation of Kelly's theory of clubs to 'pseudo' (or 'weak') double categories – to construct a pseudo-distributive law of the free symmetric strict monoidal category pseudocomonad on **Mod** over itself *qua* pseudomonad, and show that monads in the 'two-sided Kleisli bicategory' of this pseudo-distributive law are precisely symmetric polycategories.
© 2008 Elsevier Inc. All rights reserved.

## 1. Introduction

Szabo's theory of *polycategories* [1] has been the target of renewed interest over recent years. Polycategories are the 'not-necessarily-representable' cousins of the *weakly distributive categories* of [2]; their relationship mirrors that of *multicategories* to *monoidal categories*.

Though it is possible, as Szabo did, to give a 'hands on' description of a polycategory, such a description leaves a lot to be desired. For a start, the sheer quantity of data that one must check for even simple proofs quickly becomes overwhelming. Further problems arise when one wishes to address aspects of a putative 'theory of polycategories': what are the correct notions of polyfunctor or polytransformation? What is a polycategorical limit? In attempting to answer such

questions without a formal framework, one is forced into the unsatisfactory position of relying on intuition alone.

Thus far, the paper [3] has provided the only attempt to rectify this situation. Koslowski provides an abstract description of polycategories that generalises the elegant work of [4] and later [5,6] on '$T$-multicategories.' However, whilst this latter theory uses only some rather simple and obvious constructions on categories with finite limits, the structures that Koslowski uses to build his description of polycategories are rather more complicated and non-canonical. Furthermore, the generalisation from the non-symmetric to the symmetric case is not as smooth as one would like.

We therefore offer an alternative approach to the abstract description of polycategories. It is the same and not the same as Koslowski's: again, we shall build on an abstract description of multicategories, and again, composition proceeds using something like a 'distributive law.' Where we deviate from Koslowski is in the description of multicategories that we build upon.

In Section 1, we recount this alternative description: it is the approach of [7,8], based on *profunctors* rather than *spans*. We go on to describe how we may generalise this description to one for polycategories; to do this we invoke a *pseudo-distributive law* (in the sense of [9,10]) of a pseudocomonad (the 'target arity') over a pseudomonad (the 'source arity'). Polycategories now arise as monads in the 'two-sided Kleisli bicategory' of this pseudo-distributive law.

There are several advantages to this approach: it allows us to describe *symmetric* polycategories with no greater difficulty than *non-symmetric* polycategories; it will generalise easily from ordinary categories to enriched categories; and, though we do not attempt this here, it allows us to 'read off' further aspects of the theory of polycategories: the aforementioned polyfunctor, polytransformation, and so on.

In order to make this description go through, we must construct a suitable pseudo-distributive law. Now, a pseudo-distributive law is a prodigiously complicated object: it is five pieces of (complex) data subject to ten coherence laws. A bare hands construction would be both tedious and unenlightening: the genuinely interesting combinatorics involved would be obscured by a morass of trivial details.

Thus, in Section 2, we discuss how we may use the theory of *double clubs*, as developed in the companion paper [11], to reduce this Herculean task to something more manageable. Informally, the theory of double clubs tells us that it suffices to construct our pseudo-distributive law at the terminal category 1, and that we can propagate this construction elsewhere by 'labelling objects and arrows' appropriately.

Finally, in Section 3, we perform this construction at 1; and though one might think this would be an exercise in nose-following, it actually turns out to be a fairly interesting piece of categorical combinatorics. Equipped with this, we are finally able to prove the existence of our pseudo-distributive law and hence to give our preferred definition of polycategory.

Appendix A gives the definitions of *pseudomonad*, *pseudocomonad* and *pseudo-distributive law*.

## 2. Multicategories and polycategories

We begin by re-examining the theory of multicategories: the material here summarises [7,12,13], amongst others. Note that throughout, we shall only be interested in the theory of *symmetric* multicategories, and, later, of *symmetric* polycategories: that is, we allow ourselves to reorder freely the inputs and outputs of our maps. Consequently, whenever we say 'multicatego-

ry' or 'polycategory,' it may be taken that we mean the symmetric kind. The non-symmetric case for polycategories is considered in more detail by [3].

## 2.1. Multicategories

We write $X^*$ for the free monoid on a set $X$, and $\Gamma$, $\Delta$, $\Sigma$, $\Lambda$ for typical elements thereof. We will use commas to denote the concatenation operation on $X^*$, as in "$\Gamma$, $\Delta$"; and we will tend to conflate elements of $X$ with their image in $X^*$. Given $\Gamma = x_1, \ldots, x_n \in X^*$, we define $|\Gamma| = n$, and given $\sigma \in S_n$, write $\sigma \Gamma$ for the element $x_{\sigma(1)}, \ldots, x_{\sigma(n)} \in X^*$.

**Definition 1.** A *symmetric multicategory* $\mathbb{M}$ consists of:

- A set $\mathrm{ob}\,\mathbb{M}$ of *objects*.
- For every $\Gamma \in (\mathrm{ob}\,\mathbb{M})^*$ and $y \in \mathrm{ob}\,\mathbb{M}$, a set $\mathbb{M}(\Gamma; y)$ of *multimaps* from $\Gamma$ to $y$ (we write a typical element of such as $f : \Gamma \to y$); further, for every $\sigma \in S_{|\Gamma|}$, an *exchange isomorphism* $\mathbb{M}(\Gamma; y) \to \mathbb{M}(\sigma \Gamma; y)$.
- For every $x \in \mathrm{ob}\,\mathbb{M}$, an *identity map* $\mathrm{id}_x \in \mathbb{M}(x; x)$.
- For every $\Gamma, \Delta_1, \Delta_2 \in (\mathrm{ob}\,\mathbb{M})^*$ and $y, z \in \mathrm{ob}\,\mathbb{M}$, a *composition map*

$$\mathbb{M}(\Gamma; y) \times \mathbb{M}(\Delta_1, y, \Delta_2; z) \to \mathbb{M}(\Delta_1, \Gamma, \Delta_2; z).$$

This data satisfies axioms expressing the fact that exchange isomorphisms compose as expected, and that composition is associative, unital, and compatible with exchange isomorphisms: see [14] for the full details.

Now, this data expresses composition as a *binary* operation performed between two multimaps; however, there is another view, where we 'multicompose' a family of multimaps $g_i : \Gamma_i \to y_i$ with a multimap $f : y_1, \ldots, y_n \to z$.

The transit from one view to the other is straightforward: we recover the multicomposition from the binary composition by performing, in any order, the binary compositions of the $g_i$'s with $f$: the axioms for binary composition ensure that this gives a uniquely defined composite. Conversely, we can recover binary composition from multicomposition by setting all but one of the $g_i$'s to be the identity.

We can express the operation of multicomposition as follows: fix the object set $X = \mathrm{ob}\,\mathbb{M}$, and consider it as a discrete category. We write $S$ for the *free symmetric strict monoidal category* 2-monad on **Cat**, and consider the functor category $[(SX)^{\mathrm{op}} \times X, \mathbf{Set}]$. To give an object $F$ of this is to give sets of multimaps as above, together with coherent exchange isomorphisms. Further, this category has a 'substitution' monoidal structure given by

$$(G \otimes F)(\Gamma; z) = \sum_{\substack{k \in \mathbb{N} \\ y_1, \ldots, y_k \in X}} \int^{\Delta_1, \ldots, \Delta_k \in SX} G(y_1, \ldots, y_k; z) \times \prod_{i=1}^{k} F(\Delta_i; y_i) \times SX\left(\Gamma, \bigotimes_{i=1}^{k} \Delta_i\right),$$
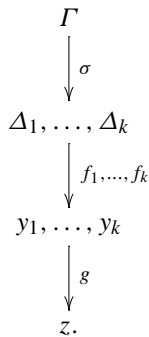
and

$$\mathbf{I}(\Gamma; x) = \begin{cases} \{*\} & \text{if } \Gamma = x, \\ \varnothing & \text{otherwise;} \end{cases}$$

and to give a multicategory is precisely to give a monoid with respect to this monoidal structure. Indeed, suppose we have a monoid $F \in [(SX)^{op} \times X, \mathbf{Set}]$. Then the unit map $j : \mathbf{I} \to F$ picks out for each $x \in X$ an element of $F(x; x)$, which will correspond to the identity multimap $\mathrm{id}_x : x \to x$. What about the multiplication map $m : F \otimes F \to F$? Unpacking the above definition, we see that $(F \otimes F)(\Gamma; z)$ can be described as follows. Let $\Delta_1, \ldots, \Delta_k \in (\mathrm{ob}\,\mathbb{M})^*$ be such that

- $|\Gamma| = n = \sum |\Delta_i|$;
- there exists $\sigma \in S_n$ such that $\sigma \Gamma = \Delta_1, \ldots, \Delta_k$,

and let $f_i : \Delta_i \to y_i$ (for $i = 1, \ldots, k$), and $g : y_1, \ldots, y_k \to z$ be multimaps in $F$. Then this gives us a typical element of $(F \otimes F)(\Gamma; z)$, which we visualise as

$$
\begin{array}{c}
\Gamma \\
\downarrow {\scriptstyle \sigma} \\
\Delta_1, \ldots, \Delta_k \\
\downarrow {\scriptstyle f_1, \ldots, f_k} \\
y_1, \ldots, y_k \\
\downarrow {\scriptstyle g} \\
z.
\end{array}
$$

The map $m : F \otimes F \to F$ sends this element to an element of $F(\Gamma; z)$; in other words, it specifies the result of this 'multicomposition.' The associativity and unitality laws for a monoid ensure that this composition process is associative and unital as required.

In fact, we may deduce the existence of the monoidal structure on $[(SX)^{op} \times X, \mathbf{Set}]$ from more abstract considerations. The key idea is to construct a bicategory $\mathcal{B}$ for which $\mathcal{B}(X, X) = [(SX)^{op} \times X, \mathbf{Set}]$, in such a way that horizontal composition in this endohom-category induces the desired substitution monoidal structure; and for this, we make use of the following result:

**Proposition 2.** *The symmetric strict monoidal category 2-monad $(S, \eta, \mu)$ on* $\mathbf{Cat}$ *lifts to a pseudomonad $(\hat{S}, \hat{\eta}, \hat{\mu}, \lambda, \rho, \tau)$ on* $\mathbf{Mod}$*, the bicategory of categories, profunctors and transformations.*

(For the definition of and notation for a pseudomonad, see Appendix A.)

**Proof.** We recount only the salient details here. For a full proof the reader may refer to [10]; but see also Section 4.1 below.

The lifted homomorphism $\hat{S} : \mathbf{Mod} \to \mathbf{Mod}$ agrees with $S : \mathbf{Cat} \to \mathbf{Cat}$ on objects; whilst on 1-cells, it sends the profunctor $F : \mathbf{D}^{op} \times \mathbf{C} \to \mathbf{Set}$ to the profunctor $\hat{S}F : (S\mathbf{D})^{op} \times S\mathbf{C} \to \mathbf{Set}$ given by:

$$
\hat{S}F\big((d_1, \ldots, d_n), (c_1, \ldots, c_m)\big) = \begin{cases} \sum_{\sigma \in S_n} \prod_{i=1}^{n} F(d_i, c_{\sigma(i)}) & \text{if } n = m; \\ 0 & \text{otherwise.} \end{cases}
$$

The components at **C** of the lifted transformations $\hat{\eta} : \mathrm{id}_{\mathbf{Mod}} \Rightarrow \hat{S}$ and $\hat{\mu} : \hat{S}\hat{S} \Rightarrow \hat{S}$ are obtained as the images of the corresponding components of $\eta$ and $\mu$ under the canonical embedding $(-)_* : \mathbf{Cat} \to \mathbf{Mod}$. Explicitly, we have:

$$\hat{\eta}_{\mathbf{C}}(\Gamma, c) = S\mathbf{C}(\Gamma, (c));$$

$$\hat{\mu}_{\mathbf{C}}(\Gamma, (\Delta_1, \ldots, \Delta_n)) = S\mathbf{C}\left(\Gamma, \bigotimes \Delta_i\right). \qquad \square$$

Now, just as each monad on a category gives rises to a Kleisli category, so each pseudomonad on a bicategory gives rise to a 'Kleisli bicategory.' This construction was first given in [15] for the special case of a pseudomonad on a 2-category; and the following is the obvious generalisation to the bicategorical case:

**Definition 3.** Let $\mathcal{B}$ be a bicategory and let $(S, \eta, \mu, \lambda, \rho, \tau)$ be a pseudomonad on $\mathcal{B}$. Then the Kleisli bicategory $Kl(S)$ of the pseudomonad $S$ has:

- *Objects* those of $\mathcal{B}$.
- *Hom-categories* given by $Kl(S)(X, Y) = \mathcal{B}(X, SY)$.
- *Identity map* at $X$ given by the component $\eta_X : X \to SX$.
- *Composition* $Kl(S)(Y, Z) \times Kl(S)(X, Y) \to Kl(S)(X, Z)$ given by

$$\mathcal{B}(Y, SZ) \times \mathcal{B}(X, SY)$$

$$\downarrow \cong$$

$$1 \times \mathcal{B}(Y, SZ) \times \mathcal{B}(X, SY)$$

$$\downarrow {}^{\ulcorner}\mu_Z{}^{\urcorner} \times S \times \mathrm{id}$$

$$\mathcal{B}(SSZ, SZ) \times \mathcal{B}(SY, SSZ) \times \mathcal{B}(X, SY)$$

$$\downarrow \otimes$$

$$\mathcal{B}(X, SZ)$$

where we use $\otimes$ to stand for some choice of order of composition for this threefold composite. Explicitly, on maps, this composition is given by

$$(Y \xrightarrow{G} SZ) \otimes (X \xrightarrow{F} SY) = X \xrightarrow{F} SY \xrightarrow{SG} SSZ \xrightarrow{\mu_Z} SZ$$

for some choice of bracketing for this composite.

The remaining data to make this a bicategory – namely, the associativity and unitality constraints – can be constructed in an obvious way using the associativity and unitality constraints for $\mathcal{B}$ and the coherence modifications for the pseudomonad $S$. The reader may easily verify that these data satisfy the bicategory axioms.

**Remark 4.** We may justify the name 'Kleisli bicategory' as follows. At the one-dimensional level, the Kleisli category of a monad $S$ on a category **C** is determined by its universality amongst

all categories **D** equipped with an embedding functor $H : \mathbf{C} \to \mathbf{D}$ and a right action $\theta : HS \Rightarrow H$ of $S$ on $H$. Similarly, we may characterise the 'Kleisli bicategory' of a pseudomonad $S$ on a bicategory $\mathcal{B}$ as universal amongst all bicategories $\mathcal{D}$ equipped with an embedding pseudofunctor $H : \mathcal{B} \to \mathcal{D}$ and a right pseudo-action $\theta : HS \Rightarrow S$ of $S$ on $H$: see [15, Theorem 4.3].

In particular, we may form the Kleisli bicategory of the pseudomonad $\hat{S}$ on **Mod**; and by substituting the data given in the proof of Proposition 2 into Definition 3, we may easily verify that horizontal composition in $Kl(\hat{S})(X, X)$ gives precisely the monoidal structure on $[(SX)^{\mathrm{op}} \times X, \mathbf{Set}]$ described above. Hence we arrive at an alternative, but equivalent, definition of multicategory:

**Definition 5.** A symmetric multicategory is a monad on a discrete object $X$ in the bicategory $Kl(\hat{S})$.

This description is well known, though not often stated in precisely this form: it is the approach of [7,8].

### 2.2. Polycategories

We recall now the notion of symmetric *polycategory*:

**Definition 6.** A *symmetric polycategory* $\mathbb{P}$ consists of

- A set $\mathrm{ob}\,\mathbb{P}$ of *objects*.
- For each pair $(\Gamma, \Delta)$ of elements of $(\mathrm{ob}\,\mathbb{P})^*$, a set $\mathbb{P}(\Gamma; \Delta)$ of *polymaps* from $\Gamma$ to $\Delta$.
- For each $\Gamma, \Delta \in (\mathrm{ob}\,\mathbb{P})^*$, each $\sigma \in S_{|\Gamma|}$ and $\tau \in S_{|\Delta|}$, *exchange isomorphisms*

$$\mathbb{P}(\Gamma; \Delta) \to \mathbb{P}(\sigma\Gamma; \tau\Delta).$$

- For each $x \in \mathrm{ob}\,\mathbb{P}$, an *identity map* $\mathrm{id}_x \in \mathbb{P}(x; x)$.
- For $\Gamma, \Delta_1, \Delta_2, \Lambda_1, \Lambda_2, \Sigma \in (\mathrm{ob}\,\mathbb{P})^*$, and $x \in \mathrm{ob}\,\mathbb{P}$, *composition maps*

$$\mathbb{P}(\Gamma; \Delta_1, x, \Delta_2) \times \mathbb{P}(\Lambda_1, x, \Lambda_2; \Sigma) \to \mathbb{P}(\Lambda_1, \Gamma, \Lambda_2; \Delta_1, \Sigma, \Delta_2),$$

subject to laws expressing the associativity and unitality of composition, expressing that the exchange isomorphisms compose as expected, and that they are compatible with composition: see [1] or [2] for the full details.

We recover the notion of a multicategory if we assert that $\mathbb{P}(\Gamma; \Delta)$ is empty unless $\Delta$ is a singleton.

Now, as before, we may shift from giving a 'binary composition' of two polymaps to giving a 'polycomposition' operation on two families of composable polymaps. First, we need to say what we mean by *composable*.

**Definition 7.** Let $\mathbf{f} := \{f_m : \Lambda_m \to \Sigma_m\}_{1 \leqslant m \leqslant j}$ and $\mathbf{g} := \{g_n : \Gamma_n \to \Delta_n\}_{1 \leqslant n \leqslant k}$ be families of polymaps, such that

$$\sum |\Sigma_m| = \sum |\Gamma_n| = l.$$

We say that a permutation $\sigma \in S_l$ is a *matching* of $\mathbf{f}$ and $\mathbf{g}$ if $\sigma(\Sigma_1, \ldots, \Sigma_j) = \Gamma_1, \ldots, \Gamma_k$.

Informally, a matching of two families **f** and **g** indicates 'which output of $f_i$ has been plugged into which input of $g_j$.' Yet not every such plugging need be obtainable from repeated binary composition; and so if our notion of polycomposition is to have the same force as our notion of binary composition, we must restrict the matchings along which we will allow polycomposition to occur.

**Definition 8.** Given a matching $\sigma$ of **f** and **g**, we define a bipartite multigraph $G_\sigma$ as follows. Its two vertex sets are labelled by $f_1, \ldots, f_m$ and $g_1, \ldots, g_n$, and we add one edge between $f_i$ and $g_j$ for every element of $\Sigma_i$ which is paired with an element of $\Gamma_j$ under the matching $\sigma$. We shall say that the matching $\sigma$ is *suitable* just when $G_\sigma$ is acyclic, connected and has no multiple edges.

**Proposition 9.** *Let there be given families* $\mathbf{f} := \{f_m : \Lambda_m \to \Sigma_m\}_{1 \leqslant m \leqslant j}$ *and* $\mathbf{g} :=$ $\{g_n : \Gamma_n \to \Delta_n\}_{1 \leqslant n \leqslant k}$ *of polymaps; together with a suitable matching* $\sigma$ *thereof. Then there is a uniquely defined polymap* $\mathbf{g} \circ_\sigma \mathbf{f} : \Lambda_1, \ldots, \Lambda_j \to \Delta_1, \ldots, \Delta_k$ *obtained by repeated binary compositions which, in some order, connect each* $x \in \Sigma_1, \ldots, \Sigma_j$ *with the corresponding* $\sigma(x) \in \Gamma_1, \ldots, \Gamma_k$.

To prove this, we will prove something slightly stronger. First, a little more notation: given a list $\Sigma = x_1, \ldots, x_k \in X^*$, by a *sublist* of $\Sigma$ we shall mean a list $\Gamma = x_{i_1}, \ldots, x_{i_j}$ where $1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant k$. Thus sublists of $\Sigma$ are in bijection with subsets of $\{1, \ldots, |\Sigma|\}$, and in particular, form a Boolean algebra; and we write $\Gamma^c$ for the complement of $\Gamma$ in this Boolean algebra. We also say that a list $\Sigma$ is an *interleaving* of two lists $\Gamma_1$ and $\Gamma_2$ if we can view $\Gamma_1$ and $\Gamma_2$ as complementary sublists of $\Sigma$.

**Definition 10.** Let $\mathbf{f} := \{f_m : \Lambda_m \to \Sigma_m\}_{1 \leqslant m \leqslant j}$ and $\mathbf{g} := \{g_n : \Gamma_n \to \Delta_n\}_{1 \leqslant n \leqslant k}$ be families of polymaps. A *partial matching* of **f** and **g** is given by a sublist $\Sigma$ of $\Sigma_1, \ldots, \Sigma_m$ and a sublist $\Gamma$ of $\Gamma_1, \ldots, \Gamma_n$ with $|\Sigma| = |\Gamma| = l$, together with a permutation $\sigma \in S_l$ satisfying $\sigma(\Sigma) = \Gamma$.

As before, we can define the notion of the associated graph $G_\sigma$ for a partial matching, and thus the notion of a *suitable* partial matching. Proposition 9 now follows *a fortiori* from the following:

**Proposition 11.** *Let there be given families of polymaps* **f** *and* **g** *as before, together with a suitable partial matching* $(\Sigma, \Gamma, \sigma)$ *thereof. Then there is a uniquely defined polymap* $\mathbf{g} \circ_\sigma \mathbf{f}$ *obtained by repeated binary compositions which, in some order, connect each* $x \in \Sigma$ *to the corresponding* $\sigma(x) \in \Gamma$. *The domain of* $\mathbf{g} \circ_\sigma \mathbf{f}$ *is an interleaving of the lists* $\Lambda_1, \ldots, \Lambda_j$ *and* $\Gamma^c$, *whilst the codomain is an interleaving of the lists* $\Delta_1, \ldots, \Delta_k$ *and* $\Sigma^c$.

**Proof.** Since the partial matching $\sigma$ is suitable, its associated graph $G_\sigma$ is a tree, and so in particular will have a vertex of degree 1. Choose any such vertex: it corresponds to one of our polymaps $f_i$ or $g_i$, without loss of generality to $f_i$, say. We begin by forming the binary composition of $f_i$ with the polymap $g_j$ which is connected to $f_i$ in $G_\sigma$. Suppose

$$f_i : \Lambda_i \to \Sigma_i, x, \Sigma_i' \quad \text{and} \quad g_j : \Gamma_j, x, \Gamma_j' \to \Delta_j,$$

where the two $x$'s are matched under $\sigma$. Then the resultant composite map will be

$$g_i \circ f_j : \Gamma_j, \Lambda_i, \Gamma_j' \to \Sigma_i, \Delta_j, \Sigma_i'.$$

Note that $f_i$ has no other outputs taking part in the partial matching $\sigma$. Thus we can now form a partial matching $\sigma'$ of $\mathbf{f} \setminus \{f_i\}$ with $\mathbf{g} \setminus \{g_j\} \cup \{g_j \circ f_i\}$, which simply matches elements in the same way as $\sigma$ except for the no-longer present matching of $x$. Now it is easy to see that the associated graph of $\sigma'$ will be the same as that of $\sigma$, but with the vertex corresponding to $f_i$ and the single adjacent edge removed. We continue by induction on the size of the tree $G_\sigma$.

Note that we may at each stage have several possible choices of vertices of degree 1 which we may take as the next binary composition to perform. However, the associativity laws for a polycategory ensure that the resultant composite will be independent of the choice we make at each stage.  □

Thus, in any polycategory, we may define the 'polycomposition' of a family $\mathbf{f}$ with a family $\mathbf{g}$ along a suitable matching $\sigma$: conversely, if we are given polycomposites along suitable matchings, we may recapture a binary composition by polycomposing with a suitable collection of identity maps. Consequently, if we are to give an abstract formulation of polycategory, it seems reasonable to do so in terms of a notion of 'polycompositional' polycategory.

In order to fully justify this last claim, we must exhibit a bijection between polycompositional polycategories and the polycategories of Definition 6. However, we do not yet have a full description of the axioms which a polycompositional polycategory should satisfy; and to write them down at this point would be very messy. Thus we postpone justification until we have given our abstract description of polycompositional polycategories, from which we will be able to extract a description of the axioms such a structure must satisfy; and hence to prove that these entities coincide with the polycategories of Definition 6.

To arrive at our abstract formulation, we imitate the methods of the previous section. Firstly, given a set $X$ of objects, we may view it as a discrete category and consider the functor category $[(SX)^{\mathrm{op}} \times SX, \mathbf{Set}]$; and to give an element of this is to give sets of polymaps together with coherent exchange isomorphisms. We would now like to set up a monoidal structure on this category such that a monoid in it is precisely a polycompositional polycategory. The unit is straightforward:

$$I(\Gamma; \Delta) = \begin{cases} \{*\} & \text{if } \Gamma = x = \Delta, \\ \emptyset & \text{otherwise;} \end{cases}$$

and we can describe what a typical element of $(F \otimes F)(\Gamma; \Delta)$ should look like. Let

$$\Psi_1, \ldots, \Psi_k, \quad \Lambda_1, \ldots, \Lambda_k, \quad \Sigma_1, \ldots, \Sigma_l \quad \text{and} \quad \Phi_1, \ldots, \Phi_l$$

be elements of $(\mathrm{ob}\,\mathbb{M})^*$, such that

- $|\Gamma| = n = \sum |\Psi_i|$;
- $\sum |\Lambda_i| = m = \sum |\Sigma_j|$;
- $\sum |\Phi_j| = p = |\Gamma|$;
- there exists $\sigma \in S_n$ such that $\sigma \Gamma = \Psi_1, \ldots, \Psi_k$;
- there exists $\tau \in S_m$ such that $\tau$ is a suitable matching of $\{\Lambda_i\}$ with $\{\Sigma_i\}$;
- there exists $\upsilon \in S_p$ such that $\upsilon(\Phi_1, \ldots, \Phi_k) = \Delta$;

and let $f_i : \Psi_i \to \Lambda_i$ (for $i = 1, \ldots, k$), and $g_j : \Sigma_j \to \Phi_j$ (for $j = 1, \ldots, l$) be polymaps in $F$. Then this gives us a typical element of $(F \otimes F)(\Gamma; \Delta)$, which we visualise as

$$\begin{array}{c}
\Gamma \\
\downarrow \sigma \\
\Psi_1, \ldots, \Psi_k \\
\downarrow f_1, \ldots, f_k \\
\Lambda_1, \ldots, \Lambda_k \\
\downarrow \tau \\
\Sigma_1, \ldots, \Sigma_l \\
\downarrow g_1, \ldots, g_l \\
\Phi_1, \ldots, \Phi_l \\
\downarrow \upsilon \\
\Delta.
\end{array}$$
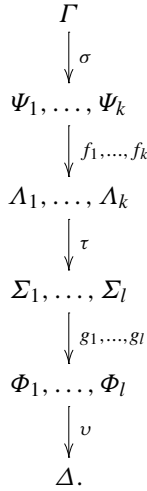
Then as for the multicategory case, the multiplication map $m : F \otimes F \to F$ should specify a composite map for this 'formal polycomposite,' and the associativity and unitality conditions for a monoid should ensure that this polycomposition is associative and unital.

So our problem is reduced to finding a suitable way of expressing this monoidal structure; and in fact we will skip straight over this stage and instead describe polycompositional polycategories as monads in a suitable bicategory. For this, we shall need the following fact:

**Proposition 12.** *The* 2-*monad* $(S, \eta, \mu)$ *on* **Cat** *lifts to a pseudocomonad* $(\hat{S}, \hat{\epsilon}, \hat{\Delta})$ *as well as a pseudomonad* $(\hat{S}, \hat{\eta}, \hat{\mu})$ *on* **Mod***.*

**Proof.** The transformations $\hat{\epsilon}$ and $\hat{\Delta}$ have respective components at **C** given by

$$\hat{\epsilon}_{\mathbf{C}} = (\eta_{\mathbf{C}})^* \quad \text{and} \quad \hat{\Delta}_{\mathbf{C}} = (\mu_{\mathbf{C}})^*.$$

We obtain the remaining data for the pseudocomonad via the calculus of mates [16], making use of the adjunctions $\hat{\eta}_{\mathbf{C}} \dashv \hat{\epsilon}_{\mathbf{C}}$ and $\hat{\mu}_{\mathbf{C}} \dashv \hat{\Delta}_{\mathbf{C}}$.   $\square$

[Since the pseudomonad $(\hat{S}, \hat{\eta}, \hat{\mu})$ and the pseudocomonad $(\hat{S}, \hat{\epsilon}, \hat{\mu})$ share the same underlying homomorphism $\hat{S} : \mathbf{Mod} \to \mathbf{Mod}$, there is some scope for confusion here. To remedy this, we will use $\hat{S}_m$ and $\hat{S}_c$ as aliases for the homomorphism $\hat{S}$; the former when we are thinking of it as part of a pseudomonad structure, and the latter, when as part of a pseudocomonad.]

The key idea is to produce a *pseudo-distributive law* $(\delta, \bar{\eta}, \bar{\epsilon}, \overline{\mu}, \overline{\Delta})$ of the pseudocomonad $\hat{S}_c$ over the pseudomonad $\hat{S}_m$; that is, there should be a pseudo-natural transformation $\delta : \hat{S}_c \hat{S}_m \Rightarrow \hat{S}_m \hat{S}_c$ satisfying the rules of a distributive law 'up to isomorphism,' as specified by the invertible modifications $\bar{\eta}, \bar{\epsilon}, \overline{\mu}$ and $\overline{\Delta}$: for full details, see Appendix A. Given such a pseudo-distributive law, polycategories will emerge as monads in its 'two-sided Kleisli bicategory.' Since this construction may not be familiar, we describe it first one dimension down:

**Definition 13.** Let **C** be a category, let $(S, \eta, \mu)$ be a monad and $(T, \epsilon, \Delta)$ a comonad on **C**, and let $\delta : TS \Rightarrow ST$ be a distributive law of the comonad over the monad; so we have the four

equalities:

$$\epsilon S = S\epsilon \circ \delta, \qquad \eta T = \delta \circ T\eta,$$

$$S\Delta \circ \delta = \delta T \circ T\delta \circ \Delta S, \quad \text{and} \quad \delta \circ T\mu = \mu T \circ S\delta \circ \delta S.$$

Then the two-sided Kleisli category $Kl(\delta)$ of the distributive law $\delta$ has:

- *Objects* those of **C**.
- *Maps* $A \to B$ in $Kl(\delta)$ given by maps $TA \to SB$ in **C**.
- *Identity maps* $\mathrm{id}_A : A \to A$ in $Kl(\delta)$ given by the map

$$TA \xrightarrow{\epsilon_A} A \xrightarrow{\eta_A} SA$$

  in **C**.
- *Composition* for maps $f : A \to B$ and $g : B \to C$ in $Kl(\delta)$ given by the map

$$TA \xrightarrow{\Delta_A} TTA \xrightarrow{Tf} TSB \xrightarrow{\delta_B} STB \xrightarrow{Sg} SSC \xrightarrow{\mu_C} SC$$

  in **C**.

Now, we can emulate such a construction one dimension up:

**Definition 14.** Let $\mathcal{B}$ be a bicategory, let $(S, \eta, \mu, \lambda, \rho, \tau)$ be a pseudomonad and $(T, \epsilon, \Delta, \lambda', \rho', \tau')$ a pseudocomonad on $\mathcal{B}$, and let $(\delta, \bar{\eta}, \bar{\epsilon}, \bar{\mu}, \overline{\Delta})$ be a pseudo-distributive law of the pseudo-comonad over the pseudomonad. Then the two-sided Kleisli bicategory $Kl(\delta)$ of the pseudo-distributive law $\delta$ has:

- *Objects* those of $\mathcal{B}$.
- *Hom-categories* given by $Kl(\delta)(X, Y) = \mathcal{B}(TX, SY)$.
- *Identity map* at $X$ given by the composite

$$TX \xrightarrow{\epsilon_X} X \xrightarrow{\eta_X} SX.$$

- *Composition* $Kl(\delta)(Y, Z) \times Kl(\delta)(X, Y) \to Kl(\delta)(X, Z)$ given by

$$\mathcal{B}(TY, SZ) \times \mathcal{B}(TX, SY)$$

$$\downarrow \cong$$

$$1 \times \mathcal{B}(TY, SZ) \times 1 \times \mathcal{B}(TX, SY) \times 1$$

$$\downarrow {}^{\ulcorner \mu_Z \urcorner \times S \times \ulcorner \delta_Y \urcorner \times T \times \ulcorner \epsilon_X \urcorner}$$

$$\mathcal{B}(SSZ, SZ) \times \mathcal{B}(STY, SSZ) \times \mathcal{B}(TSY, STY) \times$$
$$\mathcal{B}(TTX, TSY) \times \mathcal{B}(TX, TTX)$$

$$\downarrow \otimes$$

$$\mathcal{B}(TX, SZ)$$

where we use $\otimes$ to stand for some choice of order of composition for the displayed fivefold composite. Explicitly, we take $(TY \xrightarrow{G} SZ) \otimes (TX \xrightarrow{F} SY)$ to be (some choice of bracketing for) the composite

$$TX \xrightarrow{\Delta_X} TTX \xrightarrow{TF} TSY \xrightarrow{\delta_Y} STY \xrightarrow{SG} SSZ \xrightarrow{\mu_Z} SZ.$$

Again, we shall not provide the associativity and unitality constraints required to make this into a bicategory: they are now constructed from the pseudomonad structure of $S$, the pseudo-comonad structure of $T$ and the pseudo-distributive structure of $\delta$.

Returning to the case under consideration, we claim that there is a pseudo-distributive law $\delta : \hat{S}_c \hat{S}_m \Rightarrow \hat{S}_m \hat{S}_c$ given as follows. Recall that we have $\hat{S}_c = \hat{S}_m = \hat{S}$, and thus the component $\delta_{\mathbf{C}} : \hat{S}_c \hat{S}_m \mathbf{C} \to\!\!\!\to \hat{S}_m \hat{S}_c \mathbf{C}$ of $\delta$ is given by a functor $(SS\mathbf{C})^{\mathrm{op}} \times SS\mathbf{C} \to \mathbf{Set}$. So, given a *discrete* category $X$, we wish to take $\delta_X(\{\Sigma_m\}_{1 \leqslant m \leqslant j}; \{\Gamma_n\}_{1 \leqslant n \leqslant k})$ to be the set of suitable matchings of $\{\Sigma_m\}$ with $\{\Gamma_n\}$. If we unwrap the definition of two-sided Kleisli bicategory above, we now see that the desired monoidal structure on $[(SX)^{\mathrm{op}} \times SX, \mathbf{Set}]$ is given precisely by horizontal composition in $Kl(\delta)(X, X)$.

Thus we should *like* to define a polycompositional polycategory to be a monad on a discrete object $X$ in the bicategory $Kl(\delta)$; but to do this, we must first establish the existence of the pseudo-distributive law $\delta$. It is the task of the remainder of this paper to do this.

[The following alternative approach to the theory of polycategories was suggested by Robin Houston: from the paper [17], multicategories with object set $X$ can be viewed as *lax monoids* on the discrete object $X$ in **Mod**. We might hope to extend this to a notion of *lax Frobenius algebra*, following [18]; then a polycategory would be such a lax Frobenius algebra on a discrete object of **Mod**. However, we shall not pursue this further here.]

## 3. Deriving the pseudo-distributive law $\delta$

We intend to construct the pseudo-distributive law $\delta$ by exploiting the theory of double clubs, as developed in the companion paper [11]. A *double club* is a generalisation of Kelly's abstract notion of club [19] from the level of categories to that of *pseudo* (or *weak*) double categories. Let us recap briefly the details we shall need here.

A *pseudo double category* $\mathbb{K}$ is a 'pseudo-category' object in **Cat**. Explicitly, it consists of *objects $X, Y, Z, \ldots$, vertical maps $f : X \to Y$, horizontal maps $\mathbf{X} : X_s \nrightarrow X_t$* and *cells*

$$
\begin{array}{ccc}
X_s & \xrightarrow{\;\mathbf{X}\;} & X_t \\
{\scriptstyle f_s}\Big\downarrow & \Downarrow\mathbf{f} & \Big\downarrow{\scriptstyle f_t} \\
Y_s & \xrightarrow[\;\mathbf{Y}\;]{} & Y_t,
\end{array}
$$

together with notions of vertical and horizontal composition such that vertical composition is associative on the nose, whilst horizontal composition is associative up to invertible *special cells*, where a cell is said to be *special* just when its vertical source and target maps are identities. The objects and vertical maps of a pseudo double category $\mathbb{K}$ form a category $K_0$, whilst the horizontal maps and cells form a category $K_1$.

Any pseudo double category $\mathbb{K}$ contains a bicategory $\mathcal{B}\mathbb{K}$ consisting of the objects, horizontal maps and special cells of $\mathbb{K}$; and it is reasonable to think of $\mathbb{K}$ as being the bicategory $\mathcal{B}\mathbb{K}$ with 'added vertical structure.' For example, we will be concerned with the pseudo double category $\mathbb{C}at$ which has:

- *Objects* being small categories **C**.
- *Vertical maps* being functors $f : \mathbf{C} \to \mathbf{D}$.
- *Horizontal maps* being profunctors $F : \mathbf{D}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$.
- *Cells*

$$
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{H} & \mathbf{D} \\
f \downarrow & \Downarrow\alpha & \downarrow g \\
\mathbf{E} & \xrightarrow{K} & \mathbf{F}
\end{array}
$$

being natural transformations

$$
\begin{array}{ccc}
\mathbf{D}^{\mathrm{op}} \times \mathbf{C} & \xrightarrow{\;g^{\mathrm{op}} \times f\;} & \mathbf{F}^{\mathrm{op}} \times \mathbf{E} \\
& \overset{\alpha}{\Rightarrow} & \\
& \underset{H}{\searrow} \qquad \underset{K}{\swarrow} & \\
& \mathbf{Set}. &
\end{array}
$$

Following the above philosophy, we think of $\mathbb{C}at$ as being the bicategory $\mathcal{B}\mathbb{C}at = \mathbf{Mod}$ of categories, profunctors and profunctor transformations, extended with the vertical structure of honest functors.

We can now go on to give a notion of *homomorphism* of pseudo double categories, extending that for bicategories, and two notions of *transformation* between homomorphisms, namely *vertical* and *horizontal*: the former having vertical maps for its components, and the latter horizontal. The correct notion of *modification* for pseudo double categories is that of a 'cell' bordered by two horizontal and two vertical transformations. In fact, it genuinely *is* a cell in that we have:

**Proposition 15.** *Given pseudo double categories $\mathbb{K}$ and $\mathbb{L}$, there is a pseudo double category $[\mathbb{K}, \mathbb{L}]_\psi$ of homomorphisms $\mathbb{K} \to \mathbb{L}$, vertical transformations, horizontal transformations and modifications.*

Pseudo double categories, homomorphisms and vertical transformations form themselves into a 2-category $\mathbf{DblCat}_\psi$, and thus we can read off notions such as *equivalence of pseudo double categories* (equivalence in $\mathbf{DblCat}_\psi$) and *double monad* (monad in $\mathbf{DblCat}_\psi$).

We now recap very briefly the theory of double clubs developed in [11]. Given a homomorphism $S : \mathbb{K} \to \mathbb{L}$, we can construct the 'slice pseudo double category' $[\mathbb{K}, \mathbb{L}]_\psi / S\mathbf{I}$. It has

- *Objects* $(A, \alpha)$ being homomorphisms $A : \mathbb{K} \to \mathbb{L}$ together with a vertical transformation $\alpha : A \Rightarrow S$.
- *Vertical maps* $\gamma : (A, \alpha) \to (B, \beta)$ being vertical transformations $\gamma : A \Rightarrow B$ such that $\beta\gamma = \alpha$.

- *Horizontal maps* $(\mathbf{A}, \boldsymbol{\alpha}) : (A_s, \alpha_s) \longrightarrow (A_t, \alpha_t)$ being horizontal transformations $\mathbf{A} : A_s \Rightarrow A_t$ together with a modification

$$
\begin{array}{ccc}
A_s & \overset{\mathbf{A}}{\Longrightarrow} & A_t \\
\alpha_s \big\Vert & \Downarrow \boldsymbol{\alpha} & \big\Vert \alpha_t \\
S & \underset{S\mathbf{I}}{\Longrightarrow} & S.
\end{array}
$$

- *Cells*

$$
\begin{array}{ccc}
(A_s, \alpha_s) & \overset{(\mathbf{A}, \boldsymbol{\alpha})}{\longrightarrow} & (A_t, \alpha_t) \\
\gamma_s \big\downarrow & \Downarrow \boldsymbol{\gamma} & \big\downarrow \gamma_t \\
(B_s, \beta_s) & \underset{(\mathbf{B}, \boldsymbol{\beta})}{\longrightarrow} & (B_t, \beta_t)
\end{array}
$$

being modifications

$$
\begin{array}{ccc}
A_s & \overset{\mathbf{A}}{\Longrightarrow} & A_t \\
\gamma_s \big\Vert & \Downarrow \boldsymbol{\gamma} & \big\Vert \gamma_t \\
B_s & \underset{\mathbf{B}}{\Longrightarrow} & B_t
\end{array}
$$

such that $\boldsymbol{\beta}\boldsymbol{\gamma} = \boldsymbol{\alpha}$.

For a sufficiently well-behaved $S$, this has a sub-pseudo double category $\mathbb{C}oll(S)$, whose objects are cartesian vertical transformations into $S$ and whose horizontal maps are cartesian modifications into $S\mathbf{I}$. Here, a vertical transformation or modification is said to be *cartesian* just when all its naturality squares are pullbacks; and so $\mathbb{C}oll(S)$ is the pseudo double category analogue of the 'category of collections' $Coll(S)$ in Kelly's theory of clubs.

We have a strict double homomorphism $\mathsf{ev}_1 : \mathbb{C}oll(S) \to \mathbb{L}/S\mathbf{I}_1$ which evaluates at 1, where 1 is the terminal object of $\mathbb{L}$; and as in the theory of clubs, we effectively lose no information in applying this homomorphism:

**Proposition 16.** *For $\mathbb{L}$ sufficiently complete, the strict double homomorphism $\mathsf{ev}_1$ forms one side of an equivalence of pseudo double categories*

$$
\mathbb{C}oll(S) \simeq \mathbb{L}/S\mathbf{I}_1.
$$

**Proof.** See [11, Proposition 30]. $\quad\square$

In order to give a sensible definition of 'double club,' we need a notion of monoidal structure for pseudo double categories:

**Definition 17.** A *monoidal pseudo double category* is a pseudomonoid in the (cartesian) monoidal 2-category **DblCat**$_\psi$.

**Proposition 18.** *The 'endohom' pseudo double category* $[\mathbb{K}, \mathbb{K}]_\psi$ *has a canonical structure of monoidal pseudo double category; furthermore, given a double monad* $(S, \eta, \mu)$ *on* $\mathbb{K}$*, the slice pseudo double category* $[\mathbb{K}, \mathbb{K}]_\psi / S\mathbf{I}$ *has a canonical monoidal structure lifting that of* $[\mathbb{K}, \mathbb{K}]_\psi$.

**Proof.** See [11, Propositions 39 and 43]. $\square$

We now have:

**Definition 19.** A double monad $(S, \eta, \mu)$ on a pseudo double category $\mathbb{K}$ is a *double club* if $\mathbb{C}oll(S)$ is closed under the monoidal structure of $[\mathbb{K}, \mathbb{K}]_\psi / S\mathbf{I}$.

Probably the best-known (and indeed, the oldest) example of a club is that for *symmetric strict monoidal categories* on **Cat**. In [11], we show that this club extends to a double club $(S, \eta, \mu)$ on $\mathbb{C}at$; and it is this result that we shall make use of in the rest of this section.

### 3.1. Lifting to $\mathbb{C}oll(S)$

We wish to apply the theory of double clubs to simplifying the construction of our pseudo-distributive law $\delta$. Now, this pseudo-distributive law is specified in terms of certain data and axioms in the bicategory $[\mathbf{Mod}, \mathbf{Mod}]_\psi$. However, it makes sense in *any* bicategory equipped with well-behaved notions of 'whiskering' (*well-behaved* in the sense that they obey axioms formally similar to those for a **Gray**-monoid [20]).

We show in the Appendix of [11] that for any double club, $\mathbb{C}oll(S)$ is not only a monoidal pseudo double category, but is equipped with a notion of 'whiskering,' and it follows from this that $\mathcal{B}(\mathbb{C}oll(S))$ is a suitable setting for the construction of a pseudo-distributive law. Furthermore, it is easy see that there is a strict homomorphism of bicategories

$$V : \mathcal{B}(\mathbb{C}oll(S)) \to \mathcal{B}([\mathbb{C}at, \mathbb{C}at]_\psi) \to [\mathbf{Mod}, \mathbf{Mod}]_\psi$$

which first forgets the projections onto $S\mathbf{I}$, and then forgets the vertical structure; and moreover, that this homomorphism respects the 'whiskering' operations on these two bicategories. So if we can lift the pseudomonad $\hat{S}_m$ and pseudocomonad $\hat{S}_c$ along $V$, then any pseudo-distributive law we construct between their respective liftings will induce a pseudo-distributive law between $\hat{S}_m$ and $\hat{S}_c$ as desired.

At this stage, it might appear that we have only made things more complicated, by requiring ourselves to construct a pseudo-distributive law in $\mathbb{C}oll(S)$; but now we are in a position to utilise the equivalence of pseudo double categories $\mathbb{C}oll(S) \simeq \mathbb{C}at / S\mathbf{I}_1$ in order to reduce the construction of a pseudo-distributive law in $\mathbb{C}oll(S)$ to a much simpler construction 'at 1.'

So, let us begin by showing how we may lift our pseudomonad $\hat{S}_m$ and pseudocomonad $\hat{S}_c$ to $\mathcal{B}(\mathbb{C}oll(S))$. The first stage is straightforward; we lift

$$\hat{S}_m \quad \text{to} \quad \left\Vert \begin{array}{c} S_m \\ \Downarrow \text{id}_S \\ S \end{array} \right. \quad \text{and} \quad \hat{S}_c \quad \text{to} \quad \left\Vert \begin{array}{c} S_c \\ \Downarrow \text{id}_S \\ S \end{array} \right. ,$$

where, again, we are using $S_c$ and $S_m$ as aliases for $S : \mathbb{C}at \to \mathbb{C}at$. Next we must lift $\hat{\eta}$, $\hat{\mu}$, $\hat{\epsilon}$ and $\hat{\Delta}$ to horizontal transformations and cartesian modifications as follows:

$$
\begin{array}{ccc}
\mathrm{id}_{\mathbb{C}at} \xRightarrow{\;\boldsymbol{\eta}\;} S_m & & S_m S_m \xRightarrow{\;\boldsymbol{\mu}\;} S_m \\
\eta \Big\downarrow \quad \Downarrow \tilde{\boldsymbol{\eta}} \quad \Big\downarrow \mathrm{id}_S, & & \mu \Big\downarrow \quad \Downarrow \tilde{\boldsymbol{\mu}} \quad \Big\downarrow \mathrm{id}_S, \\
S \xRightarrow[S\mathbf{I}]{} S & & S \xRightarrow[S\mathbf{I}]{} S
\end{array}
$$

$$
\begin{array}{cccc}
S_c \xRightarrow{\;\boldsymbol{\epsilon}\;} \mathrm{id}_{\mathbb{C}at} & & S_c \xRightarrow{\;\boldsymbol{\Delta}\;} S_c S_c \\
\mathrm{id}_S \Big\downarrow \quad \Downarrow \tilde{\boldsymbol{\epsilon}} \quad \Big\downarrow \eta & \text{and} & \mathrm{id}_S \Big\downarrow \quad \Downarrow \tilde{\boldsymbol{\Delta}} \quad \Big\downarrow \mu \\
S \xRightarrow[S\mathbf{I}]{} S & & S \xRightarrow[S\mathbf{I}]{} S.
\end{array}
$$

Now, to give the horizontal transformation $\boldsymbol{\eta}$ we must give a 'components functor' $\mathbb{C}at_0 \to \mathbb{C}at_1$ along with 'pseudonaturality' special cells. For the former, we take the component at an object $X$ to be given by the component of $\hat{\eta}$ at $X$, and the component at a vertical map $f$ to be given by the pasting

$$
\begin{array}{ccc}
X & \xrightarrow{\;\hat{\eta}_X\;} & SX \\
f_* \Big\downarrow & \Downarrow \hat{\eta}^{-1}_{(f_*)} \quad \hat{S}(f_*) & \Big\downarrow \; (Sf)_* \\
Y & \xrightarrow[\hat{\eta}_Y]{} & SY.
\end{array}
$$

For the latter, we merely take the pseudonaturality 2-cells of $\hat{\eta}$; checking all required naturality and coherence is now routine. To give the cartesian modification $\tilde{\boldsymbol{\eta}}$, we must give components $\tilde{\boldsymbol{\eta}}_X$ as follows:

$$
\begin{array}{ccc}
X & \xrightarrow{\;\hat{\eta}_X\;} & SX \\
\eta_X \Big\downarrow & \Downarrow \tilde{\boldsymbol{\eta}}_X & \Big\downarrow \mathrm{id}_S \\
SX & \xrightarrow[S\mathbf{I}_X]{} & SX.
\end{array}
$$

But this is to give natural families of maps $\hat{\eta}_X(y; x) \to S\mathbf{I}_X(y; \langle x \rangle)$ which we do via the natural isomorphisms

$$
\hat{\eta}_X(y; x) \cong SX\big(y, \langle x \rangle\big) \cong S\mathbf{I}_X\big(y; \langle x \rangle\big).
$$

Checking naturality and cartesianness is routine. We proceed similarly to lift $\hat{\mu}$, $\hat{\epsilon}$ and $\hat{\Delta}$.

Finally, we must check that the modifications $\lambda$, $\rho$, $\tau$, $\lambda'$, $\rho'$ and $\tau'$ for $\hat{S}_m$ and $\hat{S}_c$ lift to $\mathbb{C}oll(S)$. For example, we must check that

$$\lambda : \mathrm{id}_{\hat{S}} \Rightarrow \hat{\mu} \otimes \hat{S}_m \hat{\eta} : \hat{S}_m \Rightarrow \hat{S}_m$$

lifts to a special modification

$$\boldsymbol{\lambda} : \mathbf{I}_{(S_m, \mathrm{id}_S)} \Rightarrow (\boldsymbol{\mu}, \tilde{\boldsymbol{\mu}}) \otimes (S_m, \mathrm{id}_S)(\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}) : (S_m, \mathrm{id}_S) \Rightarrow (S_m, \mathrm{id}_S).$$

This amounts to checking that the components of $\lambda$ are natural with respect to cells of $\mathbb{C}at$, and that they are compatible with the projections down to $S\mathbf{I}$; and this is merely a matter of diagram chasing.

Therefore, in order to obtain our desired pseudo-distributive law on **Mod**, it suffices to produce data and axioms for a pseudo-distributive law between $(S_m, \mathrm{id}_S)$ and $(S_c, \mathrm{id}_S)$ as detailed above. We now wish to see how we can use the theory of double clubs to reduce this to data and axioms in $\mathbb{C}at/S\mathbf{I}_1$.

### 3.2. Reducing to $\mathbb{C}at/S\mathbf{I}_1$

We begin with (PDD1), for which we must produce a horizontal arrow

$$(\boldsymbol{\delta}, \tilde{\boldsymbol{\delta}}) : (S_c S_m, \mu) \longrightarrow (S_m S_c, \mu)$$

of $\mathbb{C}oll(S)$, i.e., a horizontal transformation and a cartesian modification as follows:

$$
\begin{array}{ccc}
S_c S_m & \overset{\delta}{\Longrightarrow} & S_m S_c \\
\mu \big\Downarrow & \quad \Downarrow \tilde{\delta} \quad & \big\Downarrow \mu \\
S & \underset{S\mathbf{I}}{\Longrightarrow} & S.
\end{array}
$$

Now, suppose we have a horizontal arrow

$$
\begin{array}{ccc}
S_c S_m 1 & \overset{\delta_1}{\longmapsto} & S_m S_c 1 \\
\mu_1 \big\downarrow & \quad \Downarrow \tilde{\delta}_1 \quad & \big\downarrow \mu_1 \\
S1 & \underset{S\mathbf{I}_1}{\longmapsto} & S1
\end{array}
$$

of $\mathbb{C}at/S\mathbf{I}_1$. We should like to say that $(\delta_1, \tilde{\delta}_1)$ is the component at 1 of some horizontal arrow $(\boldsymbol{\delta}, \tilde{\boldsymbol{\delta}})$ of $\mathbb{C}oll(S)$, which amounts to asking for the double homomorphism $\mathsf{ev}_1 : \mathbb{C}oll(S) \to \mathbb{C}at/S\mathbf{I}_1$ to be 'horizontally full,' in the following sense:

**Proposition 20.** *Let $(A_s, \alpha_s)$ and $(A_t, \alpha_t)$ be objects of $\mathbb{C}oll(S)$, and suppose that we have a horizontal arrow*

$$A_s 1 \xrightarrow{\;\mathbf{a}\;} A_t 1$$

$$(\alpha_s)_1 \Big\downarrow \qquad \Downarrow \boldsymbol{\gamma} \qquad \Big\downarrow (\alpha_t)_1$$

$$S1 \xrightarrow[S\mathbf{I}_1]{} S1$$

*of* $\mathbb{C}at/S\mathbf{I}_1$*. Then there is a horizontal arrow* $(\mathbf{A}, \boldsymbol{\Gamma})$ *of* $\mathbb{C}oll(S)$*:*

$$A_s \overset{\mathbf{A}}{\Longrightarrow} A_t$$

$$\alpha_s \Big\Vert \qquad \Downarrow \boldsymbol{\Gamma} \qquad \Big\Vert \alpha_t$$

$$S \underset{S\mathbf{I}}{\Longrightarrow} S$$

*such that* $\mathsf{ev}_1(\mathbf{A}, \boldsymbol{\Gamma}) = (\mathbf{a}, \boldsymbol{\gamma})$*.*

**Proof.** Proposition 16 above tells us that $\mathsf{ev}_1 : \mathbb{C}oll(S) \to \mathbb{C}at/S\mathbf{I}_1$ forms one side of an equivalence of double categories: and the proof of this given in [11] constructs an explicit quasi-inverse $\mathbb{C}at/S\mathbf{I}_1 \to \mathbb{C}oll(S)$. The following is a simple adaptation of this construction to the problem at hand.

To give the horizontal transformation $\mathbf{A}$, we must give, amongst other things, a component profunctor $\mathbf{A_C} : A_s \mathbf{C} \nrightarrow A_t \mathbf{C}$ at each small category $\mathbf{C}$; whilst to give $\boldsymbol{\Gamma}$, we must give, for each small category $\mathbf{C}$, a cell

$$A_s \mathbf{C} \xrightarrow{\;\mathbf{A_C}\;} A_t \mathbf{C}$$

$$(\alpha_s)_\mathbf{C} \Big\downarrow \qquad \Downarrow \boldsymbol{\Gamma}_\mathbf{C} \qquad \Big\downarrow (\alpha_t)_\mathbf{C}$$

$$S\mathbf{C} \xrightarrow[S\mathbf{I_C}]{} S\mathbf{C}$$

of $\mathbb{C}at$. We may view $\boldsymbol{\gamma}$ as a morphism $\mathbf{a} \to S\mathbf{I}_1$ in the category $\mathbb{C}at_1$ of profunctors and transformations between them; and thus may form $\mathbf{A_C}$ and $\boldsymbol{\Gamma}_\mathbf{C}$ as the following pullback in $\mathbb{C}at_1$:

$$
\begin{array}{ccc}
\mathbf{A_C} & \longrightarrow & \mathbf{a} \\
\boldsymbol{\Gamma}_\mathbf{C} \Big\downarrow & & \Big\downarrow \boldsymbol{\gamma} \\
S\mathbf{I_C} & \xrightarrow[S\mathbf{I}_!]{} & S\mathbf{I}_1.
\end{array}
\qquad (*)
$$

Now, in order that $\mathbf{A_C}$ and $\boldsymbol{\Gamma}_\mathbf{C}$ should have the correct sources and targets, we must choose the pullback $(*)$ in such a way that application of the source and target functors $s, t : \mathbb{C}at_1 \to \mathbb{C}at_0$ sends it to the respective squares:

$$
\begin{array}{ccc}
A_s \mathbf{C} & \longrightarrow & A_s 1 \\
(\alpha_s)_\mathbf{C} \Big\downarrow & & \Big\downarrow (\alpha_s)_1 \\
S\mathbf{C} & \xrightarrow[S!]{} & S1
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
A_t \mathbf{C} & \longrightarrow & A_t 1 \\
(\alpha_t)_\mathbf{C} \Big\downarrow & & \Big\downarrow (\alpha_t)_1 \\
S\mathbf{C} & \xrightarrow[S!]{} & S1
\end{array}
$$

in $\mathbb{C}at_0$. That we may do this follows from two observations: firstly, that both the displayed squares are pullbacks in $\mathbb{C}at_0$, by cartesianness of $\alpha_s$ and $\alpha_t$; and secondly, that the functor $(s, t) : \mathbb{C}at_1 \to \mathbb{C}at_0 \times \mathbb{C}at_0$ (strictly) creates pullbacks.

In order that we should have $\mathsf{ev}_1(\mathbf{A}, \boldsymbol{\Gamma}) = (\mathbf{a}, \boldsymbol{\gamma})$, we make one further demand: that when $\mathbf{C} = 1$, the pullback square $(*)$ should be chosen as

$$
\begin{array}{ccc}
\mathbf{a} & \xrightarrow{\ \mathrm{id}\ } & \mathbf{a} \\
{\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \gamma} \\
S\mathbf{I}_1 & \xrightarrow[\ \mathrm{id}\ ]{} & S\mathbf{I}_1 .
\end{array}
$$

Apart from this care in choosing the pullback squares $(*)$, the remaining details in the construction of $\mathbf{A}$ and $\boldsymbol{\Gamma}$ are exactly as in the proof of Proposition 30 of [11], and hence omitted. □

Thus, given a horizontal arrow $(\boldsymbol{\delta}_1, \tilde{\boldsymbol{\delta}}_1) : (S_c S_m 1, \mu_1) \nrightarrow (S_m S_c 1, \mu_1)$ of $\mathbb{C}at/S\mathbf{I}_1$, we can produce a horizontal arrow $(\boldsymbol{\delta}, \tilde{\boldsymbol{\delta}}) : (S_c S_m, \mu) \nrightarrow (S_m S_c, \mu)$ of $\mathbb{C}oll(S)$ whose image under $\mathsf{ev}_1$ is precisely $(\boldsymbol{\delta}_1, \tilde{\boldsymbol{\delta}}_1)$.

To derive the remaining data (PDD2) and (PDD3), we observe the following: the homomorphism $F := \mathsf{ev}_1 : \mathbb{C}oll(S) \to \mathbb{C}at/S\mathbf{I}_1$ is built upon two functors $F_0 : \mathbb{C}oll(S)_0 \to (\mathbb{C}at/S\mathbf{I}_1)_0$ and $F_1 : \mathbb{C}oll(S)_1 \to (\mathbb{C}at/S\mathbf{I}_1)_1$; and since $F$ forms one side of an equivalence of pseudo double categories, it follows that $F_0$ and $F_1$ each form one side of an equivalence of ordinary categories. In particular, the functor $F_1 : \mathbb{C}oll(S)_1 \to \mathbb{C}at_1/S\mathbf{I}_1$ is full and faithful. Thus, considering $\bar{\boldsymbol{\eta}}$ for instance, we must find a special invertible cell

$$
\bar{\boldsymbol{\eta}} : (\boldsymbol{\delta}, \tilde{\boldsymbol{\delta}}) \otimes (S_m, \mathrm{id}_S)(\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}) \Rightarrow (\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}})(S_m, \mathrm{id}_S)
$$

of $\mathbb{C}oll(S)$. Since $F_1$ is full and faithful, it suffices for this to find a special invertible cell

$$
\bar{\boldsymbol{\eta}}_1 : (\boldsymbol{\delta}_1, \tilde{\boldsymbol{\delta}}_1) \otimes \big((S_m, \mathrm{id}_S)(\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}})\big)_1 \Rightarrow \big((\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}})(S_m, \mathrm{id}_S)\big)_1
$$

of $\mathbb{C}at/S\mathbf{I}_1$. We proceed similarly for the remaining data.

Finally, we must ensure that (PDA1)–(PDA10) are satisfied, which amounts to checking certain equalities of pastings in $\mathcal{B}(\mathbb{C}oll(S))$, which in turn amounts to checking certain equalities of maps in $\mathbb{C}oll(S)_1$; but since the functor $F_1 : \mathbb{C}oll(S)_1 \to \mathbb{C}at_1/S\mathbf{I}_1$ is faithful, it suffices for this to check that these equalities hold in $\mathbb{C}at/S\mathbf{I}_1$.

## 4. Constructing the pseudo-distributive law at 1

### 4.1. The double club S on $\mathbb{C}$at

In order to construct the data and axioms laid out at the end of the previous section, we will require a detailed presentation of the double club $(S, \eta, \mu)$ on $\mathbb{C}at$. Since this double club looks like the free symmetric monoidal category monad on **Cat** in the vertical direction, and like its lifting $\hat{S}$ to **Mod** in the horizontal direction, we may do this by giving a presentation of these latter two entities.

**Definition 21.** We write $S1$ for the category of finite cardinals and bijections, with:

- *Objects* the natural numbers $0, 1, 2, \ldots$;
- *Maps* $\sigma : n \to m$ bijections of $\{1, \ldots, n\}$ with $\{1, \ldots, m\}$,

and with composition and identities given in the evident way.

**Definition 22.** The *free symmetric strict monoidal category* 2-functor $S : \mathbf{Cat} \to \mathbf{Cat}$ is given as follows:

- *On objects*: Given a small category $\mathbf{C}$, we give $S\mathbf{C}$ as follows:
  - *Objects* of $S\mathbf{C}$ are pairs $(n, \langle c_i \rangle)$, where $n \in S1$ and $c_1, \ldots, c_n \in \mathrm{ob}\,\mathbf{C}$.
  - *Arrows* of $S\mathbf{C}$ are

$$\big(\sigma, \langle g_i \rangle\big) : \big(n, \langle c_i \rangle\big) \to \big(m, \langle d_i \rangle\big),$$

  where $\sigma \in S1(n, m)$ and $g_i : c_i \to d_{\sigma(i)}$ (note that necessarily $n = m$).
  Composition and identities in $S\mathbf{C}$ are given in the evident way; namely,

$$\mathrm{id}_{(n, \langle c_i \rangle)} = \big(\mathrm{id}_n, \langle \mathrm{id}_{c_i} \rangle\big)$$

$$\text{and} \quad \big(\tau, \langle g_i \rangle\big) \circ \big(\sigma, \langle f_i \rangle\big) = \big(\tau \circ \sigma, \langle g_{\sigma(i)} \circ f_i \rangle\big).$$

- *On maps*: Given a functor $F : \mathbf{C} \to \mathbf{D}$, we give $SF : S\mathbf{C} \to S\mathbf{D}$ by

$$SF\big(n, \langle c_i \rangle\big) = \big(n, \langle Fc_i \rangle\big) \quad \text{and} \quad SF\big(\sigma, \langle g_i \rangle\big) = \big(\sigma, \langle Fg_i \rangle\big).$$

- *On 2-cells*: Given a natural transformation $\alpha : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$, we give $S\alpha : SF \Rightarrow SG : S\mathbf{C} \to S\mathbf{D}$ by

$$(S\alpha)_{(n, \langle c_i \rangle)} = \big(\mathrm{id}_n, \langle \alpha_{c_i} \rangle\big).$$

Now, although the above is sufficient to describe the iterated functor $S^2 : \mathbf{Cat} \to \mathbf{Cat}$, it will be much more pleasant to work with the following alternative presentation. First note that we may describe $S^2 1$ as follows:

- *Objects* are order-preserving maps $\phi : n_\phi \to m_\phi$, where $n_\phi, m_\phi \in \mathbb{N}$. We write such an object simply as $\phi$, with the convention that $\phi$ has domain and codomain $n_\phi$ and $m_\phi$ respectively.
- *Maps* $f : \phi \to \psi$ are pairs of bijections $f_n : n_\phi \to n_\psi$ and $f_m : m_\phi \to m_\psi$ such that the following diagram commutes:

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\;f_n\;} & n_\psi \\
{\scriptstyle \phi}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\
m_\phi & \xrightarrow[\;f_m\;]{} & m_\psi\,.
\end{array}
$$

It may not be immediately obvious that this *is* a presentation of $S^2 1$. The picture is as follows: an object $\phi$ of $S^2 1$ is to be thought of as a collection of $n_\phi$ points partitioned into $m_\phi$ parts in accordance with $\phi$. Given such an object, one can permute internally any of its $m_\phi$ parts, or can in fact permute the set of $m_\phi$ parts itself; and a typical map describes such a permutation. For example, the objects

$$\phi : 5 \to 4 \qquad\qquad \psi : 5 \to 4$$
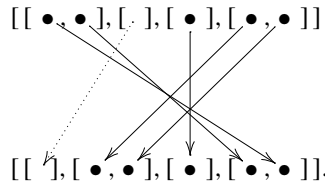$$1, 2, 3, 4, 5 \mapsto 1, 1, 3, 4, 4 \qquad 1, 2, 3, 4, 5 \mapsto 2, 2, 3, 4, 4$$

should be visualised as

$$\big[[\bullet, \bullet], [], [\bullet], [\bullet, \bullet]\big] \quad \text{and} \quad \big[[], [\bullet, \bullet], [\bullet], [\bullet, \bullet]\big]$$

respectively, whilst a typical map $\phi \to \psi$ is given by

$$f_n : 5 \to 5 \qquad\qquad f_m : 4 \to 4$$
$$1, 2, 3, 4, 5 \mapsto 5, 4, 3, 1, 2 \qquad 1, 2, 3, 4 \mapsto 4, 1, 3, 2$$

and should be visualised as



So now, given a category $\mathbf{C}$, we can present $S^2 \mathbf{C}$ as follows:

- *Objects* of $S^2 \mathbf{C}$ are pairs $(\phi, \langle c_i \rangle)$, where $\phi = n_\phi \to m_\phi \in S^2 1$ and $c_1, \ldots, c_{n_\phi} \in \mathrm{ob}\, \mathbf{C}$.
- *Arrows* of $S^2 \mathbf{C}$ are

$$\big(f, \langle g_i \rangle\big) : \big(\phi, \langle c_i \rangle\big) \to \big(\psi, \langle d_i \rangle\big),$$

  where $f = (f_n, f_m) \in S^2 1(\phi, \psi)$ and $g_i : c_i \to d_{f_n(i)}$; composition and identities are given analogously to before.

We can extend the above in the obvious way to 1- and 2-cells of **Cat** to give a presentation of the 2-functor $S^2$. Using this alternate presentation of $S^2$, we may describe the rest of the 2-monad structure of $S$:

**Definition 23.** The 2-natural transformation $\eta : \mathrm{id}_{\mathbf{Cat}} \Rightarrow S$ has component at $\mathbf{C}$ given by

$$\eta_{\mathbf{C}} : \mathbf{C} \to S\mathbf{C}$$
$$x \mapsto \big(1, \langle x \rangle\big),$$
$$f \mapsto \big(\mathrm{id}_1, \langle f \rangle\big);$$

whilst the 2-natural transformation $\mu : S^2 \Rightarrow S$ has component at $\mathbf{C}$ given by

$$\eta_{\mathbf{C}} : SS\mathbf{C} \to S\mathbf{C}$$
$$\big(\phi, \langle c_i \rangle\big) \mapsto \big(n_\phi, \langle c_i \rangle\big),$$
$$\big(f, \langle g_i \rangle\big) \mapsto \big(f_n, \langle g_i \rangle\big).$$

We will also need to make use of the threefold iterate $S^3$, and so it will be useful to present it in the above style. We first give $S^3 1$ as follows:

- *Objects* are diagrams $\phi = n_\phi \xrightarrow{\phi_1} m_\phi \xrightarrow{\phi_2} r_\phi$ in the category of finite ordinals and order preserving maps.
- *Maps* $f : \phi \to \psi$ are triples $(f_n, f_m, f_r)$ of bijections making

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\ f_n\ } & n_\psi \\
{\scriptstyle \phi_1}\big\downarrow & & \big\downarrow{\scriptstyle \psi_1} \\
m_\phi & \xrightarrow{\ f_m\ } & m_\psi \\
{\scriptstyle \phi_2}\big\downarrow & & \big\downarrow{\scriptstyle \psi_2} \\
r_\phi & \xrightarrow{\ f_r\ } & r_\psi
\end{array}
$$

commute.

Whereupon we may describe $S^3\mathbf{C}$ as follows:

- *Objects* are pairs $(\phi, \langle c_i \rangle)$, where $\phi = n_\phi \to m_\phi \to r_\phi \in S^3 1$ and $c_1, \ldots, c_{n_\phi} \in \mathrm{ob}\,\mathbf{C}$.
- *Arrows* are

$$\big(f, \langle g_i \rangle\big) : \big(\phi, \langle c_i \rangle\big) \to \big(\psi, \langle d_i \rangle\big),$$

where $f = (f_n, f_m, f_r) \in S^3 1(\phi, \psi)$ and $g_i : c_i \to d_{f_n(i)}$.

As before, we may straightforwardly extend this definition to 1- and 2-cells of $\mathbf{Cat}$.

Finally, we give a presentation of the pseudomonad $(\hat{S}, \hat{\eta}, \hat{\mu})$ on $\mathbf{Mod}$:

**Definition 24.** The homomorphism $\hat{S} : \mathbf{Mod} \to \mathbf{Mod}$ is given as follows:

- *On objects*: Given a small category $\mathbf{C}$, we take $\hat{S}\mathbf{C} = S\mathbf{C}$.
- *On maps*: Given a map $F : \mathbf{C} \nrightarrow \mathbf{D}$, the map $\hat{S}F : S\mathbf{C} \nrightarrow S\mathbf{D}$ is the following profunctor: an element of $\hat{S}F((n, \langle d_i \rangle); (m, \langle c_i \rangle))$ is given by

$$\big(\sigma, \langle g_i \rangle\big) : \big(n, \langle d_i \rangle\big) \nrightarrow \big(m, \langle c_i \rangle\big),$$

where $\sigma \in S1(n, m)$ and $g_i \in F(d_i; c_{\sigma(i)})$, whilst the action of maps $(\tau, \langle h_i \rangle) : (m, \langle c_i \rangle) \to (m', \langle c_i' \rangle)$ and $(\upsilon, \langle f_i \rangle) : (n', \langle d_i' \rangle) \to (n, \langle d_i \rangle)$ is given by

$$\big(\sigma, \langle g_i \rangle\big) \cdot \big(\upsilon, \langle f_i \rangle\big) = \big(\sigma \circ \upsilon, \langle g_{\upsilon(i)} \cdot f_i \rangle\big),$$
$$\big(\tau, \langle h_i \rangle\big) \cdot \big(\sigma, \langle g_i \rangle\big) = \big(\tau \circ \sigma, \langle h_{\sigma(i)} \cdot g_i \rangle\big).$$

- *On 2-cells*: Given a transformation $\alpha : F \Rightarrow G : \mathbf{C} \nrightarrow \mathbf{D}$, we give $S\alpha : SF \Rightarrow SG : S\mathbf{C} \nrightarrow S\mathbf{D}$ by

$$(S\alpha)\big(\sigma, \langle g_i \rangle\big) = \big(\sigma, \langle \alpha(g_i) \rangle\big).$$

Further, the pseudo-natural transformations

$$\hat{\eta} : \mathrm{id} \Rightarrow \hat{S} : \mathbf{Mod} \to \mathbf{Mod}$$

$$\text{and} \quad \hat{\mu} : \hat{S}^2 \Rightarrow \hat{S} : \mathbf{Mod} \to \mathbf{Mod}$$

have respective components

$$\hat{\eta}_X = (\eta_X)_* \quad \text{and} \quad \hat{\mu}_X = (\mu_X)_*.$$

### 4.2. Spans

We shall also need a few preliminaries about acyclic and connected graphs. We seek to capture their combinatorial essence in a categorical manner, allowing a smooth presentation of the somewhat involved proof which follows. The objects of our attention are spans in **FinCard**, i.e., diagrams $n \leftarrow k \to m$ in the category of finite cardinals and all maps. When we write 'span' in future, it should be read as 'span in **FinCard**' unless otherwise stated. We also make use without comment of the evident inclusions **FinOrd** → **FinCard** and $S1 \to$ **FinCard**. Now, each span $n \leftarrow k \to m$ determines a (categorist's) graph $k \rightrightarrows n + m$; if we forget the orientation of the edges of this graph, we get a (combinatorialist's) undirected multigraph. We say that a span $n \leftarrow k \to m$ is *acyclic* or *connected* if the associated multigraph is so. Note that the *acyclic* condition includes the assertion that there are no multiple edges.

**Proposition 25.** *Given a span* $n \xleftarrow{\theta_1} k \xrightarrow{\theta_2} m$, *the number of connected components of the graph induced by the span is given by the cardinality of* $r$ *in the pushout diagram*

$$
\begin{array}{ccc}
k & \xrightarrow{\theta_2} & m \\
{\scriptstyle \theta_1} \downarrow & & \downarrow {\scriptstyle \tau_2} \\
n & \xrightarrow{\tau_1} & r
\end{array}
$$

*in* **FinCard**.

**Proof.** Given the above pushout diagram, set $n_i = \tau_1^{-1}(i)$ and $m_i = \tau_2^{-1}(i)$ (for $i = 1, \ldots, r$). Now we observe that, for $i \neq j$, we have

$$\theta_1^{-1}(n_i) \cap \theta_2^{-1}(m_j) = \theta_1^{-1}(n_i) \cap \theta_1^{-1}(n_j) = \emptyset,$$

so that induced graph of the span has at least $r$ unconnected parts (with respective vertex sets $n_i + m_i$). On the other hand, if the induced graph $G$ had strictly more than $r$ connected components, we could find vertex sets $v_1, \ldots, v_{r+1}$ which partition $v(G)$, and for which

$$x \in v_i, \ y \in v_j \quad (\text{for } i \neq j) \quad \text{implies} \quad x \text{ is not adjacent to } y. \tag{\dagger}$$

But now define maps $\tau_1 : n \to r+1$ and $\tau_2 : m \to r+1$ by letting $\tau_i(x)$ be the $p$ for which $x \in v_p$. Then by condition (†), we have $\tau_1(\theta_1(a)) = \tau_2(\theta_2(a))$ for all $a \in k$, and so we have a commuting diagram

$$
\begin{array}{ccc}
k & \xrightarrow{\theta_2} & m \\
{\scriptstyle \theta_1}\downarrow & & \downarrow{\scriptstyle \tau_2} \\
n & \xrightarrow{\tau_1} & r+1
\end{array}
$$

for which the bottom right vertex does not factor through $r$, contradicting the assumption that $r$ was a pushout. Hence $G$ has precisely $r$ connected components. $\quad\square$

**Corollary 26.** *A span* $n \xleftarrow{\theta_1} k \xrightarrow{\theta_2} m$ *is connected if and only if the diagram*

$$
\begin{array}{ccc}
k & \xrightarrow{\theta_2} & m \\
{\scriptstyle \theta_1}\downarrow & & \downarrow \\
n & \longrightarrow & 1
\end{array}
$$

*is a pushout in* **FinCard**.

**Proposition 27.** *A span* $n \xleftarrow{\theta_1} k \xrightarrow{\theta_2} m$ *is acyclic if and only, for every monomorphism* $\iota : k' \hookrightarrow k$,

$$
\begin{array}{ccc}
k & \xrightarrow{\theta_2} & m \\
{\scriptstyle \theta_1}\downarrow & & \downarrow \\
n & \longrightarrow & r
\end{array}
\quad \text{a pushout implies} \quad
\begin{array}{ccc}
k' & \xrightarrow{\theta_2\iota} & m \\
{\scriptstyle \theta_1\iota}\downarrow & & \downarrow \\
n & \longrightarrow & r
\end{array}
\quad \text{not a pushout.}
$$

**Proof.** Suppose the left hand diagram is a pushout; then the associated graph $G$ of the span has $r$ connected components.

Suppose first that $G$ is acyclic, and $\iota : k' \hookrightarrow k$. Then the graph $G'$ associated to the span $n \xleftarrow{\theta_1\iota} k' \xrightarrow{\theta_2\iota} m$ has the same vertices as $G$ but strictly fewer edges; and since $G$ is acyclic, $G'$ must have strictly more than $r$ connected components, and hence $r$ cannot be a pushout for the right-hand diagram.

Conversely, if $G$ has a cycle, then we can remove some edge of $G$ without changing the number of connected components; and thus we obtain some monomorphism $\iota : k' \hookrightarrow k$ making the right-hand diagram a pushout. $\quad\square$

**Proposition 28.** *Suppose we have a commuting diagram*

$$
\begin{array}{ccc}
k & \xrightarrow{\ \theta_2\ } & m \\
\theta_1 \downarrow & & \downarrow \phi_2 \\
n & \xrightarrow[\ \phi_1\ ]{} & r.
\end{array}
\qquad (*)
$$

*Then the spans $m^{(i)} \leftarrow k^{(i)} \rightarrow n^{(i)}$ ( for $i = 1, \ldots, r$) induced by pulling back along elements $i : 1 \rightarrow r$ are all connected if and only if $(*)$ is a pushout.*

**Proof.** Suppose all the induced spans are connected; then each diagram

$$
\begin{array}{ccc}
k^{(i)} & \xrightarrow{\ \theta_2^{(i)}\ } & m^{(i)} \\
\theta_1^{(i)} \downarrow & & \downarrow \\
n^{(i)} & \xrightarrow{\phantom{xxxxx}} & 1
\end{array}
$$

is a pushout; hence the diagram

$$
\begin{array}{ccc}
\sum_i k^{(i)} & \xrightarrow{\ \sum_i \theta_2^{(i)}\ } & \sum_i m^{(i)} \\
\sum_i \theta_1^{(i)} \downarrow & & \downarrow \\
\sum_i n^{(i)} & \xrightarrow{\phantom{xxxxx}} & r
\end{array}
$$

is also a pushout, whence it follows that $(*)$ is itself a pushout.

Conversely, if $(*)$ is a pushout, then pulling this back along the map $i : 1 \rightarrow r$ yields another pushout in **FinCard**, so that each induced span is connected. $\quad \square$

**Proposition 29.** *Let $G$ be a graph with finite edge and vertex sets. Any two of the following conditions implies the third:*

- *$G$ is acyclic;*
- *$G$ is connected;*
- *$|v(G)| = |e(G)| + 1$.*

**Proof.**

- If $G$ is acyclic and connected, then it is a tree, and so $|v(G)| = |e(G)| + 1$;
- if $G$ is connected with $|v(G)| = |e(G)| + 1$, then it is minimally connected, hence a tree, and so acyclic;
- if $G$ is acyclic with $|v(G)| = |e(G)| + 1$, then it is maximally acyclic, hence a tree, and so connected. $\quad \square$

**Corollary 30.** *A span* $n \xleftarrow{\theta_1} k \xrightarrow{\theta_2} m$ *is acyclic and connected if and only if the diagram*

$$
\begin{array}{ccc}
k & \xrightarrow{\theta_2} & m \\
{\scriptstyle \theta_1} \downarrow & & \downarrow \\
n & \longrightarrow & 1
\end{array}
$$

*is a pushout in* **FinCard***, and* $n + m = k + 1$.

**Corollary 31.** *Let there be given a commuting diagram*

$$
\begin{array}{ccc}
k & \xrightarrow{\theta_2} & m \\
{\scriptstyle \theta_1} \downarrow & & \downarrow {\scriptstyle \phi_2} \\
n & \xrightarrow{\phi_1} & r;
\end{array}
\qquad (*)
$$

*then the induced spans* $m^{(i)} \leftarrow k^{(i)} \rightarrow n^{(i)}$ *(for* $i = 1, \ldots, r$*) are acyclic and connected if and only if* $(*)$ *is a pushout and* $m + n = k + r$.

*4.3. (PDD1)*

We are now ready to give our pseudo-distributive law at 1, and we begin with (PDD1), for which we must give a horizontal arrow

$$
\begin{array}{ccc}
S_c S_m 1 & \xrightarrow{\;\;\delta_1\;\;} & S_m S_c 1 \\
{\scriptstyle \mu_1} \downarrow & \Downarrow \tilde{\delta}_1 & \downarrow {\scriptstyle \mu_1} \\
S1 & \xrightarrow[S\mathbf{I}_1]{} & S1
\end{array}
$$

of $\mathbb{C}at/S\mathbf{I}_1$.

**Definition 32.** The *profunctor of suitable matchings*, $\delta_1 : \hat{S}_c \hat{S}_m 1 \longrightarrow \hat{S}_m \hat{S}_c 1$ is the following functor $(S^2 1)^{\mathrm{op}} \times S^2 1 \to$ **Set**:

- *On objects*: elements $f \in \delta_1(\phi; \psi)$ are bijections $f_n$ fitting into the diagram

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\;f_n\;} & n_\psi \\
{\scriptstyle \phi} \downarrow & & \downarrow {\scriptstyle \psi} \\
m_\phi & & m_\psi
\end{array}
$$

such that the span $m_\phi \xleftarrow{\phi} n_\phi \xrightarrow{\psi \circ f_n} m_\psi$ is acyclic and connected.

- *On maps*: Let $g : \psi \to \rho$ in $S^2 1$ and let $f \in \delta_1(\phi; \psi)$. Then we give $g \cdot f \in \delta_1(\phi; \rho)$ by

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{g_n \cdot f_n} & n_\rho \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \rho} \\
m_\phi & & m_\rho.
\end{array}
$$

This action is evidently functorial, but we still need to check that it really does yield an element of $\delta_1(\phi; \rho)$; that is, we need the associated span to be acyclic and connected. But this span is the top path of the diagram

$$
\begin{array}{c}
n_\phi \\
{\scriptstyle \phi}\swarrow \qquad \searrow{\scriptstyle f_n} \\
\qquad\qquad n_\psi \xrightarrow{g_n} n_\rho \\
\qquad\qquad {\scriptstyle \psi}\searrow \qquad \searrow{\scriptstyle \rho} \\
m_\phi \qquad\qquad\qquad m_\psi \xrightarrow{g_m} m_\rho;
\end{array}
$$
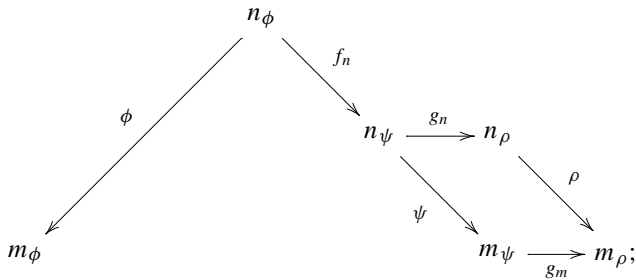
and therefore also the bottom path, since the right-hand square commutes. But since $g_m$ is an isomorphism, the graph induced by the span $m_\phi \xleftarrow{\phi} n_\phi \xrightarrow{\psi f_n} m_\psi$ is isomorphic to the graph induced by the span $m_\phi \xleftarrow{\phi} n_\phi \xrightarrow{g_m \psi f_n} m_\rho$, and hence the latter is acyclic and connected since the former is. So we have a well-defined left action of $S^2 1$ on $\delta_1$; and we proceed similarly to define an action on the right.

We now give the 2-cell $\tilde{\delta}_1$, for which we must give natural families of maps $\delta_1(\phi; \psi) \to S1(n_\phi, n_\psi)$. But this is straightforward: we simply send

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{f_n} & n_\psi \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \psi} \\
m_\phi & & m_\psi
\end{array}
$$

in $\delta_1(\phi; \psi)$ to $f_n$ in $S1(n_\phi; n_\psi)$. It is visibly the case that this satisfies the required naturality conditions.

Now, consider the transformation $\delta : \hat{S}_c \hat{S}_m \Rightarrow \hat{S}_m \hat{S}_c$ induced by this $(\delta_1, \tilde{\delta}_1)$. From Definition 32 and Proposition 20, we obtain that the component of $\delta$ at a discrete category $X$ is given by
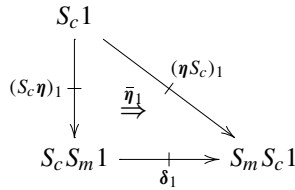
$$
\delta_X\big(\{\Sigma_m\}; \{\Gamma_n\}\big) = \big\{\sigma \;\big|\; \sigma \text{ is a suitable matching of } \{\Sigma_m\} \text{ with } \{\Gamma_n\}\big\}
$$

as desired.

*4.4. (PDD2)*

For (PDD2) we must produce the component of the invertible special modifications $\bar{\eta}$ and $\bar{\epsilon}$ at 1:

**Proposition 33.** *There is an invertible special cell*



*mediating the centre of this diagram in $\mathbb{C}oll(S)$ (where we omit the projections to $S\mathbf{I}$).*

**Proof.** With respect to the descriptions of $S1$ and $S^2 1$ given above, we observe that the functors $(S\eta)_1 : S1 \to S^2 1$ and $\eta_{S1} : S1 \to S^2 1$ are given by

$$S\eta_1 : n \mapsto (n \xrightarrow{\mathrm{id}} n), \qquad \eta_{S1} : n \mapsto (n \xrightarrow{!} 1),$$
$$f \mapsto (f, f), \qquad\qquad f \mapsto (f, !)$$

and hence $(\eta S_c)_1 : (S^2 1)^{\mathrm{op}} \times S1 \to \mathbf{Set}$ and $(S_c \eta)_1 : (S^2 1)^{\mathrm{op}} \times S1 \to \mathbf{Set}$ are given by

$$(\eta S_c)_1 (\phi; n) = (\eta_{S1})_*(\phi; n) = S^2 1\big(\phi, (n \xrightarrow{\mathrm{id}} n)\big),$$
$$(S_c \eta)_1 (\phi; n) = \hat{S}(\eta_1)_*(\phi; n) \cong (S\eta_1)_*(\phi; n) = S^2 1\big(\phi, (n \xrightarrow{!} 1)\big).$$

Thus the composite along the upper side of this diagram is given by

$$(\eta S_c)_1 (\phi; n) = S^2 1\big(\phi, (n \xrightarrow{!} 1)\big) \cong \begin{cases} S1(n_\phi, n) & \text{if } m_\phi = 1; \\ \emptyset & \text{otherwise,} \end{cases} \tag{1}$$

where the isomorphism is natural in $\phi$ and $n$; and with respect to this isomorphism, the projection down to $S\mathbf{I}$ is given simply by the inclusion

$$(\eta S_c)_1 (\phi; n) \hookrightarrow S1(n_\phi, n).$$

Now, the lower side is given by

$$\big(\delta_1 \otimes (S_c \eta)_1\big)(\phi; n) = \int^{\psi \in S^2 1} S^2 1\big(\psi, (n \xrightarrow{\mathrm{id}} n)\big) \times \delta_1(\phi; \psi),$$

which is isomorphic to $\delta_1(\phi; (n \xrightarrow{\text{id}} n))$, naturally in $\phi$ and $n$. Now, any element $f$ of $\delta_1(\phi; (n \xrightarrow{\text{id}} n))$, given by

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\ f_n\ } & n \\
\phi \downarrow & & \downarrow \text{id} \\
m_\phi & & n
\end{array}
$$

say, must satisfy $m_\phi + n = n_\phi + 1$; but since $n = n_\phi$, this can only happen if $m_\phi = 1$; and in this case, the diagram

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\ f_n\ } & n \\
\phi \downarrow & & \downarrow ! \\
m_\phi & \xrightarrow{\ !\ } & 1
\end{array}
$$

is necessarily a pushout. Hence

$$
\bigl(\delta_1 \otimes (S_c \boldsymbol{\eta})_1\bigr)(\phi; n) \cong
\begin{cases}
S1(n_\phi, n) & \text{if } m_\phi = 1; \\
\varnothing & \text{otherwise,}
\end{cases}
\tag{2}
$$

naturally in $\phi$ and $n$; and once again, the projection down to $S\mathbf{I}$ is given simply by inclusion. So, composing the isomorphisms (1) and (2), we get a special invertible cell $\bar{\boldsymbol{\eta}}_1$ which is compatible with the projections down to $S\mathbf{I}$, as required. $\quad\square$

**Proposition 34.** *There is an invertible special cell*

$$
\begin{array}{ccc}
S_c S_m 1 & \xrightarrow{\ \ \delta_1\ \ } & S_m S_c 1 \\
(\epsilon S_m)_1 \downarrow & \overset{\bar{\epsilon}_1}{\Longrightarrow} & \diagup (S_m \epsilon)_1 \\
S_m 1 & &
\end{array}
$$

*mediating the centre of this diagram in $\mathbb{C}oll(S)$ (where we omit the projections to $S\mathbf{I}$).*

**Proof.** Dual to the above. $\quad\square$

*4.5. (PDD3)*

For (PDD3) we must produce the component of the invertible special modifications $\bar{\mu}$ and $\bar{\Delta}$ at 1:

**Proposition 35.** *There is an invertible special cell*

$$
\begin{array}{ccc}
S_c S_m 1 & \xrightarrow{\ \delta_1\ } & S_m S_c 1 \\
{\scriptstyle (\boldsymbol{\Delta} S_m)_1}\big\downarrow & \quad \Downarrow \overline{\boldsymbol{\Delta}}_1 & \big\downarrow {\scriptstyle (S_m \boldsymbol{\Delta})_1} \\
S_c S_c S_m 1 & \xrightarrow[(S_c \boldsymbol{\delta})_1]{}\ S_c S_m S_c 1 \ \xrightarrow[(\boldsymbol{\delta} S_c)_1]{} & S_m S_c S_c 1
\end{array}
$$

*mediating the centre of this diagram in $\mathbb{C}\mathrm{oll}(S)$ (where we omit the projections to $S\mathbf{I}$).*

**Proof.** Let us describe explicitly the horizontal arrows involved in the above diagram. The functors $\mu_{S1} : S^3 1 \to S^2 1$ and $S\mu_1 : S^3 1 \to S^2 1$ in **Cat** are given by

$$
\mu_{S1} : (n_\phi \xrightarrow{\phi_1} m_\phi \xrightarrow{\phi_2} r_\phi) \mapsto (n_\phi \xrightarrow{\phi_1} m_\phi),
$$
$$
(f_n, f_m, f_r) \mapsto (f_n, f_m),
$$
$$
\text{and} \quad S\mu_1 : (n_\phi \xrightarrow{\phi_1} m_\phi \xrightarrow{\phi_2} r_\phi) \mapsto (n_\phi \xrightarrow{\phi_2 \phi_1} r_\phi),
$$
$$
(f_n, f_m, f_r) \mapsto (f_n, f_r)
$$

and hence $(\boldsymbol{\Delta} S_m)_1 : (S^3 1)^{\mathrm{op}} \times S^2 1 \to \mathbf{Set}$ and $(S_m \boldsymbol{\Delta})_1 : (S^3 1)^{\mathrm{op}} \times S^2 1 \to \mathbf{Set}$ are given by

$$
(\boldsymbol{\Delta} S_m)_1(\phi; \psi) = (\mu_{S1})^*(\phi; \psi) = S^2 1\big((n_\phi \xrightarrow{\phi_1} m_\phi), \psi\big),
$$
$$
(S_m \boldsymbol{\Delta})_1(\phi; \psi) = \hat{S}(\mu_1)^*(\phi; \psi) \cong (S\mu_1)^*(\phi; \psi) = S^2 1\big((n_\phi \xrightarrow{\phi_2 \phi_1} r_\phi), \psi\big).
$$

We now wish to describe $(\boldsymbol{\delta} S_c)_1$ and $(S_c \boldsymbol{\delta})_1$. It is a straightforward calculation to see that $(\boldsymbol{\delta} S_c)_1 : (S^3 1)^{\mathrm{op}} \times S^3 1 \to \mathbf{Set}$ is given as follows:

- *On objects*: elements $f \in (\boldsymbol{\delta} S_c)_1(\phi; \psi)$ are pairs of bijections $f_n$ and $f_m$ fitting in the diagram

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\ f_n\ } & n_\psi \\
{\scriptstyle \phi_1}\big\downarrow & & \big\downarrow {\scriptstyle \psi_1} \\
m_\phi & \xrightarrow{\ f_m\ } & m_\psi \\
{\scriptstyle \phi_2}\big\downarrow & & \big\downarrow {\scriptstyle \psi_2} \\
r_\phi & & r_\psi
\end{array}
$$

such that the span $r_\phi \xleftarrow{\phi_2} m_\phi \xrightarrow{\psi_2 \circ f_m} r_\psi$ is acyclic and connected.

- *On maps*: Let $g : \psi \to \rho$ in $S^3 1$ and let $f \in (\delta S_c)_1(\phi; \psi)$. Then we give an element $g \cdot f \in (\delta S_c)_1(\phi; \rho)$ by

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\; g_n \circ f_n \;} & n_\rho \\
\phi_1 \downarrow & & \downarrow \rho_1 \\
m_\phi & \xrightarrow[\; g_m \circ f_m \;]{} & m_\rho \\
\phi_2 \downarrow & & \downarrow \rho_2 \\
r_\phi & & r_\rho ;
\end{array}
$$

and we give the right action of $S^3 1$ similarly.

Likewise, it is easy to calculate that $(S_c \delta)_1 : (S^3 1)^{\mathrm{op}} \times S^3 1 \to \mathbf{Set}$ is given by:

- *On objects*: elements $f \in (S_c \delta)_1(\phi; \psi)$ are pairs of bijections $f_n : n_\phi \to n_\psi$ and $f_r : r_\phi \to r_\psi$ fitting in the diagram

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\; f_n \;} & n_\psi \\
\phi_1 \downarrow & & \downarrow \psi_1 \\
m_\phi & & m_\psi \\
\phi_2 \downarrow & & \downarrow \psi_2 \\
r_\phi & \xrightarrow[\; f_r \;]{} & r_\psi
\end{array}
$$

such that for each $i = 1, \ldots, r_\psi$, the induced spans

$$
\begin{array}{ccc}
 & n_\phi^{(i)} & \\
\phi_1^{(i)} \swarrow & & \searrow f_n^{(i)} \\
 & & n_\psi^{(i)} \\
 & & \searrow \psi_1^{(i)} \\
m_\phi^{(i)} & & m_\psi^{(i)}
\end{array}
$$

are acyclic and connected.

[Let us clarify what the induced spans referred to above actually are. We have the commuting diagram

$$
\begin{array}{ccc}
n_\phi \xrightarrow{\;f_n\;} n_\psi \xrightarrow{\;\psi_1\;} m_\psi & & \\
\phi_1 \downarrow \qquad\qquad\qquad \downarrow \psi_2 & & (*) \\
m_\phi \xrightarrow[\;\phi_2\;]{} r_\phi \xrightarrow[\;f_r\;]{} r_\psi & &
\end{array}
$$

and the induced spans are the result of pulling this diagram back along elements $i : 1 \to r_\psi$. By the results of the first section of this chapter, these spans are all acyclic and connected if and only if $(*)$ is a pushout and $r_\psi + n_\phi = m_\phi + m_\psi$.]

- *On maps*: Let $g : \psi \to \rho$ in $S^3 1$ and let $f \in (S_c \boldsymbol{\delta})_1(\phi; \psi)$. Then we give an element $g \cdot f \in (S_c \boldsymbol{\delta})_1(\phi; \rho)$ by

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\;g_n \circ f_n\;} & n_\rho \\
\phi_1 \downarrow & & \downarrow \rho_1 \\
m_\phi & & m_\rho \\
\phi_2 \downarrow & & \downarrow \rho_2 \\
r_\phi & \xrightarrow[\;g_r \circ f_r\;]{} & r_\rho ;
\end{array}
$$

and we give the right action similarly.

Now, returning to the diagram in question, the upper side is given by

$$
\big((S_m \boldsymbol{\Delta})_1 \otimes \boldsymbol{\delta}_1\big)(\phi; \rho) = \int^{\psi \in S^2 1} \boldsymbol{\delta}_1(\psi; \rho) \times S^2 1\big((n_\phi \xrightarrow{\phi_2 \phi_1} r_\phi), \psi\big),
$$

which is isomorphic to $\boldsymbol{\delta}_1((n_\phi \xrightarrow{\phi_2 \phi_1} r_\phi); \rho)$, naturally in $\phi$ and $\rho$. With respect to this isomorphism, the projection onto $S\mathbf{I}$ has component morphisms $\boldsymbol{\delta}_1((n_\phi \xrightarrow{\phi_2 \phi_1} r_\phi); \rho) \to S1(n_\phi; n_\rho)$ which send

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\;f_n\;} & n_\rho \\
\phi_2 \phi_1 \downarrow & & \downarrow \rho \\
r_\phi & & m_\rho
\end{array}
$$

to $f_n$. The lower side of this diagram, which we denote by $K$, is given by

$$
K(\phi; \rho) = \big((\boldsymbol{\delta} S_c)_1 \otimes (S_c \boldsymbol{\delta})_1 \otimes (\boldsymbol{\Delta} S_m)_1\big)(\phi; \rho)
$$

$$
= \int^{\psi, \xi \in S^3 1} S^2 1\big((n_\xi \xrightarrow{\xi_1} m_\xi), \rho\big) \times (S_c \boldsymbol{\delta})_1(\psi; \xi) \times (\boldsymbol{\delta} S_c)_1(\phi; \psi).
$$

We may represent a typical element $x \in K(\phi; \rho)$ as $x = f \otimes g \otimes h$, where $f \in (\delta S_c)_1(\phi; \psi)$, $g \in (S_c \delta)_1(\psi; \xi)$, and $h \in S^2 1((n_\xi \xrightarrow{\xi_1} m_\xi), \rho)$:

$$
\begin{array}{ccccccc}
n_\phi & \xrightarrow{f_n} & n_\psi & \xrightarrow{g_n} & n_\xi & \xrightarrow{h_n} & n_\rho \\
\phi_1 \downarrow & & \psi_1 \downarrow & & \xi_1 \downarrow & & \downarrow \rho \\
m_\phi & \xrightarrow[f_m]{} & m_\psi & & m_\xi & \xrightarrow[h_m]{} & m_\rho \\
\phi_2 \downarrow & & \psi_2 \downarrow & & \xi_2 \downarrow & & \\
r_\phi & & r_\psi & \xrightarrow[g_r]{} & r_\xi. & &
\end{array}
$$

Then the projection onto $S\mathbf{I}$ has components

$$K(\phi; \rho) \to S1(n_\phi, n_\rho),$$
$$f \otimes g \otimes h \mapsto h_n \circ g_n \circ f_n.$$

So, we need to set up an isomorphism between $K(\phi; \rho)$ and $\delta_1((n_\phi \xrightarrow{\phi_2\phi_1} r_\phi); \rho)$ which is natural in $\phi$ and $\rho$ and compatible with the projections onto $S\mathbf{I}$. In one direction, we send the element $x \in K(\phi; \rho)$:

$$
\begin{array}{ccccccc}
n_\phi & \xrightarrow{f_n} & n_\psi & \xrightarrow{g_n} & n_\xi & \xrightarrow{h_n} & n_\rho \\
\phi_1 \downarrow & & \psi_1 \downarrow & & \xi_1 \downarrow & & \downarrow \rho \\
m_\phi & \xrightarrow[f_m]{} & m_\psi & & m_\xi & \xrightarrow[h_m]{} & m_\rho \\
\phi_2 \downarrow & & \psi_2 \downarrow & & \xi_2 \downarrow & & \\
r_\phi & & r_\psi & \xrightarrow[g_r]{} & r_\xi & &
\end{array}
$$

to the element $\hat{x}$ of $\delta_1((n_\phi \xrightarrow{\phi_2\phi_1} r_\phi); \rho)$ given by

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{h_n g_n f_n} & n_\rho \\
\phi_2\phi_1 \downarrow & & \downarrow \rho \\
r_\phi & & m_\rho.
\end{array}
$$

Note that this element is independent of the representation of $x$ that we chose, that this assignation is natural in $\phi$ and $\rho$, and is compatible with the projection down to $S\mathbf{I}$; but for it to be well-defined, we need still to check that the span $r_\phi \xleftarrow{\phi_2\phi_1} n_\phi \xrightarrow{\rho h_n g_n f_n} m_\rho$ is acyclic and connected. For this, we observe first that in the following diagram

$$
\begin{array}{ccccccccc}
n_\phi & \xrightarrow{f_n} & n_\psi & \xrightarrow{g_n} & n_\xi & \xrightarrow{\xi_1} & m_\xi & \xrightarrow{h_n} & m_\rho \\
\downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\psi_1} & & & & \downarrow{\scriptstyle\xi_2} & & \downarrow \\
m_\phi & \xrightarrow{f_m} & m_\psi & \xrightarrow{\psi_2} & r_\psi & \xrightarrow{g_r} & r_\xi & & \downarrow \\
\downarrow{\scriptstyle\phi_2} & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
r_\phi & \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & 1 & &
\end{array}
$$

each of the smaller squares is a pushout; and hence the outer square is also a pushout. But the top edge is $h_n \xi_1 g_n f_n = \rho h_n g_n f_n$, so that the square

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\rho h_n g_n f_n} & n_\rho \\
\downarrow{\scriptstyle\phi_2\phi_1} & & \downarrow \\
r_\psi & \longrightarrow & 1
\end{array}
$$

is a pushout as required. Furthermore, the following equalities hold:

$$
\begin{aligned}
r_\phi + r_\psi &= m_\phi + 1, & & & m_\psi + m_\xi &= n_\psi + r_\xi, \\
m_\psi &= m_\phi, & & \text{and} & m_\rho &= m_\xi, \\
r_\psi &= r_\xi, & & & n_\psi &= n_\phi
\end{aligned}
$$

whence we have $m_\rho + r_\phi = n_\phi + 1$. So the span $r_\phi \xleftarrow{\phi_2\phi_1} n_\phi \xrightarrow{\rho h_n g_n f_n} m_\rho$ is acyclic and connected as required.

Conversely, suppose we are given an element $k$ of $\delta_1((n_\phi \xrightarrow{\phi_2\phi_1} r_\phi); \rho)$:

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{k_n} & n_\rho \\
\downarrow{\scriptstyle\phi_2\phi_1} & & \downarrow{\scriptstyle\rho} \\
r_\phi & & m_\rho;
\end{array}
$$

then we take the following pushout:

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\rho k_n} & m_\rho \\
\downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle i_2} \\
m_\phi & \xrightarrow{i_1} & r.
\end{array}
$$

Now, the map $i_1$ in this pushout square need not be order-preserving; but it has a (non-unique) factorisation as $m_\phi \xrightarrow{\alpha_1} r_1 \xrightarrow{\sigma_1} r$, where $\alpha_1$ is order-preserving and $\sigma_1$ a bijection. Similarly, we

can factorise $i_2$ as $m_\rho \xrightarrow{\alpha_2} r_2 \xrightarrow{\sigma_2} r$ with $\alpha_2$ is order-preserving and $\sigma_2$ a bijection. [Note that it follows that each of the diagrams

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\rho k_n} & m_\rho \\
{\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \sigma_1^{-1} i_2} \\
m_\phi & \xrightarrow[\alpha_1]{} & r_1
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
n_\phi & \xrightarrow{\rho k_n} & m_\rho \\
{\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\
m_\phi & \xrightarrow[\sigma_2^{-1} i_1]{} & r_2
\end{array}
$$

is also a pushout.] Now we send $k$ to the element $\hat{k}$ of $K(\phi; \rho)$ represented by the following:

$$
\begin{array}{ccccccc}
n_\phi & \xrightarrow{\text{id}} & n_\phi & \xrightarrow{k_n} & n_\rho & \xrightarrow{\text{id}} & n_\rho \\
{\scriptstyle \phi_1}\downarrow & & {\scriptstyle \phi_1}\downarrow & & {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\
m_\phi & \xrightarrow[\text{id}]{} & m_\phi & & m_\rho & \xrightarrow[\text{id}]{} & m_\rho \\
{\scriptstyle \phi_2}\downarrow & & {\scriptstyle \alpha_1}\downarrow & & {\scriptstyle \alpha_2}\downarrow & & \\
r_\phi & & r_1 & \xrightarrow[\sigma_2^{-1}\sigma_1]{} & r_2. & &
\end{array}
$$

This is visibly compatible with the projection down onto $S\mathbf{I}$, but we need to check that it is in fact a valid element of $K(\phi; \rho)$. Clearly all squares commute in the diagram above, so we need only check the acyclic and connected conditions. We start with connectedness; for the middle map, the diagram

$$
\begin{array}{ccccc}
n_\phi & \xrightarrow{k_n} & n_\rho & \xrightarrow{\rho} & m_\rho \\
{\scriptstyle \phi_1}\downarrow & & & & \downarrow{\scriptstyle \alpha_2} \\
m_\phi & \xrightarrow[\alpha_1]{} & r_1 & \xrightarrow[\sigma_2^{-1}\sigma_1]{} & r_2
\end{array}
\quad = \quad
\begin{array}{ccc}
n_\phi & \xrightarrow{\rho k_n} & m_\rho \\
{\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\
m_\phi & \xrightarrow[\sigma_2^{-1} i_1]{} & r_2
\end{array}
$$

is indeed a pushout, so the induced spans for the middle map are connected. For the left-hand map, consider the diagram

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\rho k_n} & m_\rho \\
{\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \sigma_1^{-1} i_2} \\
m_\phi & \xrightarrow[\alpha_1]{} & r_1 \\
{\scriptstyle \phi_2}\downarrow & & \downarrow \\
r_\phi & \longrightarrow & 1;
\end{array}
$$

the outer square and the upper square are both pushouts, and hence so is the lower square; so the left-hand span is connected.

And now acyclicity. For the middle map, we need that, given any monomorphism $\iota : n'_\phi \hookrightarrow n_\phi$, the diagram

$$
\begin{array}{ccc}
n'_\phi & \xrightarrow{\ \rho k_n \iota\ } & m_\rho \\
{\scriptstyle \phi_1 \iota} \downarrow & & \downarrow {\scriptstyle \alpha_2} \\
m_\phi & \xrightarrow[\ \sigma_2^{-1} i_1\ ]{} & r_2
\end{array}
$$

is no longer a pushout. But suppose it were; then in the diagram

$$
\begin{array}{ccc}
n'_\phi & \xrightarrow{\ \rho k_n \iota\ } & m_\rho \\
{\scriptstyle \phi_1 \iota} \downarrow & & \downarrow {\scriptstyle \sigma_1^{-1} i_2} \\
m_\phi & \xrightarrow[\ \alpha_1\ ]{} & r_1 \\
{\scriptstyle \phi_2} \downarrow & & \downarrow \\
r_\phi & \longrightarrow & 1
\end{array}
$$

the upper and lower squares would be pushouts, hence making the outer edge a pushout; but this contradicts the acyclicity of the span $r_\phi \leftarrow n_\phi \rightarrow m_\rho$. So the induced spans for the middle map are acyclic. Thus we now know that the following equations hold:

$$
m_\phi + m_\rho = n_\phi + r_2,
$$

$$
r_\phi + m_\rho = n_\phi + 1,
$$

$$
r_1 = r_2,
$$

and so can deduce that $r_1 + r_\phi = m_\phi + 1$, as required for the left-hand span to be acyclic.

It remains to check that these two assignations are mutually inverse. It is evident, given $k \in d_1((n_\phi \xrightarrow{\phi_2 \phi_1} r_\phi); \rho)$, that $\hat{\hat{k}} = k$. For the other direction, we send

$$
x = \begin{array}{ccccccc}
n_\phi & \xrightarrow{f_n} & n_\psi & \xrightarrow{g_n} & n_\xi & \xrightarrow{h_n} & n_\rho \\
{\scriptstyle \phi_1}\downarrow & & {\scriptstyle \psi_1}\downarrow & & {\scriptstyle \xi_1}\downarrow & & \downarrow{\scriptstyle \rho} \\
m_\phi & \xrightarrow[f_m]{} & m_\psi & & m_\xi & \xrightarrow[h_m]{} & m_\rho \\
{\scriptstyle \phi_2}\downarrow & & {\scriptstyle \psi_2}\downarrow & & {\scriptstyle \xi_2}\downarrow & & \\
r_\phi & & r_\psi & \xrightarrow[g_r]{} & r_\xi & &
\end{array}
\quad \text{to} \quad
\hat{x} = \begin{array}{ccccccc}
n_\phi & \xrightarrow{\text{id}} & n_\phi & \xrightarrow{k_n} & n_\rho & \xrightarrow{\text{id}} & n_\rho \\
{\scriptstyle \phi_1}\downarrow & & {\scriptstyle \phi_1}\downarrow & & {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\
m_\phi & \xrightarrow[\text{id}]{} & m_\phi & & m_\rho & \xrightarrow[\text{id}]{} & m_\rho \\
{\scriptstyle \phi_2}\downarrow & & {\scriptstyle \alpha_1}\downarrow & & {\scriptstyle \alpha_2}\downarrow & & \\
r_\phi & & r_1 & \xrightarrow[\sigma_2^{-1}\sigma_1]{} & r_2. & &
\end{array}
$$

We claim that these two diagrams represent the same element of $K(\phi; \rho)$. Indeed, note that in the diagram

$$
\begin{array}{ccccccccc}
n_\phi & \xrightarrow{f_n} & n_\psi & \xrightarrow{g_n} & n_\xi & \xrightarrow{\xi_1} & m_\xi & \xrightarrow{h_m} & m_\rho \\
\downarrow{\phi_1} & & \downarrow{\psi_1} & & & & \downarrow{\xi_2} & & \downarrow{g_r^{-1}\xi_2 h_m^{-1}} \\
m_\phi & \xrightarrow{f_m} & m_\psi & \xrightarrow{\psi_2} & r_\psi & \xrightarrow{g_r} & r_\xi & \xrightarrow{g_r^{-1}} & r_\psi
\end{array}
$$

each of the smaller squares is a pushout, and hence the outer edge is. But the upper edge is $h_m \xi_1 g_n f_n = \rho h_n g_n f_n = \rho k_n$, so that the diagram

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\rho k_n} & m_\rho \\
\downarrow{\phi_1} & & \downarrow{g_r^{-1}\xi_2 h_m^{-1}} \\
m_\phi & \xrightarrow{\psi_2 f_m} & r_\psi
\end{array}
$$

is a pushout. Since $r_1$ is also a pushout for this diagram, it follows that there is an isomorphism $\beta_1 : r_1 \to r_\psi$ such that $\beta_1 \alpha_1 = \psi_2 f_m$; hence the following diagram commutes:

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{f_n} & n_\psi \\
\downarrow{\phi_1} & & \downarrow{\psi_1} \\
m_\phi & \xrightarrow{f_m} & m_\psi \\
\downarrow{\alpha_1} & & \downarrow{\psi_2} \\
r_1 & \xrightarrow{\beta_1} & r_\psi.
\end{array}
$$

Similarly, we see that

$$
\begin{array}{ccc}
n_\phi & \xrightarrow{\rho k_n} & m_\rho \\
\downarrow{\phi_1} & & \downarrow{\xi_2 h_m^{-1}} \\
m_\phi & \xrightarrow{g_r \psi_2 f_m} & r_\xi
\end{array}
$$

is a pushout, and so there is an isomorphism $\beta_2 : r_\xi \to r_2$ such that $\beta_2 \xi_2 h_m^{-1} = \alpha_2$, i.e., $\beta_2 \xi_2 = \alpha_2 h_m$. Hence the following diagram commutes:

$$\begin{array}{ccc}
n_\xi & \xrightarrow{h_n} & n_\rho \\
\xi_1 \downarrow & & \downarrow \rho \\
m_\xi & \xrightarrow{h_m} & m_\rho \\
\xi_2 \downarrow & & \downarrow \alpha_2 \\
r_\xi & \xrightarrow{\beta_2} & r_2.
\end{array}$$

Furthermore, we have $r_1 \xrightarrow{\beta_1} r_\psi \xrightarrow{g_r} r_\xi \xrightarrow{\beta_2} r_2 = r_1 \xrightarrow{\sigma_1} r \xrightarrow{\sigma_2^{-1}} r_2$, since each of these objects is a pushout of the same span, and the isomorphisms between them are isomorphisms of pushouts. Thus, using an evident notation for the internal actions, we have

$$x = \begin{array}{ccccccc}
n_\phi & \xrightarrow{f_n} & n_\psi & \xrightarrow{g_n} & n_\xi & \xrightarrow{h_n} & n_\rho \\
\phi_1\downarrow & & \psi_1\downarrow & & \xi_1\downarrow & & \downarrow\rho \\
m_\phi & \xrightarrow{f_m} & m_\psi & & m_\xi & \xrightarrow{h_m} & m_\rho \\
\phi_2\downarrow & & \psi_2\downarrow & & \xi_2\downarrow & & \\
r_\phi & & r_\psi & \xrightarrow{g_r} & r_\xi & &
\end{array}
\equiv
\begin{array}{ccccccccc}
n_\phi & \xrightarrow{id} & n_\phi & \xrightarrow{f_n} & n_\psi & \xrightarrow{g_n} & n_\xi & \xrightarrow{h_n} & n_\rho & \xrightarrow{id} & n_\rho \\
\phi_1\downarrow & & \phi_1\downarrow & & \psi_1\downarrow & & \xi_1\downarrow & & \rho\downarrow & & \downarrow\rho \\
m_\phi & \xrightarrow{id} & m_\phi & \xrightarrow{f_m} & m_\psi & & m_\xi & \xrightarrow{h_m} & m_\rho & \xrightarrow{id} & m_\rho \\
\phi_2\downarrow & & \alpha_1\downarrow & & \psi_2\downarrow & & \xi_2\downarrow & & \alpha_2\downarrow & & \\
r_\phi & & r_1 & \xrightarrow{\beta_1} & r_\psi & \xrightarrow{g_r} & r_\xi & \xrightarrow{\beta_2} & r_2 & &
\end{array}$$

$$\equiv
\begin{array}{ccccccc}
n_\phi & \xrightarrow{id} & n_\phi & \xrightarrow{k_n} & n_\rho & \xrightarrow{id} & n_\rho \\
\phi_1\downarrow & & \phi_1\downarrow & & \rho\downarrow & & \downarrow\rho \\
m_\phi & \xrightarrow{id} & m_\phi & & m_\rho & \xrightarrow{id} & m_\rho \\
\phi_2\downarrow & & \alpha_1\downarrow & & \alpha_2\downarrow & & \\
r_\phi & & r_1 & \xrightarrow{\sigma_2^{-1}\sigma_1} & r_2 & &
\end{array} = \hat{\hat{x}}.$$

So the assignments $x \mapsto \hat{x}$ and $k \mapsto \hat{k}$ are mutually inverse as required. It now follows that the assignment $\delta_1((n_\phi \xrightarrow{\phi_2\phi_1} r_\phi); \rho) \to K(\phi; \rho)$ is natural in $\phi$ and $\rho$, since its inverse is. $\quad\square$

**Proposition 36.** *There is an invertible special cell*

$$\begin{array}{ccc}
S_c S_m S_m 1 & \xrightarrow{(\delta S_m)_1} S_m S_c S_m 1 \xrightarrow{(S_m \delta)_1} & S_c S_c S_m 1 \\
(S_c \mu)_1 \downarrow & \quad\Downarrow\bar{\mu}_1\quad & \downarrow(\mu S_c)_1 \\
S_c S_m 1 & \xrightarrow[\delta_1]{} & S_m S_c 1
\end{array}$$

*mediating the centre of this diagram in $\mathbb{C}oll(S)$ (where we omit the projections to $S\mathbf{I}$).*

**Proof.** Dual to the above. $\quad\square$

### 4.6. (PDA1)–(PDA10)

It remains only to show that the data produced above satisfies the ten coherence axioms (PDA1)–(PDA10). At first this may appear somewhat forbidding, but our job is made rather simple by the following argument.

**Definition 37.** We say that a cell

$$
\begin{array}{ccc}
X_s & \xrightarrow{\;\mathbf{X}\;} & X_t \\
f_s \downarrow & \Downarrow \mathbf{f} & \downarrow f_t \\
Y_s & \xrightarrow[\;\mathbf{Y}\;]{} & Y_t
\end{array}
$$

of $\mathbb{C}at$ is *locally monomorphic* if it is a monomorphism when viewed as a map of $[X_t^{\mathrm{op}} \times X_s, \mathbf{Set}]$:

$$
\begin{array}{ccc}
X_t^{\mathrm{op}} \times X_s & \xrightarrow{\; f_t^{\mathrm{op}} \times f_s \;} & Y_t^{\mathrm{op}} \times Y_s \\
& {}_{X}\searrow \quad \overset{f}{\Rightarrow} \quad \swarrow_{Y} & \\
& \mathbf{Set}. &
\end{array}
$$

Now, local monomorphisms admit a limited form of 'left cancellation.' Indeed, suppose we are given objects $\mathbf{X} = X : X_s \nrightarrow X_t$ and $\mathbf{X}' = X' : X_s \nrightarrow X_t$ of $\mathbb{C}at_1$, and *special* maps $\mathbf{g}_1$ and $\mathbf{g}_2 : \mathbf{X}' \to \mathbf{X}$; then given a local monomorphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, we have that

$$
\mathbf{f} \circ \mathbf{g}_1 = \mathbf{f} \circ \mathbf{g}_2 \quad \text{implies} \quad \mathbf{g}_1 = \mathbf{g}_2,
$$

since to give a special map $g_i : \mathbf{X}' \to \mathbf{X}$ is equivalently to give a natural transformation $g_i : X' \Rightarrow X$; therefore the result follows from the fact that $f : X \Rightarrow (Y \circ f_t^{\mathrm{op}} \times f_s)$ is a monomorphism in $[X_t^{\mathrm{op}} \times X_s, \mathbf{Set}]$.

Observe also that, given a special isomorphism $\mathbf{g} : \mathbf{X}' \to \mathbf{X}$ and a local monomorphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, the map $\mathbf{f} \circ \mathbf{g}$ is again a local monomorphism.

**Proposition 38.** *Consider each of the pasting diagrams in the axioms* (PDA1)–(PDA10) *as a diagram in* $\mathbb{C}at/S\mathbf{I}_1$. *Then the projection map from each 'source' and 'target' face down onto* $S\mathbf{I}_1$ *is a local monomorphism.*

**Proof.** Observe that every special cell in the pasting diagrams for (PDA1)–(PDA10) is invertible, and therefore, for each pasting diagram it suffices to show for *any one* path through it that the projection onto $S\mathbf{I}_1$ is a local monomorphism; it then follows, by the discussion preceding this proposition, that the same is true for all other paths. We now work our way through the ten axioms:

- (PDA1): Let us write $K$ for the composite $S_c 1 \xrightarrow{\epsilon_1} \mathrm{id} \xrightarrow{\eta_1} S_m 1$; then we have

$$K(m; n) = \begin{cases} \{*\} & \text{if } m = n = 1; \\ \varnothing & \text{otherwise} \end{cases}$$

  and the projection down onto $S\mathbf{I}_1$ simply sends the unique element of $K(1; 1)$ to the unique element of $S1(1; 1)$, and thus is a local monomorphism as required.

- (PDA2)–(PDA5): For each of these we look at the path $\delta_1 : S_c S_m 1 \to S_m S_c 1$, and from the definitions, the projection onto $S\mathbf{I}_1$ is visibly a local monomorphism.

- (PDA6): Let us write $K$ for the composite

$$S_c S_m S_m S_m 1 \xrightarrow{(S_c S_m \mu)_1} S_c S_m S_m 1 \xrightarrow{(S_c \mu)_1} S_c S_m 1 \xrightarrow{\delta_1} S_m S_c 1.$$

  Then we have an isomorphism

$$K(\phi; \psi) \cong \delta_1\big(\phi; (n_\psi \xrightarrow{\psi_3 \psi_2 \psi_1} s_\psi)\big)$$

  natural in $\phi$ and $\psi$, where we are writing a typical element of $S_c S_m S_m S_m 1$ as $\psi = n_\psi \xrightarrow{\psi_1} m_\psi \xrightarrow{\psi_2} r_\psi \xrightarrow{\psi_3} s_\psi$ in the evident way. With respect to this isomorphism, the projection down onto $S\mathbf{I}_1$ is given simply by the value of $\tilde{\delta}_1$ there, which is a monomorphism as required.

- (PDA7): Dual to (PDA6).

- (PDA8): Let us write $K$ for the composite

$$S_c S_m S_m 1 \xrightarrow{(S_c \mu)_1} S_c S_m 1 \xrightarrow{\delta_1} S_m S_c 1 \xrightarrow{(S_m \epsilon)_1} S_m 1;$$

  then we have

$$K(m; \phi) \cong \delta_1\big((m \xrightarrow{\mathrm{id}} m); (n_\phi \xrightarrow{\phi_2 \phi_1} r_\phi)\big)$$

  and again the projection down onto $S\mathbf{I}_1$ is simply given by the value of $\tilde{\delta}_1$ there; and so a local monomorphism.

- (PDA9): Dual to (PDA8).

- (PDA10): Let us write $K$ for the composite

$$S_c S_m S_m 1 \xrightarrow{(S_c \mu)_1} S_c S_m 1 \xrightarrow{\delta_1} S_c S_m 1 \xrightarrow{(S_m \Delta)_1} S_m S_c S_c 1;$$

  then we have

$$K(\psi; \phi) \cong \delta_1\big((n_\psi \xrightarrow{\psi_2 \psi_1} r_\psi); (n_\phi \xrightarrow{\phi_2 \phi_1} r_\phi)\big).$$

  Once more, the projection down onto $S\mathbf{I}_1$ is just the value of $\tilde{\delta}_1$ there, and so a local monomorphism. $\square$

**Corollary 39.** *The pasting equalities* (PDA1)–(PDA10)*, when viewed as diagrams in* $\mathbb{C}at/S\mathbf{I}_1$*, hold for the data* (PDD1)–(PDD5) *given above.*

**Proof.** Consider (PDA1) for example. The two pasting diagrams under consideration pick out two arrows **f** and **g** of $\mathbb{C}at_1/S\mathbf{I}_1$:

$$(\boldsymbol{\epsilon} S_m)_1 \otimes (S_c \boldsymbol{\eta})_1 \xrightarrow{\ \ \mathbf{f}\ \ } (S_m \boldsymbol{\epsilon})_1 \otimes (\boldsymbol{\eta} S_c)_1$$

with $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ mapping down to $S\mathbf{I}_1$

and

$$(\boldsymbol{\epsilon} S_m)_1 \otimes (S_c \boldsymbol{\eta})_1 \xrightarrow{\ \ \mathbf{g}\ \ } (S_m \boldsymbol{\epsilon})_1 \otimes (\boldsymbol{\eta} S_c)_1$$

with $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ mapping down to $S\mathbf{I}_1$,

where both the above diagrams commute. But by the previous proposition, the projections $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are local monomorphisms, and since **f** and **g** are special maps, we have

$$\boldsymbol{\pi}_2 \circ \mathbf{f} = \boldsymbol{\pi}_1 = \boldsymbol{\pi}_2 \circ \mathbf{g} \quad \text{implying} \quad \mathbf{f} = \mathbf{g}.$$

We argue similarly for the other nine diagrams.   $\square$

This completes the definition of our pseudo-distributive law in $\mathcal{B}(\mathbb{C}at/S\mathbf{I}_1)$; so now, by the arguments of Section 2, we can produce from this a pseudo-distributive law in $\mathcal{B}(\mathbb{C}oll(S))$, and thence, via the strict homomorphism $V : \mathcal{B}(\mathbb{C}oll(S)) \to [\mathbf{Mod}, \mathbf{Mod}]_\psi$, our desired pseudo-distributive law $\delta : \hat{S}_c \hat{S}_m \Rightarrow \hat{S}_m \hat{S}_c$ in **Mod**.

We are now finally able to state our abstract description of polycategories:

**Definition 40.** A *polycompositional polycategory* with object set $X$ is a monad on the discrete object $X$ in the bicategory $Kl(\delta)$.

There is one loose end to tie up: we must complete the argument begun in Proposition 9, and show that the polycompositional polycategories we have just defined are equivalent to polycategories equipped with a binary composition.

**Proposition 41.** *There is a bijection between polycompositional polycategories with object set $X$; and polycategories with object set $X$ in the sense of Definition* 6.

**Proof.** From the arguments which conclude Section 2.2, together with the explicit description of $\delta_X$ given at the end of Section 4.3, we see that the basic data for a polycompositional polycategory with object set $X$ are: sets of polymaps, equipped with actions by the symmetric groups; identity maps $x \to x$ for each element $x \in X$; and polycomposites for each pair of families of polymaps equipped with a suitable matching.

The axioms which a polycompositional category will satisfy are associativity and unitality laws, which may be extracted from the axioms for the corresponding monad in $Kl(\delta)$; and compatibility laws between polycomposition and exchange isomorphisms, which may be deduced from an examination of the coend composition in $Kl(\delta)$.

It thus follows from Proposition 9 that we may derive the basic data for a polycompositional polycategory from the data for a standard polycategory, and vice versa; and it is now a matter of straightforward verification to check that the axioms for the one entail the axioms for the other. Thus we have assignations in both directions between standard polycategories to polycompositional polycategories; and further verification shows these assignations to be mutually inverse.  □

And so we conclude with the main result of this paper:

**Theorem 42.** *To give a polycategory with object set $X$ is to give a monad on the discrete object $X$ in the bicategory $Kl(\delta)$.*

## Appendix A.  Pseudo notions

We give here definitions of *pseudomonad*, *pseudocomonad* and of a *pseudo-distributive law* of the latter over the former.

**Definition 43.** A *pseudomonad* on a bicategory $\mathcal{B}$ consists of the following data:

(PMD1)  A homomorphism $S : \mathcal{B} \to \mathcal{B}$.
(PMD2)  Pseudonatural transformations $\eta : \mathrm{id}_\mathcal{B} \Rightarrow S$ and $\mu : SS \Rightarrow S$.
(PMD3)  Invertible modifications
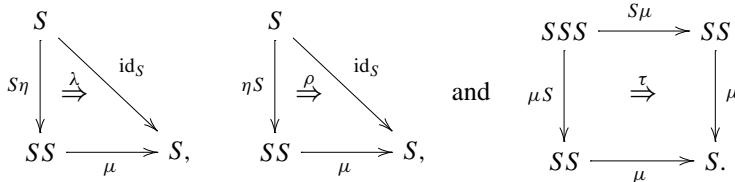


All subject to the following two axioms:

(PMA1)  The following pastings agree:

(PMA2)  The following pastings agree:

$$
\begin{array}{ccc}
& S^3 \xrightarrow{\;\mu S\;} S^2 & \\
S\eta S \nearrow \;\; \Downarrow S\rho \;\searrow \;\;\Downarrow \tau\;\; \mu \searrow & & \\
S^2 \xrightarrow{\;\;\mathrm{id}\;\;} S^2 \xrightarrow{\;\;\mu\;\;} S & = & S\eta S \nearrow \;\;S^3 \;\;\mu S \searrow \\
& & S^2 \xrightarrow{\;\;\mathrm{id}\;\;} S^2 \xrightarrow{\;\;\mu\;\;} S.
\end{array}
$$

Dually, we have the notion of a pseudocomonad on a bicategory:

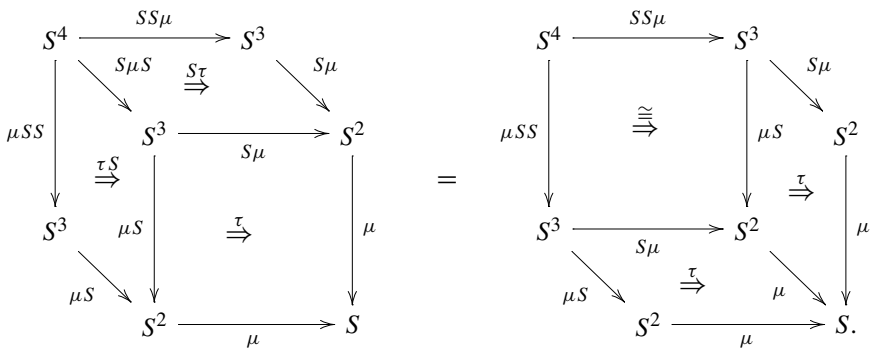**Definition 44.** A *pseudocomonad* on a bicategory $\mathcal{B}$ consists of the following data:

(PCD1)  A homomorphism $T : \mathcal{B} \to \mathcal{B}$;

(PCD2)  Pseudonatural transformations $\epsilon : T \Rightarrow \mathrm{id}_{\mathcal{B}}$ and $\Delta : T \Rightarrow TT$;

(PCD3)  Invertible modifications

$$
\begin{array}{ccccc}
T \xrightarrow{\;\Delta\;} T^2 & \quad & T \xrightarrow{\;\Delta\;} T^2 & \quad & T \xrightarrow{\;\Delta\;} T^2 \\
\Big\downarrow{\scriptstyle\mathrm{id}_T} \overset{\lambda'}{\Rightarrow} \Big\downarrow{\scriptstyle T\epsilon} & & \Big\downarrow{\scriptstyle\mathrm{id}_T} \overset{\rho'}{\Rightarrow} \Big\downarrow{\scriptstyle \epsilon T} & \text{and} & {\scriptstyle\Delta}\Big\downarrow \overset{\tau'}{\Rightarrow} \Big\downarrow{\scriptstyle \Delta T} \\
T, & & T, & & T^2 \xrightarrow{\;T\Delta\;} T^3.
\end{array}
$$

Subject to the two axioms:

(PCA1)  The following pastings agree:

$$
\qquad = \qquad
$$

(PCA2)  The following pastings agree:

**Definition 45.** Let $(S, \eta, \mu, \lambda, \rho, \tau)$ be a pseudomonad and $(T, \epsilon, \Delta, \lambda', \rho', \tau')$ a pseudocomonad on a bicategory $\mathcal{B}$. Then a *pseudo-distributive law* $\delta$ of $T$ over $S$ is given by the following data:

(PDD1)  A pseudo-natural transformation $\delta : TS \Rightarrow ST$.

(PDD2)  Invertible modifications

$$
\begin{array}{ccc}
& T & \\
T\eta \downarrow & \overset{\bar\eta}{\Rightarrow} & \nearrow \eta T \\
TS & \underset{\delta}{\longrightarrow} & ST
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
TS & \overset{\delta}{\longrightarrow} & ST \\
\epsilon S \downarrow & \overset{\bar\epsilon}{\Rightarrow} & \swarrow S\epsilon \\
& S. &
\end{array}
$$

(PDD3)  Invertible modifications

$$
\begin{array}{ccccc}
TSS & \overset{\delta S}{\longrightarrow} & STS & \overset{S\delta}{\longrightarrow} & SST \\
T\mu \downarrow & & \Downarrow\bar\mu & & \downarrow \mu T \\
TS & & \underset{\delta}{\longrightarrow} & & ST
\end{array}
\quad \text{and} \quad
\begin{array}{ccccc}
TS & & \overset{\delta}{\longrightarrow} & & ST \\
\Delta S \downarrow & & \Downarrow\bar\Delta & & \downarrow S\Delta \\
TTS & \underset{T\delta}{\longrightarrow} & TST & \underset{\delta T}{\longrightarrow} & STT,
\end{array}
$$

subject to the following axioms

$$
\begin{array}{ccc}
TS & \overset{\epsilon S}{\longrightarrow} & S \\
T\eta \uparrow \ \ \Downarrow\bar\eta & \overset{\delta}{\nearrow}\ \Downarrow\bar\epsilon & \uparrow S\epsilon \\
T & \underset{\eta T}{\longrightarrow} & ST
\end{array}
\ = \
\begin{array}{ccc}
TS & \overset{\epsilon S}{\longrightarrow} & S \\
T\eta \uparrow\ \Downarrow\cong & \text{id}_{\mathcal{B}}\ \nearrow^{\eta}\ \Downarrow\cong & \uparrow S\epsilon \\
T & \underset{\eta T}{\longrightarrow} & ST,
\end{array}
\qquad \text{(PDA1)}
$$

$$
\begin{array}{ccccc}
TSS & \overset{\delta S}{\longrightarrow} & STS & \overset{S\delta}{\longrightarrow} & SST \\
T\eta S \uparrow\ \Downarrow T\rho & \overset{T\mu}{\searrow}\ \Downarrow\bar\mu & & & \downarrow \mu T \\
TS & \underset{\text{id}_{TS}}{\longrightarrow} & TS & \underset{\delta}{\longrightarrow} & ST
\end{array}
\ = \
\begin{array}{ccccc}
TSS & \overset{\delta S}{\longrightarrow} & STS & \overset{S\delta}{\longrightarrow} & SST \\
T\eta S \uparrow\ \Downarrow\bar\eta S & \nearrow^{\eta TS}\ \Downarrow\cong\ \nearrow^{\eta ST} & & \Downarrow\rho T & \downarrow \mu T \\
TS & \underset{\delta}{\longrightarrow} & ST & \underset{\text{id}_{ST}}{\longrightarrow} & ST,
\end{array}
\qquad \text{(PDA2)}
$$

$$
\begin{array}{ccccc}
TSS & \overset{\delta S}{\longrightarrow} & STS & \overset{S\delta}{\longrightarrow} & SST \\
TS\eta \uparrow\ \Downarrow T\lambda & \overset{T\mu}{\searrow}\ \Downarrow\bar\mu & & & \downarrow \mu T \\
TS & \underset{\text{id}_{TS}}{\longrightarrow} & TS & \underset{\delta}{\longrightarrow} & ST
\end{array}
\ = \
\begin{array}{ccccc}
TSS & \overset{\delta S}{\longrightarrow} & STS & \overset{S\delta}{\longrightarrow} & SST \\
TS\eta \uparrow\ \Downarrow\cong & ST\eta \uparrow\ \Downarrow S\bar\eta\ \nearrow^{S\eta T}\ \Downarrow\lambda T & & & \downarrow \mu T \\
TS & \underset{\delta}{\longrightarrow} & ST & \underset{\text{id}_{ST}}{\longrightarrow} & ST,
\end{array}
\qquad \text{(PDA3)}
$$

$$
\begin{array}{c}
TS \xrightarrow{\ \delta\ } ST \xrightarrow{\ \mathrm{id}_{ST}\ } ST \\[4pt]
\Delta S \downarrow \quad \Downarrow \overline{\Delta} \quad \Downarrow S\rho' \quad \uparrow S\epsilon T \\[4pt]
TTS \xrightarrow{\ T\delta\ } TST \xrightarrow{\ \delta T\ } STT
\end{array}
\; = \;
\begin{array}{c}
TS \xrightarrow{\ \mathrm{id}_{TS}\ } TS \xrightarrow{\ \delta\ } ST \\[4pt]
\Delta S \downarrow \quad \Downarrow \rho' S \quad \Downarrow \cong \quad \uparrow S\epsilon T \\[4pt]
TTS \xrightarrow{\ T\delta\ } TST \xrightarrow{\ \delta T\ } STT,
\end{array}
$$

with $\epsilon T S$, $\epsilon S T$, $\Downarrow \bar\epsilon T$, $S\Delta$ labels.

$$\text{(PDA4)}$$

$$
\begin{array}{c}
TS \xrightarrow{\ \delta\ } ST \xrightarrow{\ \mathrm{id}_{ST}\ } ST \\[4pt]
\Delta S \downarrow \quad \Downarrow \overline{\Delta} \quad \Downarrow S\lambda' \quad \uparrow ST\epsilon \\[4pt]
TTS \xrightarrow{\ T\delta\ } TST \xrightarrow{\ \delta T\ } STT
\end{array}
\; = \;
\begin{array}{c}
TS \xrightarrow{\ \mathrm{id}_{TS}\ } TS \xrightarrow{\ \delta\ } ST \\[4pt]
\Delta S \downarrow \quad \Downarrow \lambda' S \quad \Downarrow T\bar\epsilon \quad \Downarrow \cong \quad \uparrow ST\epsilon \\[4pt]
TTS \xrightarrow{\ T\delta\ } TST \xrightarrow{\ \delta T\ } STT,
\end{array}
$$

with $T\epsilon S$, $TS\epsilon$, $S\Delta$ labels.

$$\text{(PDA5)}$$

$$
\begin{array}{c}
TSSS \xrightarrow{\ \delta SS\ } STSS \xrightarrow{\ S\delta S\ } SSTS \xrightarrow{\ SS\delta\ } SSST \\
TS\mu \quad T\mu S \quad \Downarrow \overline{\mu} S \quad \mu TS \quad \Downarrow \cong \quad \mu ST \\
TSS \quad \overset{T\tau}{\Rightarrow} \quad TSS \xrightarrow{\ \delta S\ } STS \xrightarrow{\ S\delta\ } SST \\
T\mu \quad T\mu \quad \Downarrow \overline{\mu} \quad \mu T \\
TS \xrightarrow{\hspace{4cm} \delta \hspace{4cm}} ST
\end{array}
$$

$$||$$

$$\text{(PDA6)}$$

$$
\begin{array}{c}
TSSS \xrightarrow{\ \delta SS\ } STSS \xrightarrow{\ S\delta S\ } SSTS \xrightarrow{\ SS\delta\ } SSST \\
TS\mu \quad \Downarrow \cong \quad ST\mu \quad \Downarrow S\overline{\mu} \quad S\mu T \quad \mu ST \\
TSS \xrightarrow{\ \delta S\ } STS \xrightarrow{\ S\delta\ } SST \quad \overset{\tau T}{\Rightarrow} \quad SST \\
T\mu \quad \Downarrow \overline{\mu} \quad \mu T \quad \mu T \\
TS \xrightarrow{\hspace{5cm} \delta \hspace{5cm}} ST,
\end{array}
$$

$$
\begin{array}{c}
TS \xrightarrow{\ \delta\ } ST \\
\Delta S \quad \Delta S \quad \Downarrow \overline{\Delta} \quad S\Delta \\
TTS \xRightarrow{\tau'S} TTS \xrightarrow{T\delta} TST \xrightarrow{\delta T} STT \\
\Delta TS \quad T\Delta S \quad \Downarrow T\overline{\Delta} \quad TS\Delta \quad \Downarrow \cong \quad ST\Delta \\
TTTS \xrightarrow{TT\delta} TTST \xrightarrow{T\delta T} TSTT \xrightarrow{\delta TT} STTT
\end{array}
$$

(PDA7)

$$
||
$$

$$
\begin{array}{c}
TS \xrightarrow{\ \delta\ } ST \\
\Delta S \quad \Downarrow \overline{\Delta} \quad S\Delta \quad S\Delta \\
TTS \xrightarrow{T\delta} TST \xrightarrow{\delta T} STT \xRightarrow{S\tau'} STT \\
\Delta TS \quad \Downarrow \cong \quad \Delta ST \quad \Downarrow \overline{\Delta}T \quad S\Delta T \quad ST\Delta \\
TTTS \xrightarrow{TT\delta} TTST \xrightarrow{T\delta T} TSTT \xrightarrow{\delta TT} STTT,
\end{array}
$$

$$
\begin{array}{c}
TSS \xrightarrow{\epsilon SS} SS \\
\delta S \quad \Downarrow \overline{\epsilon}S \quad S\epsilon S \\
T\mu \quad STS \quad \Downarrow S\overline{\epsilon} \quad SS\epsilon \\
\quad \Downarrow \overline{\mu} \quad S\delta \quad SST \quad \Downarrow \cong \quad \mu \\
\quad \mu T \\
TS \xrightarrow{\delta} ST \xrightarrow{S\epsilon} S
\end{array}
\quad = \quad
\begin{array}{c}
TSS \xrightarrow{\epsilon SS} SS \\
T\mu \quad \Downarrow \cong \quad \mu \\
TS \\
\delta \quad \Downarrow \overline{\epsilon} \quad \epsilon S \\
ST \xrightarrow{S\epsilon} S,
\end{array}
$$

(PDA8)

$$
\begin{array}{c}
T \xrightarrow{T\eta} TS \xrightarrow{\delta} ST \\
\Downarrow \cong \quad \Delta S \quad \Downarrow \overline{\Delta} \\
\Delta \quad TTS \xrightarrow{T\delta} \quad S\Delta \\
TT\eta \quad \Downarrow T\overline{\mu} \quad TST \\
\quad T\eta T \quad \Downarrow \overline{\eta}T \quad \delta T \\
TT \xrightarrow{\eta TT} STT
\end{array}
\quad = \quad
\begin{array}{c}
T \xrightarrow{T\eta} TS \\
\Downarrow \overline{\eta} \quad \delta \\
\Delta \quad \eta T \quad ST \\
\quad \Downarrow \cong \quad S\Delta \\
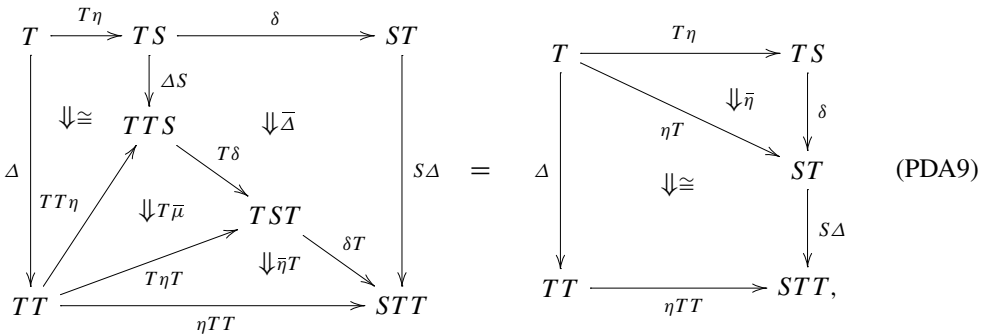TT \xrightarrow{\eta TT} STT,
\end{array}
$$

(PDA9)

$$
\begin{array}{ccccc}
TSS & \xrightarrow{\delta S} & STS & \xrightarrow{S\delta} & SST
\end{array}
$$

(diagram PDA10, top)

Nodes and arrows: $TSS \xrightarrow{\delta S} STS \xrightarrow{S\delta} SST$; $T\mu$, $\Delta SS$, $\Downarrow\overline{\Delta}S$, $S\Delta S$, $\Downarrow S\overline{\Delta}$, $SS\Delta$; $STTS$, $\delta TS$, $ST\delta$; $TS$, $\xLeftarrow{\cong}$, $TTSS \xrightarrow{T\delta S} TSTS$, $\cong$, $STST \xrightarrow{S\delta T} SSTT$; $\Delta S$, $TT\mu$, $\Downarrow T\overline{\mu}$, $TS\delta$, $\delta ST$, $\Downarrow\overline{\mu}T$, $\mu TT$; $TSST$, $T\mu T$; $TTS \xrightarrow{T\delta} TST \xrightarrow{\delta T} STT,$

$$
\|
$$

(PDA10)

$$
\begin{array}{ccccc}
TSS & \xrightarrow{\delta S} & STS & \xrightarrow{S\delta} & SST
\end{array}
$$

(diagram PDA10, bottom)

Nodes and arrows: $TSS \xrightarrow{\delta S} STS \xrightarrow{S\delta} SST$; $T\mu$, $\Downarrow\overline{\mu}$, $\mu T$, $SS\Delta$; $TS \xrightarrow{\delta} ST$, $\xLeftarrow{\cong}$, $SSTT$; $\Delta S$, $\Downarrow\overline{\Delta}$, $S\Delta$, $\mu TT$; $TTS \xrightarrow{T\delta} TST \xrightarrow{\delta T} STT.$

## References

[1] M.E. Szabo, Polycategories, Comm. Algebra 3 (8) (1975) 663–689.

[2] J.R.B. Cockett, R.A.G. Seely, Weakly distributive categories, J. Pure Appl. Algebra 114 (2) (1997) 133–173.

[3] J. Koslowski, A monadic approach to polycategories, in: Electron. Notes Theor. Comput. Sci., vol. 69, 2003, pp. 1–26.

[4] A. Burroni, *T*-catégories (catégories dans un triple), Cah. Topol. Géom. Différ. Catég. 12 (1971) 215–321.

[5] C. Hermida, Representable multicategories, Adv. Math. 151 (2) (2000) 164–225.

[6] T. Leinster, Higher Operads, Higher Categories, London Math. Soc. Lecture Note Ser., vol. 298, Cambridge Univ. Press, Cambridge, 2004.

[7] J.C. Baez, J. Dolan, Higher-dimensional algebra III. *n*-Categories and the algebra of opetopes, Adv. Math. 135 (2) (1998) 145–206.

[8] M.M. Clementino, W. Tholen, Metric, topology and multicategory—A common approach, J. Pure Appl. Algebra 179 (1–2) (2003) 13–47.

[9] F. Marmolejo, Distributive laws for pseudomonads, Theory Appl. Categ. 5 (5) (1999) 91–147 (electronic).

[10] M. Tanaka, Pseudo-distributive laws, PhD thesis, Edinburgh, 2004.

[11] R. Garner, Double clubs, Cah. Topol. Géom. Différ. Catég. 47 (4) (2006) 261–317.

[12] J.M.E. Hyland, Proof theory in the abstract, Ann. Pure Appl. Logic 114 (1–3) (2002) 43–78, commemorative Symposium Dedicated to Anne S. Troelstra (Noordwijkerhout, 1999).

[13] T. Leinster, Operads in higher-dimensional category theory, Theory Appl. Categ. 12 (3) (2004) 73–194 (electronic).

[14] J. Lambek, Deductive systems and categories. II. Standard constructions and closed categories, in: Category Theory, Homology Theory and their Applications, I, Battelle Institute Conference, Seattle, Wash., 1968, vol. one, Springer, Berlin, 1969, pp. 76–122.

[15] E. Cheng, M. Hyland, J. Power, Pseudo-distributive laws, in: Stephen Brookes, Prakash Panangaden (Eds.), Proceedings of Mathematical Foundations of Programming Semantics, in: Electron. Notes Theor. Comput. Sci., vol. 83, 2003, 12 pages.

[16] G.M. Kelly, R. Street, Review of the elements of 2-categories, in: Category Seminar, Proc. Sem., Sydney, 1972/1973, in: Lecture Notes in Math., vol. 420, Springer, Berlin, 1974, pp. 75–103.

[17] B. Day, R. Street, Lax monoids, pseudo-operads, and convolution, in: Diagrammatic Morphisms and Applications, San Francisco, CA, 2000, in: Contemp. Math., vol. 318, Amer. Math. Soc., Providence, RI, 2003, pp. 75–96.

[18] R. Street, Frobenius monads and pseudomonoids, J. Math. Phys. 45 (10) (2004) 3930–3948.

[19] G.M. Kelly, On clubs and data-type constructors, in: Applications of Categories in Computer Science, Durham, 1991, in: London Math. Soc. Lecture Note Ser., vol. 177, Cambridge Univ. Press, Cambridge, 1992, pp. 163–190.

[20] B. Day, R. Street, Monoidal bicategories and Hopf algebroids, Adv. Math. 129 (1) (1997) 99–157.