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MONOIDAL KLEISLI BICATEGORIES AND THE ARITHMETIC PRODUCT OF COLOURED SYMMETRIC SEQUENCES

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Abstract. We extend the arithmetic product of species of structures and symmetric sequences studied by Maia and Méndez and by Dwyer and Hess to coloured symmetric sequences and show that it determines a normal oplax monoidal structure on the bicategory of coloured symmetric sequences. In order to do this, we establish general results on extending monoidal structures to Kleisli bicategories. Our approach uses monoidal double categories, which help us to attack the difficult problem of verifying the coherence conditions for a monoidal bicategory in an efficient way.

Contents

1. Introduction

Context, aim and motivation. Joyal's theory of species of structures [Joy81] provides an illuminating and powerful approach to enumerative combinatorics, as amply illustrated in [BLL98], and finds applications also in algebra [AM10]. By definition, a species of structures F is simply a functor from the category $\mathfrak{B}$ of finite sets and bijections to the category of sets and functions, mapping a finite set U of 'labels' to a set $F[U]$ of 'F-structures' (*e.g.* binary rooted trees) labelled by elements of U. Importantly for applications, species of structures support a calculus of operations (which includes substitution, sum, product and differentiation) that has a combinatorial interpretation and provides a 'categorification' of the calculus of exponential power series widely used in combinatorics [Wil06]. This point of view is supported by the introduction of the so-called analytic functor associated to a species of structures [Joy86], which is defined by the formula

$$F(X) = \sum_{n \in \mathbb{N}} \frac{F[n] \times X^n}{\mathfrak{S}_n},$$

where, for $n \in \mathbb{N}$, $[n] = \{1, \ldots, n\}$ and the fraction denotes the quotient of $F[n] \times X^n$ by the evident action of the n-th symmetric group $\mathfrak{S}_n$. The passage from species of structures to analytic functors goes via symmetric sequences, which are defined as functors from $\mathfrak{S}$, the skeleton of $\mathfrak{B}$ whose objects are finite cardinals, into the category of sets. Under the equivalence between species of structures and symmetric sequences, the substitution operation of species of structures corresponds to the substitution monoidal structure on symmetric sequences defined in [Kel05], which is of interest since monoids with respect to it are precisely symmetric operads [BV73, May72].

In [MM08], Maia and Méndez introduced a new operation on species of structures, baptized *arithmetic product*, and provided a combinatorial interpretation for it. Given species F and G, their arithmetic product $F \boxtimes G$ is defined by letting

$$(F_1 \boxtimes F_2)[U] = \sum_{(\pi_1, \pi_2) \in \mathcal{R}[U]} F_1[\pi_1] \times F_2[\pi_2],$$

where $\mathcal{R}[U]$ denotes a certain set of partitions of U, called rectangles. Independently of the work of Maia and Méndez, Dwyer and Hess rediscovered this operation[1] in the context of symmetric sequences [DH14] in order to extend the Boardman–Vogt tensor product of symmetric operads [BV73] to operadic bimodules. For symmetric sequences, the arithmetic product is defined from the product of natural numbers on $\mathfrak{S}$ (which is functorial, even if it is not the cartesian product) by Day convolution [Day70], via the coend formula

$$(F_1 \boxtimes F_2)[m] = \int^{m_1, m_2} \mathfrak{S}[m, m_1 \cdot m_2] \times F_1[m_1] \times F_2[m_2]. \tag{1}$$

The connection between the arithmetic product for species in [MM08] and symmetric sequences in [DH14] seems to have been first noted by Bremner and Dotsenko in [BD20].

In [DH14], the authors also observed that the arithmetic product of symmetric sequences interacts with the substitution monoidal structure in an interesting way, in that there is a natural transformation with components

$$(G_1 \circ F_1) \boxtimes (G_2 \circ F_2) \to (G_1 \boxtimes G_2) \circ (F_1 \boxtimes F_2) \tag{2}$$

which are not necessarily invertible. Dwyer and Hess conjectured that this transformation underlies what is usually called a *duoidal* or *2-monoidal* structure, in which two monoidal structures interact by means of an interchange law [AM10]. The conjecture was settled positively by the second-named author and López Franco, who also showed that this duoidal structure is *normal*, in the sense that the units of the two monoidal structures essentially coincide. This was done as part of their general study of commutative operations [GL16], which involves introducing a general notion of *commuting tensor product* of $\circ$-monoids in a normal duoidal category $(\mathcal{V}, \boxtimes, \circ)$. When this notion is instantiated at the normal duoidal category of symmetric sequences, it re-finds the Boardman–Vogt tensor product $P \otimes_{\mathrm{BV}} Q$ of symmetric operads P and Q. In particular, to express that the operations of P and Q *commute* with each other in $P \otimes_{\mathrm{BV}} Q$ one uses a diagram of the form

$$P \boxtimes Q \rightrightarrows (P \otimes_{\mathrm{BV}} Q) \circ (P \otimes_{\mathrm{BV}} Q) \longrightarrow P \otimes_{\mathrm{BV}} Q \tag{3}$$

which involves both the arithmetic product and the substitution monoidal structures. Furthermore, [FV20] uses this duoidal structure to establish an enrichment of symmetric operads in symmetric cooperads.

The aim of this paper is to generalise the definition of the arithmetic product, and the key results concerning it, from symmetric sequences to coloured symmetric sequences: this corresponds to the passage from symmetric operads to coloured symmetric operads, *i.e.* from the single-object to the many-object case. We will show that such generalisation not only is possible, but actually determines a new kind of low-dimensional categorical structure.

The motivation for this work is manifold. First, it is part of a wider research programme aimed at understanding the structure of the bicategory of coloured symmetric sequences and related bicategories, with applications to logic and theoretical computer science, *cf.* [CW05, Hyl10]. In particular, it provides the basis to extend the Garner–López Franco theory of commutativity, to re-find the Boardman–Vogt tensor product of coloured symmetric operads (*cf.* (3)), and to develop a corresponding tensor product of bimodules between them, generalising the results of Dwyer and Hess, a project that we leave for future work. The results presented here are also useful to extend the study of enrichment in [Vas19] to relate coloured (co)operads and their (co)modules. Finally, we hope that our results may eventually be of interest in combinatorics, since the arithmetic product of coloured symmetric sequences defined here induces a corresponding operation on coloured species of structures [FGHW08], which extends the arithmetic product of Maia and Méndez to variants of Joyal's species of structures that are particular instances of coloured species of structures.

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[1]Dwyer and Hess called it matrix multiplication. Here we prefer to say arithmetic product in order to avoid potential confusion with the composition operation of the bicategory of matrices.

Main results. Whereas the original arithmetic product is extra structure on the substitution monoidal category of symmetric sequences, our generalised arithmetic product for coloured symmetric sequences will be extra structure on the bicategory Sym of coloured symmetric sequences of [FGHW08]. Recall that, for sets X and Y, an (X, Y)-coloured symmetric sequence is a functor $M\colon SY^{\mathrm{op}} \times X \to \mathsf{Set}$, where SY is the free symmetric strict monoidal category on Y. Such a functor M assigns a set $M(\vec{y}, x)$ to each $\vec{y} = (y_1, \ldots, y_n) \in SY$ and $x \in X$, the elements of which can be thought of as operations $f\colon y_1, \ldots, y_n \to x$, with inputs of sorts $y_1, \ldots, y_n$ and output of sort x, typically pictured as corollas. Taking $X = Y = 1$ recovers the notion of ordinary symmetric sequence since $S1 = \mathfrak{S}$; and as shown in [Fio05, FGHW08], the calculus of symmetric sequences can be extended to their coloured counterparts. In particular, the substitution monoidal structure can be generalised to a composition operation, which is the composition of the bicategory Sym whose objects are sets, and whose maps from X to Y are the symmetric (X, Y)-coloured sequences [FGHW08, FGHW18]. The monads in this bicategory are then symmetric coloured operads [BD98].

Given an (X_1, Y_1)-coloured symmetric sequence M and a (X_2, Y_2)-coloured symmetric sequence M_2, we will define their arithmetic product as the $(X_1 \times X_2, Y_1 \times Y_2)$-coloured symmetric sequence $M_1 \boxtimes M_2$ given by

$$(M_1 \boxtimes M_2)(\vec{y}, (x_1, x_2)) = \int^{\vec{y}^1, \vec{y}^2 \in SY} S(Y_1 \times Y_2)[\vec{y}, \vec{y}_1 \boxtimes \vec{y}_2] \times M_1(\vec{y}_1, x_1) \times M_2(\vec{y}_2, x_2), \qquad (4)$$

where

$$\boxtimes\colon SY_1 \times SY_2 \to S(Y_1 \times Y_2) \qquad (5)$$

is an operation determined by lexicographic ordering of pairs. As expected, when $X_1 = Y_1 = X_2 = Y_2 = 1$, we obtain the arithmetic product in (1) of Dwyer and Hess.

Our main result, Theorem 10.9, asserts that the said arithmetic product determines a *normal oplax monoidal structure* on the bicategory Sym of coloured symmetric sequences. The notion of a normal oplax monoidal bicategory appears to be novel and is introduced here in Definition 4.8 as the natural 'many-object' generalisation of the normal duoidal structure in (2). The key challenge to be overcome to obtain our main result is the verification of the axioms for a normal oplax monoidal bicategory, which are of the same daunting complexity as those for a monoidal bicategory [GPS95, Gur13]. As such, attempting a direct verification seems hopelessly complicated and unlikely to result in any insight. Instead, we solve the problem developing ideas of 2-dimensional monad theory [BKP89], obtaining some general results that are of independent interest.

Our approach exploits crucially the notion of a (pseudo) double category, which adeptly handles the bookwork around dealing with structures involving two kinds of morphisms. More specifically, in a double category one has objects, two kinds of 1-cells (called horizontal and vertical), and squares (which help to relate horizontal and vertical 1-cells). Of key importance for our development are the double category of profunctors $\mathbb{P}\mathsf{rof}$, which has categories as objects, profunctors [Bén73, Bén00] (also known as distributors or bimodules) as horizontal 1-cells and functors as vertical 1-cells; and the double category of matrices $\mathbb{M}\mathsf{at}$, which is the full double subcategory of $\mathbb{P}\mathsf{rof}$ spanned by sets (viewed as discrete categories).

Double categories are important for us because they provide an efficient way of constructing three-dimensional structures such as monoidal bicategories and, as we shall see, their oplax variants. The basic insight, as explained in [GG09, Shu10, HS19], is that, in order to obtain a monoidal structure on a bicategory $\mathcal{E}$, it is sufficient to represent $\mathcal{E}$ as the horizontal bicategory of a double category and then construct a monoidal structure on this double category. Since the coherence data and axioms for a monoidal double category are of the same character as those of a monoidal category, rather than a monoidal bicategory, this significantly reduces the volume and complexity of the checks required to establish the structure.

The relevance of this to our situation is that the bicategory of coloured symmetric sequences Sym can be represented as the horizontal bicategory of a double category $\mathbb{S}\mathsf{ym}$ in which vertical 1-cells are functions between sets. In fact, building on [CS10, FGHW18], this $\mathbb{S}\mathsf{ym}$ can be seen as a full sub-double category (on the discrete objects) of the double category $\mathbb{C}\mathsf{atSym}$ of *categorical symmetric sequences*; which can, in turn, be constructed as the Kleisli double category of a double monad on the double category $\mathbb{P}\mathsf{rof}$ of profunctors. The double monad in question maps a category X to its symmetric strict monoidal completion SX, extending the corresponding 2-monad on the 2-category of categories.

Given the above, the desired normal oplax monoidal structure on the bicategory of coloured symmetric sequences can be obtained as follows. Firstly (Theorem 4.10), we extend the results of [GG09, Shu10, HS19]

to establish that such a structure can be obtained from a normal oplax monoidal structure on $\mathbb{S}$ym, or more generally, $\mathbb{C}$atSym. Secondly, to obtain this, we prove and apply a result (Theorem 9.4) which isolates sufficient conditions on a double monad T on a double category $\mathbb{C}$ under which a monoidal structure on $\mathbb{C}$ will extend to an oplax monoidal structure on the Kleisli double category $\mathbb{K}l(T)$. Pleasingly, this condition on T turns out to be a natural one, namely a suitably adapted form of the *pseudo-commutativity* of [HP02]. This is satisfied by the 2-monad S used in our application, and indeed, the operation (5) featured in the definition of the arithmetic product is part of this pseudo-commutative structure on $S \colon$ Cat $\to$ Cat.

Thus, using this result, the monoidal structure on $\mathbb{P}$rof given by the cartesian product of sets extends to give the arithmetic product oplax monoidal structure on the double category $\mathbb{C}$atSym of categorical symmetric sequences, which in turn induces the desired oplax monoidal structure on the double category $\mathbb{S}$ym of coloured symmetric sequences. This general method leads exactly to the formula in (4), which is a natural generalisation of that in (1).

While our approach offers a clear pathway to prove our main results, and others besides, we still have to overcome significant technical challenges, dealing with coherence conditions at the double categorical level, keeping track of strictness and weakness of the structures involved. Roughly speaking, vertical structure tends to be stricter than horizontal one, but the two are closely related under the assumption that the double categories under consideration are fibrant, in the sense of [Shu08]. This allows us to induce a lot of the horizontal, weaker, structure that we need for applications from vertical, stricter, one, that is already known, thereby keeping some control of the complexity of our calculations.

While it undoubtedly requires more groundwork to set up the abstract approach that we take, the end result is a modular framework which is easily applicable to other, related situations. For example, although we shall not do so here, it is entirely straightforward to adapt our results from the setting appropriate for studying symmetric coloured operads to the setting appropriate for many-sorted algebraic theories: it is simply a matter of replacing the double monad S for symmetric strict monoidal categories with a corresponding double monad F for categories with strictly associative finite products, and verifying that everything still carries through. In this setting, we obtain an oplax monoidal structure on the appropriate Kleisli bicategory—which is essentially the bicategory of sifted-cocontinuous functors between presheaf categories—which extends the duoidal structure on the functor category $[F(1)^{\mathrm{op}}, \mathsf{Set}]$ used in [GL16] to study the commuting tensor product of single-sorted algebraic theories.

For expository convenience, we outlined our results above for Set-valued coloured symmetric sequences, but in fact our development will be carried out in a more general enriched context. For any symmetric monoidal closed cocomplete $\mathcal{V}$ there is an analogue of the 2-monad S leading to a bicategory of $\mathcal{V}$-enriched symmetric sequences. However, to obtain an oplax monoidal structure on this bicategory, we will restrict to the case where the tensor product of $\mathcal{V}$ is in fact *cartesian* product. The reason for this restriction, which was also made in [DH14, GL16], is that the 2-monad S is only pseudo-commutative when $\mathcal{V}$ is cartesian, since the structure maps in (5) for this pseudo-commutativity involve a 'duplication' of objects that is not available in the general symmetric monoidal closed setting.

Outline of the paper. Sections 2 and 3 recall the notions of double category, double functor, horizontal and vertical transformation, and modification. We pay particular attention to companions, leading to the notion of a special vertical transformation, and establish a few useful lemmas about them. Sections 4 and 5 focus on monoidal double categories, monoidal double functors, monoidal horizontal and vertical transformations and monoidal modifications. In particular, we show that, for monoidal double categories $\mathbb{C}$ and $\mathbb{D}$, monoidal double functors between them are the objects of a functor double category (Proposition 5.6). Section 6 considers monoids in monoidal double categories. We use this notion in Sections 7 and 8 to define double monads and monoidal double monads and obtain results on them in a homogeneous manner. To do this, we show that the double category of monoidal double endofunctors on a monoidal double category admits a monoidal structure, given by composition (Proposition 8.1). In Section 9, we consider Kleisli double categories and establish sufficient conditions for a double monad on a monoidal double category to determine a monoidal structure on the Kleisli double category. We apply these results to coloured symmetric sequences in Section 10, leading up to our main results on the existence of oplax monoidal structures on the relevant double category (Theorem 10.8) and bicategory (Theorem 10.9).

2. Double categories

By double category, we will mean a horizontally weak double category, also known as pseudo double category; relevant material can be found in [GP99, GP04, Gar06a, HS19].

Definition 2.1 (Double category)**.** A *double category* $\mathbb{C}$ consists of the following data:

- a category $\mathbb{C}_0$, whose objects are 0-*cells*, and whose arrows are *vertical 1-cells* $f \colon X \to X'$;
- a category $\mathbb{C}_1$, whose objects are *horizontal 1-cells* $M \colon X \longrightarrow Y$, and whose arrows are *2-morphisms*

$$
\begin{array}{ccc}
X & \xrightarrow{M} & Y \\
f \downarrow & \Downarrow \phi & \downarrow g \\
X' & \xrightarrow[M']{} & Y';
\end{array}
\tag{6}
$$

- two functors $\mathbf{s}, \mathbf{t} \colon \mathbb{C}_1 \to \mathbb{C}_0$ called *source* and *target* respectively;
- *composition* and *identity* functors $\circ \colon \mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 \to \mathbb{C}_1$ and $\mathrm{id} \colon \mathbb{C}_0 \to \mathbb{C}_1$;
- natural families of globular isomorphisms in $\mathbb{C}_1$:

$$
a_{M,N,P} \colon (M \circ N) \circ P \Rightarrow M \circ (N \circ P), \quad \ell_M \colon \mathrm{id}_Y \circ M \Rightarrow M \quad \text{and} \quad r_M \colon M \circ \mathrm{id}_X \Rightarrow M.
\tag{7}
$$

These data are required to satisfy coherence axioms analogous to those for a bicategory; see [GP99, §7.1].

In the last item of the preceding definition, we use the notion of a *globular* 2-morphism in a double category $\mathbb{C}$; this is a 2-morphism ϕ for which $\mathbf{s}(\phi)$ and $\mathbf{t}(\phi)$ are identities:

$$
\begin{array}{ccc}
X & \xrightarrow{M} & X \\
\| & \Downarrow \phi & \| \\
X' & \xrightarrow[M']{} & Y.
\end{array}
$$

Given a double category $\mathbb{C}$, we write $\mathcal{H}(\mathbb{C})$ for the its *horizontal bicategory*, comprising the 0-cells, horizontal 1-cells and globular 2-morphisms; and we write $\mathsf{V}(\mathbb{C})$ for its *vertical 2-category*, whose objects and morphisms are the 0-cells and vertical 1-cells of $\mathbb{C}$, and where a 2-cell $f \Rightarrow g$ is a 2-morphism in $\mathbb{C}$ of the form

$$
\begin{array}{ccc}
X & \xrightarrow{\mathrm{id}_X} & X \\
g \downarrow & \Downarrow \phi & \downarrow f \\
X' & \xrightarrow[\mathrm{id}_{X'}]{} & X'.
\end{array}
$$

Notice that vertical composition of 2-cells in $\mathsf{V}(\mathbb{C})$ is given by pasting in the horizontal direction in $\mathbb{C}$, and vice versa.

Example 2.2 (Bicategories and monoidal categories as double categories)*.* Any bicategory can be seen as a double category with only identity vertical arrows. In particular, any monoidal category $(\mathcal{V}, \circ, J)$ can be regarded as a double category with a single object and only the identity vertical arrow.

Example 2.3 (The double category of matrices)*.* Fix a monoidal category $(\mathcal{V}, \circ, J)$ with small coproducts, such that the tensor product preserves coproducts in each variable; in particular, this holds whenever the monoidal structure is closed. Given sets X and Y, an (X, Y)-*matrix* $M \colon X \longrightarrow Y$ is a family of objects $(M(y, x) \in \mathcal{V} : x \in X, y \in Y)$. The double category of $\mathcal{V}$-*matrices* $\mathbb{M}\mathrm{at}_\mathcal{V}$ has objects and vertical 1-cells being sets and functions, respectively; horizontal 1-cells $M \colon X \longrightarrow Y$ being (X, Y)-matrices; and 2-morphisms $\phi \colon M \Rightarrow N$ as in (6) being families of $\mathcal{V}$-arrows $(\phi_{yx} \colon M(y, x) \to N(gy, fx) : x \in X, y \in Y)$. Horizontal composition $N \circ M \colon X \longrightarrow Y \longrightarrow Z$ is given by

$$
(N \circ M)(z, x) = \sum_{y \in Y} N(z, y) \circ M(y, x),
$$

while the horizontal identities $\mathrm{id}_X \colon X \longrightarrow X$ are defined by $\mathrm{id}_X(x, x') = J$ if $x = x'$ and $\mathrm{id}_X(x, x') = 0$ if $x \neq x'$. The horizontal bicategory of this double category is the usual bicategory of enriched matrices; see [BCSW83].

The choice of a different notation for Example 2.4 below is needed in view of Example 4.5.

Example 2.4 (The double category of profunctors). Fix a braided monoidal cocomplete category $(\mathcal{V}, \otimes, I)$ in which the tensor product preserves colimits in each variable. Below, we use freely the standard notions of $\mathcal{V}$-category, $\mathcal{V}$-functor and $\mathcal{V}$-natural transformation, for which we invite readers to refer to [Kel82]. Recall that, for small $\mathcal{V}$-categories X and Y, a (Y, X)-*profunctor* $M \colon X \longrightarrow Y$ is a $\mathcal{V}$-functor $M \colon Y^{\mathrm{op}} \otimes X \to \mathcal{V}$. We may now define the double category $\mathbb{P}\mathrm{rof}_\mathcal{V}$ of $\mathcal{V}$-*profunctors* as follows. The objects and vertical 1-cells of $\mathbb{P}\mathrm{rof}_\mathcal{V}$ are small $\mathcal{V}$-categories and $\mathcal{V}$-functors; the horizontal 1-cells are $\mathcal{V}$-profunctors, and squares as in (6) (writing now F and G instead of f and g) are $\mathcal{V}$-natural transformations $\phi \colon M \Rightarrow M'(G \otimes F)$, where

$$Y^{\mathrm{op}} \otimes X \xrightarrow{G \otimes F} Y'^{\mathrm{op}} \otimes X' \xrightarrow{M'} \mathcal{V}.$$

Horizontal composition of $M \colon X \longrightarrow Y$ and $N \colon Y \longrightarrow Z$ is given by the coend

$$(N \circ M)(z, x) = \int^{y \in Y} N(z, y) \otimes M(y, x). \tag{8}$$

The horizontal identity $\mathrm{id}_X \colon X \longrightarrow X$ is defined by $\mathrm{id}_X(x', x) = X[x', x]^{[2]}$. Note that $\mathbb{M}\mathrm{at}_\mathcal{V}$ can be regarded as a sub-double category of $\mathbb{P}\mathrm{rof}_\mathcal{V}$ by considering sets as discrete $\mathcal{V}$-categories. The horizontal bicategory of $\mathbb{P}\mathrm{rof}_\mathcal{V}$ is the familiar bicategory of profunctors, while the vertical 2-category is the 2-category of small $\mathcal{V}$-categories.

The examples illustrate a phenomenon that pervades the paper: the vertical structure is strict, while the horizontal structure is weaker, and thus requires careful consideration of coherence issues. The management of these coherence issues is simplified when the horizontal and vertical structures can be universally related in the following way.

Definition 2.5. Let $\mathbb{C}$ be a double category. A *companion* for a vertical 1-cell $f \colon X \to X'$ in $\mathbb{C}$ is given by a horizontal 1-cell $\widehat{f} \colon X \longrightarrow X'$ along with 2-morphisms

$$
\begin{array}{ccc}
X \xrightarrow{\widehat{f}} X' & & X \xrightarrow{\mathrm{id}_X} X \\
f\downarrow \quad \Downarrow p_1 \quad \| & \text{and} & \| \quad \Downarrow p_2 \quad \downarrow f \\
X' \xrightarrow[\mathrm{id}_{X'}]{} X' & & X \xrightarrow[\widehat{f}]{} X'
\end{array}
\tag{9}
$$

such that

$$
\begin{array}{ccc}
\begin{array}{c}
X \xrightarrow{\mathrm{id}_X} X \\
\| \quad \Downarrow p_2 \quad \downarrow f \\
X \xrightarrow{\widehat{f}} X' \\
f\downarrow \quad \Downarrow p_1 \quad \| \\
X' \xrightarrow[\mathrm{id}_{X'}]{} X'
\end{array} = \mathrm{id}_f
& \text{and} &
\begin{array}{c}
X \xrightarrow{\mathrm{id}_X} X \xrightarrow{\widehat{f}} X' \\
\| \quad \Downarrow p_2 \quad \downarrow f \quad \Downarrow p_1 \quad \| \\
X \xrightarrow{\widehat{f}} X' \xrightarrow{\mathrm{id}_{X'}} X' \\
\| \quad \quad \Downarrow \ell \quad \quad \| \\
X \xrightarrow{\widehat{f}} X'
\end{array}
=
\begin{array}{c}
X \xrightarrow{\mathrm{id}_X} X \xrightarrow{\widehat{f}} X' \\
\| \quad \quad \Downarrow r \quad \quad \| \\
X \xrightarrow[\widehat{f}]{} X'.
\end{array}
\end{array}
\tag{10}
$$

Although companions are defined algebraically, they also have a universal characterisation:

Lemma 2.6. *Let $f \colon X \to X'$ be vertical 1-cell of a double category $\mathbb{C}$. Giving a companion $(\widehat{f}, p_1, p_2)$ for f is equivalent to giving either of the following:*

- *A horizontal 1-cell $\widehat{f}$ and 2-morphism p_1 as in (9) such that, for every horizontal 1-cell $M \colon X' \longrightarrow Y'$, the composite 2-morphism to the left below is cartesian with respect to $(\mathbf{s}, \mathbf{t}) \colon \mathbb{C}_1 \to \mathbb{C}_0 \times \mathbb{C}_0$; or*
- *A horizontal 1-cell $\widehat{f}$ and 2-morphism p_2 as in (9) such that, for every horizontal 1-cell $M \colon W \longrightarrow X$, the composite 2-morphism to the right below is opcartesian with respect to $(\mathbf{s}, \mathbf{t}) \colon \mathbb{C}_1 \to \mathbb{C}_0 \times \mathbb{C}_0$:*

---

[2]We use brackets for hom-objects of categories, and parentheses for applications of (pro)functors and other maps.

$$
\begin{array}{ccc}
X \xrightarrow{\;\hat{f}\;} X' \xrightarrow{\;M\;} Y' & \qquad & W \xrightarrow{\hspace{3em}M\hspace{3em}} X \\
f\downarrow \quad \Downarrow p_1 \quad \| \qquad \| & & \| \qquad \Downarrow \ell^{-1} \qquad \| \\
X' \xrightarrow{\mathrm{id}_{X'}} X' \xrightarrow{\;M\;} Y' & & W \xrightarrow{\;M\;} X \xrightarrow{\mathrm{id}_X} X \\
\| \quad \Downarrow r \quad \| & & \| \qquad \| \quad \Downarrow p_2 \quad \downarrow f \\
X' \xrightarrow{\hspace{4em}M\hspace{4em}} Y' & & W \xrightarrow{\;M\;} X \xrightarrow{\;\hat{f}\;} X'.
\end{array}
$$

By virtue of this result, companions of a vertical 1-cell are unique up to a unique globular 2-isomorphism, so that by the usual abuse of notation we may refer simply to *the* companion. In what follows we will often require the existence of certain companions, but in many examples of interest we have all companions and also all *conjoints*—the dual notion to companion, which associates to a vertical 1-cell $f\colon X \to X'$ a horizontal 1-cell $\check{f}\colon X' \rightarrow X$ along with unit and counit 2-morphisms. In this case, we may speak of a *framed bicategory* in the sense of [Shu08] or *fibrant double category* in other references. By the above lemma and an appropriate dual lemma for conjoints, a double category is a framed bicategory if and only if $(\mathbf{s},\mathbf{t})\colon \mathbb{C}_1 \to \mathbb{C}_0 \times \mathbb{C}_0$ is a Grothendieck fibration, or equivalently, a Grothendieck opfibration.

Example 2.7. The double category $\mathbb{P}\mathsf{rof}_{\mathcal{V}}$ of $\mathcal{V}$-profunctors is a fibrant double category. The companion and conjoint of a $\mathcal{V}$-functor $F\colon X \to X'$ are the $\mathcal{V}$-profunctors $\hat{F}\colon X \rightarrow X'$ and $\check{F}\colon X' \rightarrow X$ given by

$$
\hat{F}(x',x) = X'[x',Fx] \quad \text{and} \quad \check{F}(x,x') = X'[Fx,x']. \tag{11}
$$

It follows *a fortiori* that the double category $\mathbb{M}\mathsf{at}_{\mathcal{V}}$ of matrices is also fibrant, where for a function $f\colon X \to X'$ its companion and conjoint $\hat{f}\colon X \rightarrow X'$ and $\check{f}\colon X' \rightarrow X$ are the $\mathcal{V}$-matrices given by:

$$
\hat{f}(x',x) = \check{f}(x,x') = \begin{cases} I & \text{if } f(x) = x', \\ 0 & \text{otherwise.} \end{cases}
$$

Note that, in these examples, we have that $\hat{F} \dashv \check{F}$ and $\hat{f} \dashv \check{f}$ in the horizontal bicategory. In fact, it is *always* true that the companion and conjoint of a vertical 1-cell are adjoint in this way.

The universality of companions in Lemma 2.6 immediately implies the following omnibus proposition.

Proposition 2.8. *Let $\mathbb{C}$ be a double category.*

(i) *The vertical identity 1-cell $1_X\colon X \to X$ has the horizontal identity $\mathrm{id}_X\colon X \rightarrow X$ as a companion.*

(ii) *If the vertical 1-cells $f\colon X \to X'$ and $g\colon X' \to X''$ have the companions $\hat{f}$ and $\hat{g}$, then $g \circ f$ has the companion $\hat{g} \circ \hat{f}$.*

(iii) *If the vertical 1-cells $f\colon X \to X'$ and $g\colon Y \to Y'$ admit companions, then pasting with the companion 2-morphisms for f and g gives a bijection between 2-morphisms ϕ as in (6) and globular 2-morphisms*

$$
\begin{array}{ccc}
X \xrightarrow{\;M\;} Y \xrightarrow{\;\hat{g}\;} Y' \\
\| \qquad \Downarrow \hat{\phi} \qquad \| \\
X \xrightarrow{\;\hat{f}\;} X' \xrightarrow{\;M'\;} Y'.
\end{array} \tag{12}
$$

If f and g are invertible in $\mathbb{C}_0$, then under this correspondence ϕ is invertible in $\mathbb{C}_1$ if and only if $\hat{\phi}$ is.

(iv) *If $\mathsf{V}(\mathbb{C})'$ denotes the locally full sub-2-category of $\mathsf{V}(\mathbb{C})$ with the same objects, and as morphisms just those the vertical 1-cells which admit companions, then taking companions underlies an identity-on-objects homomorphism of bicategories $\mathsf{V}(\mathbb{C})' \to \mathcal{H}(\mathbb{C})$.*

(v) *If $f\colon X \to X'$ is a vertical 1-isomorphism in $\mathbb{C}$ and both f and f^{-1} admit companions, then $\hat{f}$ is an equivalence in $\mathcal{H}(\mathbb{C})$ with pseudoinverse $\widehat{f^{-1}}$;*

(vi) *If $\phi\colon f \Rightarrow g$ is an invertible 2-cell in $\mathsf{V}(\mathbb{C})$, and f and g admit companions, then $\hat{\phi}$ is an invertible globular 2-morphism $\hat{f} \Rightarrow \hat{g}$ in $\mathbb{C}$.*

3. Maps of double categories

In this section, we recall the various kinds of maps existing between double categories, starting with the notions of double functor and oplax double functor. In most of the paper we will work with double functors, which preserve composition and identities up to isomorphism. However, we will also need oplax

double functors, which preserve composition and identities only up to a non-invertible 2-cell, in one important situation, namely when we define oplax monoidal structure in Definition 4.1.

Definition 3.1 (Oplax double functor, double functor). Let $\mathbb{C}$ and $\mathbb{D}$ be double categories. An *oplax double functor* $F\colon \mathbb{C} \to \mathbb{D}$ consists of the following data:

- two ordinary functors $F_0\colon \mathbb{C}_0 \to \mathbb{D}_0$, $F_1\colon \mathbb{C}_1 \to \mathbb{D}_1$, denoted by the same letter F below, such that $\mathbf{s}F_1 = F_0\mathbf{s}$ and $\mathbf{t}F_1 = F_0\mathbf{t}$, as displayed in:

$$
\begin{array}{ccc}
X \xrightarrow{M} Y & & FX \xrightarrow{FM} FY \\
f\downarrow \quad \Downarrow\phi \quad \downarrow g & \mapsto & Ff\downarrow \quad \Downarrow F\phi \quad \downarrow Fg \\
X' \xrightarrow[M']{} Y' & & FX' \xrightarrow[FM']{} FY';
\end{array}
$$

- two natural transformations with components

$$
\begin{array}{ccc}
FX \xrightarrow{F(N\circ M)} FZ & & FX \xrightarrow{F(\mathrm{id}_X)} FX \\
\| \quad \Downarrow\xi_{M,N} \quad \| & & \| \quad \Downarrow\xi_X \quad \| \\
FX \xrightarrow[FM]{} FY \xrightarrow[FN]{} FZ & & FX \xrightarrow[\mathrm{id}_{FX}]{} FX.
\end{array}
\tag{13}
$$

These data are required to satisfy coherence conditions similar to those for an oplax functor between bicategories, one regarding associativity and two for unitality; see [GP99, §7.2].

A *(pseudo) double functor* $F\colon \mathbb{C} \to \mathbb{D}$ is an oplax double functor for which the 2-cells $\xi_{M,N}$ and ξ_X are invertible.

Lemma 3.2. *Let $F\colon \mathbb{C} \to \mathbb{D}$ be an oplax (pseudo) double functor. Then F induces an oplax (pseudo) functor of bicategories $\mathcal{H}(F)\colon \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{D})$. This assignment extends to an ordinary functor from the category of double categories and oplax (pseudo) double functors to the category of bicategories and oplax (pseudo) functors.*

Proof. This follows from the definition, and is an oplax analogue of [HS19, Theorem 4.1]. Recall that functors of double categories and bicategories compose strictly associatively. $\qquad\square$

Lemma 3.3. *Let $F\colon \mathbb{C} \to \mathbb{D}$ be a double functor and $f\colon X \to X'$ a vertical 1-cell of $\mathbb{C}$. Assume that f admits a companion $\widehat{f}$ with structure cells p_1, p_2. Then then vertical 1-cell $Ff\colon FX \to FX'$ of $\mathbb{D}$ admits the companion $F\widehat{f}$ via the structure cells*

$$
\begin{array}{ccc}
FX \xrightarrow{F\widehat{f}} FX' & & FX \xrightarrow{\mathrm{id}_{FX}} FX \\
Ff\downarrow \quad \Downarrow Fp_1 \quad \| & & \| \quad \Downarrow\xi_X^{-1} \quad \| \\
FX' \xrightarrow{F(\mathrm{id}_{X'})} FX' & and & FX \xrightarrow{F(\mathrm{id}_X)} FX \\
\| \quad \Downarrow\xi_{X'} \quad \| & & \| \quad \Downarrow Fp_2 \quad \downarrow Ff \\
FX' \xrightarrow[\mathrm{id}_{FX'}]{} FX'. & & FX' \xrightarrow[\mathrm{id}_{FX'}]{} FX'.
\end{array}
$$

In future, we will tend to suppress the unit coherence cells ξ appearing above, and simply write Fp_1 and Fp_2 for these pasting composites.

Since a double category has two kinds of 1-cell, vertical and horizontal, there are two kinds of natural transformations between double functors, vertical and horizontal, depending on the directions of their components. For our applications, it is the *horizontal* transformations, recalled in Definition 3.4, which will be most important, since these induce structure on the horizontal bicategory. However, the *vertical* transformations, recalled in Definition 3.5, are simpler to construct and work with, and so fundamental to our development will be the possibility of turning a vertical natural transformation into a horizontal one in the presence of well-behaved companions. The precise conditions needed are isolated in the the notion of a special vertical transformation (Definition 3.6), and are justified in Proposition 3.10, where we show that the special vertical transformations are exactly the vertical 1-cells of the functor double category (Proposition 3.9) which admit companions.

The following definition can be found *e.g.* in [Gar06b, §2.4] or [GP99, §7.4] under the name 'strong vertical transformation'.

Definition 3.4 (Horizontal transformation). Let $F, G\colon \mathbb{C} \to \mathbb{D}$ be oplax double functors. A *horizontal transformation* $\beta\colon F \Rightarrow G$ consists of

- horizontal 1-cell components $\beta_X\colon FX \rightarrowtail GX$ in $\mathbb{D}$ for each object $X \in \mathbb{C}$;
- 2-morphism components

$$
\begin{array}{ccc}
FX & \xrightarrow{\beta_X} & GX \\
{\scriptstyle Ff}\downarrow & \Downarrow\beta_f & \downarrow{\scriptstyle Gf} \\
FX' & \xrightarrow[\beta_{X'}]{} & GX'
\end{array}
$$

in $\mathbb{D}$ for each vertical 1-cell $f\colon X \to X'$ in $\mathbb{C}$;

- invertible globular coherence 2-morphisms

$$
\begin{array}{ccccc}
FX & \xrightarrow{FM} & FY & \xrightarrow{\beta_Y} & GY \\
\| & & \Downarrow\beta_M & & \| \\
FX & \xrightarrow[\beta_X]{} & GX & \xrightarrow[GM]{} & GY
\end{array}
$$

in $\mathbb{D}$ for each horizontal 1-cell $M\colon X \rightarrowtail Y$ in $\mathbb{C}$.

These data are required to satisfy, firstly, the two axioms

$$
\begin{array}{ccc}
FX & \xrightarrow{\beta_X} & GX \\
{\scriptstyle Ff}\downarrow \ \Downarrow\beta_f & & \downarrow{\scriptstyle Gf} \\
FX' & \xrightarrow{\beta_{X'}} & GX' \\
{\scriptstyle Fg}\downarrow \ \Downarrow\beta_g & & \downarrow{\scriptstyle Gg} \\
FX'' & \xrightarrow{\beta_{X''}} & GX''
\end{array}
=
\begin{array}{ccc}
FX & \xrightarrow{\beta_X} & GX \\
{\scriptstyle F(gf)}\downarrow & \Downarrow\beta_{gf} & \downarrow{\scriptstyle G(gf)} \\
& & \\
FX'' & \xrightarrow{\beta_{X''}} & GX''
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
FX & \xrightarrow{\beta_X} & GX \\
{\scriptstyle F(1_X)}\| \ \Downarrow\beta_{1_X} & & \|{\scriptstyle G(1_X)} \\
FX & \xrightarrow[\beta_X]{} & FX
\end{array}
=
\begin{array}{ccc}
FX & \xrightarrow{\beta_X} & GX \\
\| \ \Downarrow 1_{\beta_X} & & \| \\
FX & \xrightarrow[\beta_X]{} & GX
\end{array}
\tag{14}
$$

expressing that $\beta_{(-)}\colon \mathbb{C}_0 \to \mathbb{D}_1$ is a functor; then the axiom

$$
\begin{array}{ccccc}
FX & \xrightarrow{FM} & FY & \xrightarrow{\beta_Y} & GY \\
\| & & \Downarrow\beta_M & & \| \\
FX & \xrightarrow[\beta_X]{} & GX & \xrightarrow[GM]{} & GY \\
{\scriptstyle Ff}\downarrow \ \Downarrow\beta_f & & \downarrow{\scriptstyle Gf} \ \Downarrow G\phi & & \downarrow{\scriptstyle Gg} \\
FX' & \xrightarrow[\beta_{X'}]{} & GX' & \xrightarrow[GM']{} & GY'
\end{array}
=
\begin{array}{ccccc}
FX & \xrightarrow{FM} & FY & \xrightarrow{\beta_Y} & GY \\
{\scriptstyle Ff}\downarrow \ \Downarrow F\phi & & \downarrow{\scriptstyle Fg} \ \Downarrow\beta_g & & \downarrow{\scriptstyle Gg} \\
FX' & \xrightarrow[FM']{} & FY' & \xrightarrow[\beta_{Y'}]{} & GY' \\
\| & & \Downarrow\beta_{M'} & & \| \\
FX' & \xrightarrow[\beta_{X'}]{} & GX' & \xrightarrow[GM']{} & GY'
\end{array}
\tag{15}
$$

expressing that the 2-morphisms β_M are components of a natural transformation; and finally, the axioms

$$
\begin{array}{ccccccc}
FX & \xrightarrow{\hspace{2em}F(N\circ M)\hspace{2em}} & & & FZ & \xrightarrow{\beta_Z} & GZ \\
\| & & \Downarrow\xi_{M,N} & & \| & & \| \\
FX & \xrightarrow[FM]{} & FY & \xrightarrow[FN]{} & FZ & \xrightarrow[\beta_Z]{} & GZ \\
\| & & \| & & \Downarrow\beta_N & & \| \\
FX & \xrightarrow[FM]{} & FY & \xrightarrow[\beta_Y]{} & GY & \xrightarrow[GM]{} & GZ \\
\| & & \Downarrow\beta_M & & \| & & \| \\
FX & \xrightarrow[\beta_X]{} & GX & \xrightarrow[GM]{} & GY & \xrightarrow[GN]{} & GZ
\end{array}
=
\begin{array}{ccccc}
FX & \xrightarrow{\hspace{1em}F(N\circ M)\hspace{1em}} & FZ & \xrightarrow{\beta_Z} & GZ \\
\| & \Downarrow\beta_{N\circ M} & & & \| \\
FX & \xrightarrow[\beta_X]{} & GX & \xrightarrow{\hspace{1em}G(N\circ M)\hspace{1em}} & GZ \\
\| & & \| & \Downarrow\xi_{M,N} & \| \\
FX & \xrightarrow[\beta_X]{} & GX & \xrightarrow[GM]{} GY \xrightarrow[GN]{} & GZ
\end{array}
$$

$$
\begin{array}{c}
FX \xrightarrow{F(\mathrm{id}_X)} FX \xrightarrow{\beta_X} GX \\
\Vert \qquad \Downarrow\beta_{\mathrm{id}} \qquad \Vert \\
FX \xrightarrow[\beta_X]{} GX \xrightarrow[G(\mathrm{id}_X)]{} GX \\
\Vert \qquad \Vert \quad \Downarrow\xi_X \quad \Vert \\
FX \xrightarrow[\beta_X]{} GX \xrightarrow[\mathrm{id}_{GX}]{} GX \\
\Vert \qquad \qquad \Vert \\
FX \xrightarrow[\qquad\beta_X\qquad]{} GX
\end{array}
\quad = \quad
\begin{array}{c}
FX \xrightarrow{F(\mathrm{id}_X)} FX \xrightarrow{\beta_X} GX \\
\Vert \quad \Downarrow\xi_X \quad \Vert \qquad \Vert \\
FX \xrightarrow[\mathrm{id}_{FX}]{} FX \xrightarrow[\beta_X]{} GX \\
\Vert \qquad \qquad \Vert \\
FX \xrightarrow[\qquad\beta_X\qquad]{} GX
\end{array}
$$

expressing compatibility of β with the double structure of F and G.

The following definition can be found *e.g.* in [Gar06b, §2.3] or [HS19, Definition 2.8] under the name 'tight transformation'.

Definition 3.5 (Vertical transformation). Let $F, F' \colon \mathbb{C} \to \mathbb{D}$ be oplax double functors. A *vertical transformation* $\sigma \colon F \Rightarrow F'$ consists of the following data:
- vertical 1-cell components $\sigma_X \colon FX \to F'X$ in $\mathbb{D}$ for each object $X \in \mathbb{C}$;
- 2-morphism components

$$
\begin{array}{ccc}
FX & \xrightarrow{FM} & FY \\
\sigma_X \downarrow & \Downarrow\sigma_M & \downarrow \sigma_Y \\
F'X & \xrightarrow[F'M]{} & F'Y
\end{array}
\tag{16}
$$

in $\mathbb{D}$ for each horizontal 1-cell $M \colon X \nrightarrow Y$ in $\mathbb{C}$.

These data are required to satisfy, firstly, the axiom

$$
\begin{array}{ccc}
FX & \xrightarrow{FM} & FY \\
\sigma_X \downarrow & \Downarrow\sigma_M & \downarrow \sigma_Y \\
F'X & \xrightarrow{F'M} & F'Y \\
F'f \downarrow & \Downarrow F'\phi & \downarrow F'g \\
F'X' & \xrightarrow[F'M']{} & F'Y'
\end{array}
\quad = \quad
\begin{array}{ccc}
FX & \xrightarrow{FM} & FY \\
Ff \downarrow & \Downarrow F\phi & \downarrow Fg \\
FX' & \xrightarrow{FM'} & FY' \\
\sigma_{X'} \downarrow & \Downarrow\sigma_{M'} & \downarrow \sigma_{Y'} \\
F'X' & \xrightarrow[F'M']{} & F'Y'
\end{array}
\tag{17}
$$

expressing that we have two ordinary natural transformations $F_0 \Rightarrow F'_0$ and $F_1 \Rightarrow F'_1$; then the axiom

$$
\begin{array}{c}
FX \xrightarrow{F(N \circ M)} FZ \\
\Vert \quad \Downarrow\xi_{M,N} \quad \Vert \\
FX \xrightarrow{FM} FY \xrightarrow{FN} FZ \\
\sigma_X \downarrow \; \Downarrow\sigma_M \; \sigma_Y \downarrow \; \Downarrow\sigma_N \; \downarrow\sigma_Z \\
F'X \xrightarrow[F'M]{} F'Y \xrightarrow[F'N]{} F'Z
\end{array}
\quad = \quad
\begin{array}{c}
FX \xrightarrow{F(N \circ M)} FZ \\
\sigma_X \downarrow \quad \Downarrow\sigma_{N \circ M} \quad \downarrow\sigma_Z \\
F'X \xrightarrow{F'(N \circ M)} F'Z \\
\Vert \quad \Downarrow\xi_{M,N} \quad \Vert \\
F'X \xrightarrow[F'M]{} F'Y \xrightarrow[F'N]{} F'Z
\end{array}
\tag{18}
$$

expressing compatibility with horizontal composition; and finally, the axiom

$$
\begin{array}{c}
FX \xrightarrow{F(\mathrm{id}_X)} FX \\
\Vert \quad \Downarrow\xi_X \quad \Vert \\
FX \xrightarrow{\mathrm{id}_{FX}} FX \\
\sigma_X \downarrow \; \Downarrow\mathrm{id}_{\sigma_X} \; \downarrow\sigma_X \\
F'X \xrightarrow[\mathrm{id}_{F'X}]{} F'X
\end{array}
\quad = \quad
\begin{array}{c}
FX \xrightarrow{F(\mathrm{id}_X)} FX \\
\sigma_X \downarrow \; \Downarrow\sigma_{\mathrm{id}_X} \; \downarrow\sigma_X \\
F'X \xrightarrow{F'(\mathrm{id}_X)} F'X \\
\Vert \quad \Downarrow\xi_X \quad \Vert \\
F'X \xrightarrow[\mathrm{id}_{F'X}]{} F'X
\end{array}
\tag{19}
$$

expressing compatibility with horizontal identities.

It is easy to see that a horizontal transformation $\beta \colon F \Rrightarrow F'$ between double functors induces a pseudo-natural transformation $\mathcal{H}(\beta) \colon \mathcal{H}(F) \Rightarrow \mathcal{H}(F')$ between the associated homomorphisms of bicategories. On the other hand, from a vertical transformation $\sigma \colon F \Rightarrow F'$, there is no direct way of inducing anything

$\mathcal{H}(F) \Rightarrow \mathcal{H}(F')$. However, there is an indirect way of doing so, if we can first turn the vertical transformation $\sigma\colon F \Rightarrow F'$ into a horizontal one $\widehat{\sigma}\colon F \Rrightarrow F'$. The following definition isolates the properties required of σ for this to be possible.

Definition 3.6 (Special vertical transformation)**.** Let $\sigma\colon F \Rightarrow F'$ be a vertical transformation between oplax double functors $\mathbb{C} \to \mathbb{D}$. We say that σ is *special* if for every $X \in \mathbb{C}$, the vertical 1-cell component $\sigma_X\colon FX \to F'X$ has a companion $\widehat{\sigma}_X\colon FX \rightarrowtail F'X$ in $\mathbb{D}$, and the companion transposes

$$
\begin{array}{ccccc}
FX & \xrightarrow{FM} & FY & \xrightarrow{\widehat{\sigma}_Y} & F'Y \\
\Big\| & & \Downarrow \widehat{\sigma}_M & & \Big\| \\
FX & \xrightarrow[\widehat{\sigma}_X]{} & F'X & \xrightarrow[F'M]{} & F'Y
\end{array}
\tag{20}
$$

of the 2-morphism components (16) of σ are invertible.

A special vertical transformation was called a transformation with *loosely strong companions* in [HS19, Definition 4.10], characterised precisely by the following proposition; when considered in the setting of fibrant categories, it was called a *horizontally strong* transformation in [CS10, Definition A.4].

Proposition 3.7. *Let* $\sigma\colon F \Rightarrow F'$ *be a special vertical transformation between oplax double functors. The companion 1-cells* $\widehat{\sigma}_X\colon FX \rightarrowtail F'X$ *are the horizontal 1-cell components of a horizontal transformation* $\widehat{\sigma}\colon F \Rrightarrow F'$, *whose 2-morphism components* $\widehat{\sigma}_f$ *are the companion transposes of the equalities of vertical 1-cells* $\sigma_{X'} \circ Ff = F'f \circ \sigma_X$ *as in:*

$$
\begin{array}{ccc}
FX & \xrightarrow{\widehat{\sigma}_X} & F'X \\
Ff \Big\downarrow & \Downarrow \widehat{\sigma}_f & \Big\downarrow F'f \\
FX' & \xrightarrow[\widehat{\sigma}_{X'}]{} & F'X'
\end{array}
$$

and whose globular coherence 2-morphisms are given by (20)*. In particular,* σ *induces a pseudonatural transformation* $\mathcal{H}(\widehat{\sigma})\colon \mathcal{H}(F) \Rightarrow \mathcal{H}(F')$ *between the induced oplax functors of horizontal bicategories.*

Proof. The horizontal transformation axioms are a straightforward diagram chase using the universal property of companions. $\qquad\square$

We could now proceed to verify by hand further desirable properties of the construction $\sigma \mapsto \widehat{\sigma}$ (for example, its functoriality); however, this turns out to be unneccessary, as we can in fact characterise $\widehat{\sigma}$ as a companion for σ in a suitable *functor double category*, and then apply results such as Proposition 2.8. We first define these functor double categories.

Definition 3.8 (Modification)**.** Let $\beta\colon F \Rrightarrow G$ and $\beta'\colon F' \Rrightarrow G'$ be horizontal transformations and let $\sigma\colon F \Rightarrow F'$ and $\tau\colon G \Rightarrow G'$ be vertical transformations between oplax double functors $\mathbb{C} \to \mathbb{D}$. A *modification*

$$
\begin{array}{ccc}
F & \xRightarrow{\ \beta\ } & G \\
\sigma \Big\| & \Downarrow\!\!\Downarrow \gamma & \Big\| \tau \\
F' & \xRightarrow[\ \beta'\ ]{} & G'
\end{array}
$$

consists of 2-morphisms

$$
\begin{array}{ccc}
FX & \xrightarrow{\beta_X} & GX \\
\sigma_X \Big\downarrow & \Downarrow \gamma_X & \Big\downarrow \tau_X \\
F'X & \xrightarrow[\beta'_X]{} & G'X
\end{array}
$$

in $\mathbb{D}$ for every object $X \in \mathbb{C}$, subject to the naturality axiom:

$$
\begin{array}{ccccccc}
FX & \xrightarrow{\beta_X} & GX & & FX & \xrightarrow{\beta_X} & GX \\
\sigma_X \Big\downarrow & \Downarrow \gamma_X & \Big\downarrow \tau_X & & Ff \Big\downarrow & \Downarrow \beta_f & \Big\downarrow Gf \\
F'X & \xrightarrow[\beta'_X]{} & G'X & = & FX' & \xrightarrow[\beta_{X'}]{} & GX' \\
F'f \Big\downarrow & \Downarrow \beta'_f & \Big\downarrow G'f & & \sigma_{X'} \Big\downarrow & \Downarrow \gamma_{X'} & \Big\downarrow \tau_{X'} \\
F'X' & \xrightarrow[\beta'_{X'}]{} & G'X' & & F'X' & \xrightarrow[\beta'_{X'}]{} & G'X'
\end{array}
\ ;
$$

and the following axiom expressing compatibility with β, β', σ, τ:

$$
\begin{array}{ccc}
FX \xrightarrow{FM} FY \xrightarrow{\beta_Y} GY & & FX \xrightarrow{FM} FY \xrightarrow{\beta_Y} GY \\
\| \quad \Downarrow\beta_M \quad \| & & \sigma_X\downarrow \quad \Downarrow\sigma_M \ \ \sigma_Y\downarrow \quad \Downarrow\gamma_Y \quad \downarrow\tau_Y \\
FX \xrightarrow{\beta_X} GX \xrightarrow{GM} GY & = & F'X \xrightarrow{F'M} F'Y \xrightarrow{\beta'_Y} G'Y \\
\sigma_X\downarrow \ \Downarrow\gamma_X \ \tau_X\downarrow \ \Downarrow\tau_M \ \downarrow\tau_Y & & \| \quad \Downarrow\beta'_M \quad \| \\
F'X \xrightarrow{\beta'_X} G'X \xrightarrow{G'M} G'Y. & & F'X \xrightarrow{\beta'_X} G'X \xrightarrow{G'M} G'Y.
\end{array}
$$

Proposition 3.9 (Functor double categories). *Let $\mathbb{C}$ and $\mathbb{D}$ be double categories. There is a double category $\mathsf{DblCat}[\mathbb{C}, \mathbb{D}]$ (resp., $\mathsf{DblCat}_{\mathrm{oplax}}[\mathbb{C}, \mathbb{D}]$) composed of double functors from $\mathbb{C}$ to $\mathbb{D}$ (resp., oplax double functors), vertical transformations, horizontal transformations and modifications.*

Proof. Each of the forms of vertical and horizontal composition is given by composing the relevant component data in the same direction; verifying the axioms is routine. The only point requiring any further note is that, for composable horizontal transformations $\beta\colon F \Rightarrow G$ and $\delta\colon G \Rightarrow H$, the globular coherence 2-isomorphisms of the composite $\delta \circ \beta\colon F \Rightarrow H$ are given by

$$
\begin{array}{ccccc}
FX & \xrightarrow{FM} & FY & \xrightarrow{\beta_Y} & GY & \xrightarrow{\delta_Y} & HY \\
\| & & \Downarrow\beta_M & & \| & & \| \\
FX & \xrightarrow{\beta_X} & GX & \xrightarrow{GM} & GY & \xrightarrow{\delta_Y} & HY \\
\| & & & \| & & \Downarrow\delta_M & \| \\
FX & \xrightarrow{\beta_X} & GX & \xrightarrow{\delta_X} & HX & \xrightarrow{HM} & HY.
\end{array} \qquad \square
$$

Proposition 3.10. *Let $\sigma\colon F \Rightarrow F'$ be a vertical transformation between double functors (resp., oplax double functors). Then σ has a companion as a vertical 1-cell in the double category $\mathsf{DblCat}[\mathbb{C}, \mathbb{D}]$ (resp., $\mathsf{DblCat}_{\mathrm{oplax}}[\mathbb{C}, \mathbb{D}]$) if and only if it is special in the sense of Definition 3.6.*

Proof. If σ is special, then we have an associated horizontal transformation $\widehat{\sigma}\colon F \Rightarrow F'$ via Proposition 3.7. Moreover, we can define modifications

$$
\begin{array}{ccc}
F \xRightarrow{\widehat{\sigma}} F' & & F \xRightarrow{\mathrm{id}_F} F \\
\sigma\Downarrow \ \ \Downarrow p_1 \ \ \| & \text{and} & \| \ \ \Downarrow p_2 \ \ \Downarrow\sigma \\
F' \xRightarrow[\mathrm{id}_{F'}]{} F' & & F \xRightarrow[\widehat{\sigma}]{} F'
\end{array} \tag{21}
$$

whose component 2-morphisms are those witnessing that each $\widehat{\sigma}_X$ is a companion of σ_X; the modification axioms of Definition 3.8 are now easily verified, and it is clear that these modifications satisfy the companion axioms since they do so componentwise.

Suppose conversely that σ has a companion in the functor double category, namely a horizontal transformation $\widehat{\sigma}$ with the modifications witnessing this given as in (21). The components of these modifications witness that each horizontal 1-cell $\widehat{\sigma}_X$ is a companion for the vertical 1-cell σ_X. Furthermore, the second modification axiom for p_1 ensures that the invertible coherence 2-morphism $\widehat{\sigma}_M$ is the companion transpose of the 2-morphism component σ_M; in particular, this says that σ is special as required. $\square$

As a sample application of the utility of this result, let us use it to give an efficient proof of:

Proposition 3.11. *Let $\sigma\colon F \Rightarrow F'\colon \mathbb{C} \to \mathbb{D}$ be an invertible vertical transformation between oplax double functors. If the 1-cell components of σ and σ^{-1} have companions, then they induce a horizontal equivalence $\widehat{\sigma}\colon F \Rightarrow F'$ and so a pseudonatural equivalence $\widehat{\sigma}\colon \mathcal{H}(F) \Rightarrow \mathcal{H}(F')$ between oplax functors of bicategories.*

Proof. Since σ is invertible and its components have companions, it is special; likewise, σ^{-1} is special. So by Proposition 3.10, both σ and σ^{-1} admit companions in $\mathsf{DblCat}_{\mathrm{oplax}}[\mathbb{C}, \mathbb{D}]$. It follows by Proposition 2.8(v) that $\widehat{\sigma}$ is an equivalence in $\mathcal{H}(\mathsf{DblCat}_{\mathrm{oplax}}[\mathbb{C}, \mathbb{D}])$ as desired. $\square$

We conclude this section with a miscellaneous technical lemma concerning components of a vertical transformation, which will be used in Sections 6, 9 and 10.

Lemma 3.12. *Let $\sigma\colon F \Rightarrow F'$ be a vertical transformation and $f\colon X \to X'$ be a vertical 1-cell in $\mathbb{C}$. If f has a companion $\widehat{f}$, then the component $\sigma_{\widehat{f}}$ is the transpose of the naturality vertical identity as in*

$$
\begin{array}{ccc}
FX \xrightarrow{F\widehat{f}} FX' & & \\
\sigma_X \downarrow \quad \Downarrow\sigma_{\widehat{f}} \quad \downarrow \sigma_{X'} & = & \\
F'X \xrightarrow[F'\widehat{f}]{} F'X' & &
\end{array}
$$

Proof. It suffices to show these two 2-morphisms have the same companion transposes, which follows by the calculation (in which we again suppress unit coherence 2-morphisms for F and F'):

using naturality of σ; the companion axiom (10); and axiom (19) for a vertical transformation. $\square$

4. MONOIDAL DOUBLE CATEGORIES

The aim of this section is to introduce the notions of *monoidal double category* and *oplax monoidal double category*, and to prove some useful facts about them. Both notions describe double categories endowed with a monoidal product: the key difference is that in the former case, this tensor product is a double functor, while in the latter case, it is merely an oplax double functor as in Definition 3.1. In particular, it should be emphasised that 'oplax' only modifies the functoriality of the tensor product, rather than the nature of the associativity and unit constraints for this tensor, which for us will always be invertible.

In what follows, we will be concerned with the the question of extending a *monoidal* structure on a double category to an *oplax monoidal* structure on an associated Kleisli double category. Since we need both notions, we here define them simultaneously.

Definition 4.1 (Oplax monoidal double category, monoidal double category)**.** Let $\mathbb{C}$ be a double category. An *oplax monoidal structure* on $\mathbb{C}$ consists of the following data:

- an oplax double functor $\otimes\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$;
- an oplax double functor $I\colon \mathbf{1} \to \mathbb{C}$;
- invertible vertical transformations $\alpha\colon \otimes \circ (1 \times \otimes) \Rightarrow \otimes \circ (\otimes \times 1)$, $\lambda\colon \otimes \circ (I \times 1) \Rightarrow 1$ and $\rho\colon \otimes \circ (1 \times I) \Rightarrow 1$

satisfying the usual Mac Lane coherence axioms for α, λ and ρ. Said another way, this structure amounts to the following:

- monoidal structures $(\otimes_0, I_0)$ and $(\otimes_1, I_1)$ on the categories $\mathbb{C}_0$ and $\mathbb{C}_1$;
- *strict* monoidality of $\mathbf{s}, \mathbf{t}\colon \mathbb{C}_1 \rightrightarrows \mathbb{C}_0$. For example, the associativity constraint for $\mathbb{C}_1$ has components

$$
\begin{array}{ccc}
(X_1 \otimes X_2) \otimes X_2 & \xrightarrow{(M_1 \otimes M_2) \otimes M_3} & (Y_1 \otimes Y_2) \otimes Y_3 \\
\alpha_{X_1,X_2,X_3} \downarrow & \Downarrow \alpha_{M_1,M_2,M_3} & \downarrow \alpha_{Y_1,Y_2,Y_3} \\
X_1 \otimes (X_2 \otimes X_3) & \xrightarrow[M_1 \otimes (M_2 \otimes M_3)]{} & Y_1 \otimes (Y_2 \otimes Y_3);
\end{array}
\tag{22}
$$

- globular 2-morphisms

$$
\begin{array}{ccc}
X_1 \otimes X_2 \xrightarrow{(N_1 \circ M_1) \otimes (N_2 \circ M_2)} Z_1 \otimes Z_2 & & X_1 \otimes X_2 \xrightarrow{\mathrm{id}_{X_1} \otimes \mathrm{id}_{X_2}} X_1 \otimes X_2 \\
\parallel \quad\quad \Downarrow\tau \quad\quad \parallel & \text{and} & \parallel \quad\quad \Downarrow\eta \quad\quad \parallel \\
X_1 \otimes X_2 \xrightarrow[M_1 \otimes M_2]{} Y_1 \otimes Y_2 \xrightarrow[N_1 \otimes N_2]{} Z_1 \otimes Z_2 & & X_1 \otimes X_2 \xrightarrow[\mathrm{id}_{X_1 \otimes X_2}]{} X_1 \otimes X_2,
\end{array} \tag{23}
$$

subject to axioms that make $\otimes$ into an oplax double functor;
- globular 2-morphisms

$$
\begin{array}{ccc}
I_0 \xrightarrow{I_1} I_0 & & I_0 \xrightarrow{I_1} I_0 \\
\parallel \quad \Downarrow\delta \quad \parallel & \text{and} & \parallel \quad \Downarrow\iota \quad \parallel \\
I_0 \xrightarrow[I_1]{} I_0 \xrightarrow[I_1]{} I_0 & & I_0 \xrightarrow[\mathrm{id}_{I_0}]{} I_0,
\end{array} \tag{24}
$$

subject to axioms that make I into an oplax double functor;
- two axioms ensuring that the associativity constraint α is a vertical transformation between oplax double functors;
- four axioms ensuring that the unit constraints λ and ρ are vertical transformations between oplax double functors.

The above axioms are written explicitly in Appendix A.1. We have a *monoidal double category* when the tensor and unit are specified by double functors, rather than oplax double functors; said another way, when each of the 2-morphisms in (23) and (24) is invertible.

What we call here an oplax monoidal double category is what is called simply a monoidal double category in [GP04, §5.5]; it is equally well a pseudomonoid in the cartesian monoidal 2-category of double categories, oplax double functors and vertical transformations.

Remark 4.2. We defined a monoidal double category to be an oplax monoidal double category satisfying some additional properties; but these additional properties in fact allow us to simplify the structure further, as explained in [HS19, Page 8]. Indeed, in a monoidal double category, the monoidal unit I_1 of $\mathbb{C}_1$ is always canonically isomorphic to id_{I_0} via ι; and it does no harm to assume that, in fact, I_1 is id_{I_0} and ι is the identity 2-morphism—which in turn forces $\delta = \ell_{\mathrm{id}_{I_0}} = r_{\mathrm{id}_{I_0}}$ for the globular isomorphisms (7) for horizontal identities in $\mathbb{C}$. As such, if in specifying a monoidal double category we follow these conventions, then we need only provide the invertible structure 2-morphisms τ and η satisfying the appropriate coherence axioms. By contrast, in an oplax monoidal double category, none of the data are redundant: indeed, δ and ι as in (24) now specify a *comonad* structure on I_1 in $\mathcal{H}(\mathbb{C})$, see (78).

Moreover, notice that just as a double category is an internal pseudocategory in the 2-category of small categories, functors and natural transformations, a monoidal double category is an internal pseudocategory in the 2-category of monoidal categories, lax monoidal functors and monoidal transformations for which the source and target functors are strict monoidal.

As mentioned in the introduction, the notion of oplax monoidal structure will be exploited in future work in order to provide a general notion of *commuting tensor product*, generalising the theory of [GL16], which will in particular recover the Boardman–Vogt tensor product of symmetric coloured operads and its extension to operadic bimodules in [DH14]. For these applications, it will be important that the oplax monoidal structure is *normal* in the sense of the following definition.

Definition 4.3 (Normal oplax monoidal double category). An oplax monoidal double category $\mathbb{C}$ is said to be *normal* if:

(i) $I \colon \mathbf{1} \to \mathbb{C}$ is a (pseudo) double functor;
(ii) for all objects $X_1, X_2 \in \mathbb{C}$, the following restricted oplax double functors are (pseudo) double functors:

$$
X_1 \otimes (-) \colon \mathbb{C} \cong \mathbf{1} \times \mathbb{C} \xrightarrow{X_1 \times \mathbb{C}} \mathbb{C} \times \mathbb{C} \xrightarrow{\otimes} \mathbb{C} \quad \text{and} \quad (-) \otimes X_2 \colon \mathbb{C} \cong \mathbb{C} \times \mathbf{1} \xrightarrow{\mathbb{C} \times X_2} \mathbb{C} \times \mathbb{C} \xrightarrow{\otimes} \mathbb{C}. \tag{25}
$$

Said another way, both 2-morphisms δ and ι in (24) are invertible; while in (23), η is invertible, and each τ for which $M_1 = N_1 = \mathrm{id}_{X_1}$ or $M_2 = N_2 = \mathrm{id}_{X_2}$ is also invertible.

We give now some examples of monoidal double categories and oplax monoidal double categories. The example of profunctors in Example 4.6 will be fundamental for our application in Section 10.

Example 4.4 (Duoidal categories). Recall from Example 2.2 that a monoidal category $(\mathcal{V}, \circ, J)$ is the same thing as a double category with a single object and only identity vertical arrow. To equip this double category with an oplax monoidal structure in the sense of Definition 4.1 is the same thing as equipping $\mathcal{V}$ with additional structure making it into a duoidal category [AM10]. Explicitly, this amounts to providing a second monoidal structure $(\otimes, I)$ on $\mathcal{V}$, along with maps

$$\xi \colon (Y_1 \circ X_1) \otimes (Y_2 \circ X_2) \to (Y_1 \otimes Y_2) \circ (X_1 \otimes X_2),$$
$$\mu \colon J \otimes J \to J, \qquad \gamma \colon I \to I \circ I, \qquad \nu \colon I \to J,$$

satisfying appropriate axioms. Here, the interchange law ξ corresponds to the square τ in (23).

This oplax monoidal structure is a genuine monoidal structure whenever all of ξ, μ, γ and ν are invertible. In this case, by the Eckmann–Hilton argument, the identity functor underlies a monoidal isomorphism $(\mathcal{V}, \otimes, I) \to (\mathcal{V}, \circ, J)$, and the two isomorphic monoidal structures are each *braided*. Loosely, then, we may say that in this situation, $\mathcal{V}$ 'is' a braided monoidal category. In particular, if we merely start with a braided monoidal category $(\mathcal{V}, \otimes, I)$, then it becomes duoidal on taking $\circ = \otimes$, $J = I$, $\nu = \mathrm{id}$, $\mu = r_I$, $\gamma = r_I^{-1}$, and ξ the canonical constraint built from associativity and braiding maps.

Returning to the general situation, the oplax monoidal structure on $\mathcal{V}$ *qua* double category is normal if, and only if, the duoidal structure on $\mathcal{V}$ is *normal* meaning that ν, γ and μ are all invertible. The only non-obvious point is that $\mathcal{V}$ being normal duoidal implies Definition 4.3(ii); but this latter condition amounts to the invertibility of ξ when $X_1 = Y_1 = J$ or $X_2 = Y_2 = J$, and these follow from the oplax monoidality of the unit constraints for $\otimes$. The notion of normal duoidal category as an enrichment base was central to the theory of commuting tensor products developed in [GL16], and in future work, we will see that normal oplax monoidal double categories are a suitable setting for a generalisation of that theory.

Example 4.5 (Oplax monoidal structure on $\mathbb{M}\mathsf{at}_\mathcal{V}$). Recall from Example 2.3 that, for a monoidal category $(\mathcal{V}, \circ, J)$ in which the tensor product preserves coproducts in each variable, we have a double category $\mathbb{M}\mathsf{at}_\mathcal{V}$ of $\mathcal{V}$-matrices. If $\mathcal{V}$ is further equipped with a second monoidal structure $(\otimes, I)$ which also preserves coproducts in each variable, and data as above making it into a duoidal category, then $\mathbb{M}\mathsf{at}_\mathcal{V}$ acquires an oplax monoidal structure. The tensor product is given by the cartesian product of sets and functions on the vertical level, and for $\mathcal{V}$-matrices $M_1 \colon X_1 \longrightarrow Y_1$ and $M_2 \colon X_2 \longrightarrow Y_2$ the tensor $M_1 \otimes M_2 \colon X_1 \times X_2 \longrightarrow Y_1 \times Y_2$ defined by letting

$$(M_1 \otimes M_2)((y_1, y_2), (x_1, x_2)) = M_1(y_1, x_1) \otimes M_2(y_2, x_2).$$

The monoidal unit is $I \colon 1 \longrightarrow 1$ with unique component $I_{*,*} = I$. The structure cells (23) and (24) are formed using the duoidal structure maps, with the most complex case being that the 2-morphism $\tau \colon (N_1 \circ M_1) \otimes (N_2 \circ M_2) \to (N_1 \otimes N_2) \circ (M_1 \otimes M_2)$ has components:

$$\left(\textstyle\sum_{y_1} N_1(z_1, y_1) \circ M_1(y_1, x_1) \right) \otimes \left(\textstyle\sum_{y_2} N_2(z_2, y_2) \circ M_2(y_2, x_2) \right)$$
$$\downarrow \cong$$
$$\textstyle\sum_{y_1, y_2} \left(N_1(z_1, y_1) \circ M_1(y_1, x_1) \right) \otimes \left(N_2(z_2, y_2) \circ M_2(y_2, x_2) \right)$$
$$\downarrow \scriptstyle\sum_{y_1, y_2} \xi$$
$$\textstyle\sum_{y_1, y_2} \left(N_1(z_1, y_1) \otimes N_2(z_2, y_2) \right) \circ \left(M_1(y_1, x_1) \otimes M_2(y_2, x_2) \right).$$

It is not hard to see that this oplax monoidal structure is normal precisely when $\mathcal{V}$ is normal as a duoidal category in the sense of the preceding example, and that it is genuinely monoidal just when the duoidal structure of $\mathcal{V}$ comes from a braided monoidal structure.

Example 4.6 (Monoidal structure on $\mathbb{P}\mathsf{rof}_\mathcal{V}$). Let $\mathcal{V}$ be a braided monoidal category in which the tensor product preserves colimits in each variable and recall the double category $\mathbb{P}\mathsf{rof}_\mathcal{V}$ of Example 2.4. This double category admits a monoidal structure extending that of $\mathbb{M}\mathsf{at}_\mathcal{V}$ in the braided case. On objects and vertical 1-cells, this is simply the monoidal structure of the 2-category $\mathsf{Cat}_\mathcal{V}$. On horizontal 1-cells, given $\mathcal{V}$-profunctors $M_1 \colon X_1 \longrightarrow Y_1$ and $M_2 \colon X_2 \longrightarrow Y_2$, the $\mathcal{V}$-profunctor $M_1 \otimes M_2 \colon X_1 \otimes Y_2 \longrightarrow Y_1 \otimes Y_2$ is given by

$$(M_1 \otimes M_2)\big((y_1, y_2)(x_1, x_2)\big) = M_1(y_1, x_1) \otimes M_2(y_2, x_2), \tag{26}$$

with a corresponding definition on 2-morphisms. The key structure isomorphism τ in (23) is formed using the braiding of the tensor product on $\mathcal{V}$ and the fact that it preserves colimits in each variable.

One can carry out the construction above also when $\mathcal{V}$ is merely a duoidal category, thereby extending Example 4.5, but we shall not need this level of generality for our application in Section 10.

One of the main results of [HS19], building on [Shu10, GG09], is that under suitable assumptions, the horizontal bicategory of a monoidal double category is a monoidal bicategory in the sense of [GPS95]; see Theorem 1.1 of *op. cit.* In view of our application in Section 10, we would like a generalisation of this result which endows the horizontal bicategory of an oplax monoidal double category with an oplax monoidal structure.

The first obstacle to be faced is the definition of oplax monoidal structure on a bicategory $\mathcal{K}$. As before, 'oplax' refers to the strictness of the tensor product functor, and so we might attempt the following naive adaptation of the usual notion of monoidal bicategory. Firstly, we would require oplax functors of bicategories $\otimes\colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and $I\colon \mathbf{1} \to \mathcal{K}$; then pseudonatural equivalences

$$\tag{27}$$

giving the associativity and unit constraints; then invertible modifications

$$\tag{28}$$

$$\tag{29}$$

$$\tag{30}$$

witnessing the pseudo-coherence of the constraint cells; and, finally, the appropriate coherence axioms for these pseudo-coherences, as found, for example, in [McC99, §A.1].

The problem with this definition can be seen in (28) above. In the domain of π we have, among other things, the pseudonatural transformation $\alpha \times 1$ whiskered by the oplax functor $\otimes\colon \mathcal{K}^2 \to \mathcal{K}$. However, such a composition does *not* yield another pseudonatural transformation, nor even a lax or oplax transformation. So π is not well-posed; and similar issues arise for μ, L and R in (29) and (30).

We will resolve this issue by imposing a further constraint on the components of the pseudonatural equivalences α, λ, ρ of (27) which we will term *centrality*, loosely inspired by the nomenclature of [PR97]. Centrality of the components of a pseudonatural transformation γ will ensure that composites of the form $\gamma \otimes 1 := \otimes \circ (\gamma \times 1)$ and $1 \otimes \gamma := \otimes \circ (1 \times \gamma)$ are well-posed pseudonatural transformations; and, furthermore, that the same pseudonaturality holds for any iterated tensorings such as $((1 \otimes \gamma) \otimes 1) \otimes 1$. Applied to the case of α, λ and ρ, this will ensure that the transformations appearing in (28) to (30), as well as all of those appearing in the corresponding coherence axioms, make sense.

Definition 4.7. Let $\mathcal{K}$ be a bicategory endowed with an oplax functor $(\otimes, \tau, \eta)\colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$, an oplax functor $(I, \delta, \iota)\colon 1 \to \mathcal{K}$ and pseudonatural equivalences as in (27). A 1-cell $f\colon X \to Y$ of $\mathcal{K}$ is said to be

central when for all maps $g\colon X' \to X$, $h\colon Y \to Y'$, $k\colon W \to W'$ and $\ell\colon Z \to Z'$, the following composite oplax structure cells are invertible:

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
\tag{31}
$$

Note that in (31), we consider three-fold tensor products bracketed to the *left*. We could equally have chosen to bracket to the *right*, but this would make no difference, since composing (31) with the components of α and their pseudoinverses would yield invertibility of the corresponding cells for the other bracketing.

Now, by taking $k = \mathrm{id}_I$ or $\ell = \mathrm{id}_I$ in (31), and composing with the components of λ or ρ and their pseudoinverses, we obtain the invertibility of oplax constraints of the following forms:

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
$$

Because of this, if $\gamma\colon F \Rightarrow G\colon \mathcal{L} \to \mathcal{K}$ is a pseudonatural transformation with central components, then both $\gamma \otimes 1$ and $1 \otimes \gamma$ will also be pseudonatural; for example, in the case of $\gamma \otimes 1$, the pseudonaturality of the object components $\gamma_X \otimes \mathrm{id}_C \colon FX \otimes C \to GX \otimes C$ is witnessed by the invertible 2-cells

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
$$

In a similar way, the general form of (31) implies that $(1 \otimes \gamma) \otimes 1$ is also pseudonatural; note that this does *not* seem to follow from the pseudonaturality of $1 \otimes \gamma$ and $\gamma \otimes 1$. However, once we have pseudonaturality of $(1 \otimes \gamma) \otimes 1$, we obtain *a fortiori* that of, say, $(1 \otimes \gamma) \otimes (1 \otimes 1)$ and so by composing with the equivalence components of α, the pseudonaturality of $((1 \otimes \gamma) \otimes 1) \otimes 1$. By following this pattern, we see that *any* tensoring of γ with identity pseudonatural transformations will again be pseudonatural.

In particular, if we require the pseudonatural transformations α, λ and ρ to themselves have central components, then we see that every 2-cell pasting which appears in the axioms (28) to (30) will be a well-posed pseudonatural transformation, and likewise for the pastings appearing in the coherence axioms. Thus, we are justified in giving:

Definition 4.8. Let $\mathcal{K}$ be a bicategory. An *oplax monoidal structure* on $\mathcal{K}$ consists of:

- an oplax functor of bicategories $\otimes\colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$;
- an oplax homomorphism $I\colon \mathbf{1} \to \mathcal{K}$;
- pseudonatural equivalences α, λ, ρ as in (27), whose components are central;
- invertible modifications π, μ, L, R as in (28)–(30);

satisfying the coherence axioms for a monoidal bicategory as found, for example, in [McC99, §A.1]. Like in Definition 4.3, we say that the oplax monoidal structure on $\mathcal{K}$ is *normal* if $I\colon \mathbf{1} \to \mathcal{K}$ is a homomorphism of bicategories, and $\otimes$ is pseudo in each variable, *i.e.* for each $X, Y \in \mathcal{K}$ the oplax functors $X \otimes (-)\colon \mathcal{K} \to \mathcal{K}$ and $(-) \otimes Y\colon \mathcal{K} \to \mathcal{K}$ are homomorphisms of bicategories.

We now explain how a normal oplax monoidal double category $\mathbb{C}$ gives rise to a normal oplax monoidal bicategory. First of all, the oplax double functor $\otimes$ and the pseudo double functor I induce functors on the horizontal bicategory

$$
\otimes\colon \mathcal{H}(\mathbb{C}) \times \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C}), \quad I\colon \mathbf{1} \to \mathcal{H}(\mathbb{C})
\tag{32}
$$

which we denote with the same symbol. Here, as per Lemma 3.2, $\otimes$ is an oplax functor of bicategories (which is pseudo in each variable) and I is a homomorphism of bicategories. If now we assume that the components of the invertible vertical transformations α, λ and ρ associated to the monoidal structure on $\mathbb{C}$ have companions, then they will induce pseudonatural equivalences

$$\widehat{\alpha}\colon \otimes \circ (1 \times \otimes) \Rightarrow \otimes \circ (\otimes \times 1) \qquad \widehat{\lambda}\colon \otimes \circ (I \times 1) \Rightarrow 1 \qquad \widehat{r}\colon 1 \Rightarrow \otimes \circ (1 \times I) \qquad (33)$$

between oplax functors of bicategories, according to Proposition 3.11 and since $\mathcal{H}$ is functorial. To proceed further, we need the components of $\widehat{\alpha}, \widehat{\lambda}$ and $\widehat{\rho}$ to be central. This will be a consequence of the following lemma. Note that *normality* of the oplax monoidal structure on $\mathbb{C}$ is important for the proof. It is not clear to us if the corresponding result without it would hold; however, since normality will be present in our applications, we have not pursued this point any further.

Lemma 4.9. *Let $\mathbb{C}$ be a normal oplax monoidal double category in which the vertical 1-cells giving associativity, left and right unit constraints have companions. If $f\colon X \to X'$ is any vertical isomorphism in $\mathbb{C}$ that has a companion, then $\widehat{f}\colon X \nrightarrow X'$ is central in the horizontal bicategory $\mathcal{H}(\mathbb{C})$ with respect to the structure of (32) and (33).*

Proof. We must show that each structure cell (31), as displayed left below, is invertible. Because $\otimes$ is pseudo in each variable and f has companion $\widehat{f}$, it follows that $(1 \otimes f) \otimes 1$ has companion $(\mathrm{id} \otimes \widehat{f}) \otimes \mathrm{id}$. Thus, $\tau(\tau \otimes 1)$ as displayed to the left will be invertible just when its companion transpose, as to the right, is also invertible:

$$
\begin{array}{ccc}
(X \otimes Y) \otimes Z \xrightarrow{(k\otimes(g\circ\widehat{f}))\otimes\ell} (W \otimes U) \otimes V & \qquad & (X \otimes Y) \otimes Z \xrightarrow{(k\otimes(g\circ\widehat{f}))\otimes\ell} (W \otimes U) \otimes V \\
\| \qquad \Downarrow\tau(\tau\otimes1) \qquad \| & \qquad & {\scriptstyle(1\otimes f)\otimes1}\downarrow \quad \Downarrow\tau\widehat{(\tau\otimes1)} \quad \| \\
(X \otimes Y) \otimes Z \xrightarrow[(\mathrm{id}\otimes\widehat{f})\otimes\mathrm{id}]{} (X \otimes Y') \otimes Z \xrightarrow[(k\otimes g)\otimes\ell]{} (W \otimes U) \otimes V & \qquad & (X \otimes Y') \otimes Z \xrightarrow[(k\otimes g)\otimes\ell]{} (W \otimes U) \otimes V\ .
\end{array}
$$

We claim that this companion transpose is, in fact, the tensor product in $\mathbb{C}$ of the three 2-morphisms displayed below, and hence invertible:

$$
\left(
\begin{array}{ccc}
X \xrightarrow{k} W \\
{\scriptstyle(1\otimes f)\otimes1}\downarrow \quad \Downarrow1_k \quad \| \\
X \xrightarrow[k]{} W
\end{array}
\right)
\otimes
\left(
\begin{array}{ccccc}
Y \xrightarrow{\widehat{f}} Y \xrightarrow{g} U \\
{\scriptstyle f}\downarrow \quad \Downarrow p_1 \quad \| \quad \Downarrow1_g \quad \| \\
Y' \xrightarrow[\mathrm{id}]{} Y' \xrightarrow[g]{} U
\end{array}
\right)
\otimes
\left(
\begin{array}{ccc}
Z \xrightarrow{\ell} V \\
{\scriptstyle1}\downarrow \quad \Downarrow1_\ell \quad \| \\
Z \xrightarrow[\ell]{} V
\end{array}
\right)\ .
$$

This claim can be verified by pasting both 2-morphisms with $(1 \otimes p_2) \otimes 1$ and using the axioms (10) and naturality of the components of τ. In an analogous way, but pasting with $(1 \otimes p_1^{-1}) \otimes 1$ in place of $(1 \otimes p_2) \otimes 1$, we can verify that the other composite structure cell of (31) is also invertible. $\square$

Theorem 4.10. *If $\mathbb{C}$ is a normal oplax monoidal double category in which the vertical 1-cells giving associativity, left and right unit constraints have companions, then the horizontal bicategory $\mathcal{H}(\mathbb{C})$ inherits a normal oplax monoidal structure with underlying data (32) and (33).*

Proof. Since the pseudonatural transformations $\widehat{\alpha}, \widehat{\lambda}$ and $\widehat{\rho}$ of (33) have as their components the horizontal companions of vertical isomorphisms, we can apply Lemma 4.9 to see that these components are all central in the sense of Definition 4.7. We now need to provide the four invertible modifications of Definition 4.8 for $\mathcal{H}(\mathbb{C})$ to have the structure of a normal oplax monoidal bicategory. The components of (28)–(30) are of the form

$$
\begin{array}{c}
(X_1 \otimes X_2) \otimes (X_3 \otimes X_4) \\
{}^{\widehat{\alpha}}\nearrow \qquad \searrow{}^{\widehat{\alpha}} \\
((X_1 \otimes X_2) \otimes X_3) \otimes X_4 \qquad X_1 \otimes (X_2 \otimes (X_3 \otimes X_4)) \\
\Downarrow\pi \\
{}_{\widehat{\alpha}\otimes\mathrm{id}}\searrow \qquad \nearrow{}_{\mathrm{id}\otimes\widehat{\alpha}} \\
(X_1 \otimes (X_2 \otimes X_3)) \otimes X_4 \xrightarrow[\widehat{\alpha}]{} X_1 \otimes ((X_2 \otimes X_3) \otimes X_4)
\end{array}
$$

$$(X_1 \otimes I) \otimes X_2 \xrightarrow{\quad \hat{\alpha} \quad} X_1 \otimes (I \otimes X_2)$$

$$\Downarrow \mu$$

$$\hat{\rho} \otimes \mathrm{id} \searrow \qquad X_1 \otimes X_2 \qquad \swarrow \mathrm{id} \otimes \hat{\lambda}$$

$$(X_1 \otimes X_2) \otimes I \xrightarrow{\quad \hat{\alpha} \quad} X_1 \otimes (X_2 \otimes \lambda)$$

$$\Downarrow R$$

$$\hat{\rho} \searrow \qquad X_1 \otimes X_2 \qquad \swarrow \mathrm{id} \otimes \hat{\rho}$$

$$(I \otimes X_1) \otimes X_2 \xrightarrow{\quad \hat{\alpha} \quad} I \otimes (X_1 \otimes X_2)$$

$$\Downarrow L$$

$$\hat{\lambda} \otimes \mathrm{id} \searrow \qquad X_1 \otimes X_2 \qquad \swarrow \hat{\lambda}$$

Notice that the two sides in each case are companions of the corresponding sides of the pentagon axiom, the triangle axiom and two known equations for the ordinary monoidal category $\mathbb{C}_0$, due to Proposition 2.8 and Lemma 3.3. For example, since $\otimes$ is a pseudo double functor in each variable, each $(-) \otimes X$ and $Y \otimes (-)$ preserves companions thus $\hat{\alpha} \otimes \mathrm{id}_X$ is canonically a companion of $\alpha \otimes 1_X$. As a result, we take π, μ, L, R to be the unique isomorphisms between companions of the same vertical 1-cells. It can then be verified that these invertible cells form a modification between pseudonatural transformations of oplax double functors by [Shu10, Lemma 4.8][3].

Finally, the three equations that relate those π, μ, L, R can be checked in exactly the same way as in the proof of [Shu10, Theorem 5.1]. In more detail, the domain and codomain of the pasted 2-cells involved in the equations are companions of the same isomorphism in $\mathbb{C}_0$, namely the unique $(((X_1 \otimes X_2) \otimes X_3) \otimes X_4) \otimes X_5 \cong X_1 \otimes (X_2 \otimes (X_3 \otimes (X_4 \otimes X_5)))$ as well as the associator $(X_1 \otimes X_2) \otimes X_3 \cong X_1 \otimes (X_2 \otimes X_3)$. Using a collection of technical lemmas [Shu10, Lemma 3.11, 3.14, 3.15, 3.19, 4.10] concerning the composition as well as the tensoring of the unique isomorphisms between companions (the latter adjusted in the normal oplax monoidal case in a straightforward way), we deduce that there can only be a unique invertible 2-cell inside each one of the diagrams, hence the equations must hold. $\square$

5. Maps of monoidal double categories

For our development in Sections 7 and 8, we will need results concerning both double monads and pseudomonoidal double monads. It turns out that many of these results can be proved uniformly across the two cases, by exhibiting both kind of structure as monoids in suitable endofunctor double categories. This is much as ordinary monads and monoidal monads can be seen as monoids in appropriate endofunctor categories. In order to do this for the case of pseudomonoidal double monads, we need to construct a suitable double category of (lax) monoidal double functors and monoidal transformations. While the notions of lax monoidal double functor and monoidal horizontal transformation (recalled in Definition 5.1 and Definition 5.2 below) are as expected, it turns out that in our motivating examples, the vertical transformations which we need are not monoidal in the obvious way but only *pseudomonoidal*. While this may seem an innocuous change, it adds an additional layer of subtlety to our development, very much in analogy with what happens in the purely 2-categorical setting [HP02].

We being with the notion of lax monoidal double functor. If we view monoidal double categories as pseudomonoids in a 2-category of double categories, double functors and vertical transformations, then the lax monoidal functors are simply the lax morphisms of pseudomonoids. This definition can also be found in [HS19, Definition 2.14], although with the orientations of the (invertible) τ and η reversed.

Definition 5.1 (Lax monoidal double functor). Let $\mathbb{C}$ and $\mathbb{D}$ be monoidal double categories. A *lax monoidal double functor* $F \colon \mathbb{C} \to \mathbb{D}$ is a (pseudo) double functor equipped with:

---

[3]Although the cited result refers to vertical transformations between *pseudo* double functors, the proof is identical in the oplax setting.

- a vertical transformation $F^2 \colon \otimes \circ (F \times F) \Rightarrow F \circ \otimes$, whose vertical 1-cell components we denote by $F^2_{X_1,X_2} \colon FX_1 \otimes FX_2 \to F(X_1 \otimes X_2)$, and whose 2-morphism components we denote by:

$$
\begin{array}{ccc}
FX_1 \otimes FX_2 & \xrightarrow{FM \otimes FN} & FY_1 \otimes FY_2 \\
{\scriptstyle F^2_{X_1,X_2}}\downarrow & {\scriptstyle \Downarrow F^2_{M,N}} & \downarrow{\scriptstyle F^2_{Y_1,Y_2}} \\
F(X_1 \otimes X_2) & \xrightarrow[F(M \otimes N)]{} & F(Y_1 \otimes Y_2);
\end{array}
\tag{34}
$$

- a vertical transformation $F^0 \colon I_{\mathbb{D}} \Rightarrow F \circ I_{\mathbb{C}}$, whose vertical 1-cell component is $F^0 \colon I \to FI$ and whose 2-morphism component is:

$$
\begin{array}{ccc}
I & \xrightarrow{\mathrm{id}_I} & I \\
{\scriptstyle F^0}\downarrow & \Downarrow & \downarrow{\scriptstyle F^0} \\
FI & \xrightarrow[F(\mathrm{id}_I)]{} & FI;
\end{array}
\tag{35}
$$

subject to axioms expressing that the vertical 1-cells $F^2_{X_1,X_2} \colon FX_1 \otimes FX_2 \to F(X_1 \otimes X_2)$ and $F^0 \colon I \to FI$ endow $F_0 \colon \mathbb{C}_0 \to \mathbb{D}_0$ with the structure of a lax monoidal functor, and that the 2-morphisms of (34) and (35) do the same for $F_1 \colon \mathbb{C}_0 \to \mathbb{D}_0$.

The reader will notice that we have not named the 2-morphism in (35). This is because its definition is forced: for indeed, since F^0 is a vertical transformation between double functors, the axiom (18) causes (35) to be equal to id_{F^0} followed by the unit structure isomorphism of F.

We now turn to monoidal transformations between lax monoidal double functors. We begin with the horizontal case, which is as expected, though we could not find it in the literature.

Definition 5.2 (Monoidal horizontal transformation). Let $F, G \colon \mathbb{C} \to \mathbb{D}$ be lax monoidal double functors. A *monoidal horizontal transformation* $\beta \colon F \Rightarrow G$ is a horizontal transformation endowed with cells

$$
\begin{array}{ccc}
FX_1 \otimes FX_2 & \xrightarrow{\beta_{X_1} \otimes \beta_{X_2}} & GX_1 \otimes GX_2 \\
{\scriptstyle F^2_{X_1,X_2}}\downarrow & {\scriptstyle \Downarrow \beta^2_{X_1,X_2}} & \downarrow{\scriptstyle G^2_{X,X_2}} \\
F(X_1 \otimes X_2) & \xrightarrow[\beta_{X_1 \otimes X_2}]{} & G(X_1 \otimes X_2)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
I & \xrightarrow{\mathrm{id}_I} & I \\
{\scriptstyle F^0}\downarrow & {\scriptstyle \Downarrow \beta^0} & \downarrow{\scriptstyle G^0} \\
FI & \xrightarrow[\beta_I]{} & GI
\end{array}
\tag{36}
$$

which, firstly, make $\beta_{(-)} \colon \mathbb{C}_0 \to \mathbb{D}_1$ into a lax monoidal functor; in other words, such that the naturality condition

$$
\begin{array}{ccc}
FX_1 \otimes FX_2 & \xrightarrow{\beta_X \otimes \beta_{X_2}} & GX \otimes GX_2 \\
{\scriptstyle F^2_{X_1,X_2}}\downarrow & {\scriptstyle \Downarrow \beta^2_{X_1,X_2}} & \downarrow{\scriptstyle G^2_{X_1,X_2}} \\
F(X_1 \otimes X_2) & \xrightarrow{\beta_{X_1 \otimes X_2}} & G(X_1 \otimes X_2) \\
{\scriptstyle F(f \otimes g)}\downarrow & {\scriptstyle \Downarrow \beta_{f \otimes g}} & \downarrow{\scriptstyle G(f \otimes g)} \\
F(X'_1 \otimes X'_2) & \xrightarrow[\beta_{X'_1 \otimes X'_2}]{} & G(X'_1 \otimes X'_2)
\end{array}
\; = \;
\begin{array}{ccc}
FX_1 \otimes FX_2 & \xrightarrow{\beta_{X_1} \otimes \beta_{X_2}} & GX_1 \otimes GX_2 \\
{\scriptstyle Ff \otimes Fg}\downarrow & {\scriptstyle \Downarrow \beta_f \otimes \beta_g} & \downarrow{\scriptstyle Gf \otimes Gg} \\
FX'_1 \otimes FX'_2 & \xrightarrow{\beta_{X'_1} \otimes \beta_{X'_2}} & GX'_1 \otimes GX'_2 \\
{\scriptstyle F^2_{X'_1,X'_2}}\downarrow & {\scriptstyle \Downarrow \beta^2_{X'_1,X'_2}} & \downarrow{\scriptstyle G^2_{X'_1,X'_2}} \\
F(X'_1 \otimes X'_2) & \xrightarrow[\beta_{X'_1 \otimes X'_2}]{} & G(X'_1 \otimes X'_2)
\end{array}
$$

is satisfied, along with the usual associativity and unitality conditions, identifying the two evident 2-morphisms $(\beta_{X_1} \otimes \beta_{X_2}) \otimes \beta_{X_3} \Rightarrow \beta_{X_1 \otimes (X_2 \otimes X_3)}$, the two 2-morphisms $\beta_{X_1} \otimes \mathrm{id}_I \Rightarrow \beta_{X_1 \otimes I}$ and the two 2-morphisms $\mathrm{id}_I \otimes \beta_{X_2} \Rightarrow \beta_{I \otimes X_1}$. We moreover require the equality of the pastings:

$$
\begin{array}{ccccc}
FX_1 \otimes FX_2 & \xrightarrow{FM \otimes FN} & FY_1 \otimes FY_2 & \xrightarrow{\beta_Y \otimes \beta_W} & GY_2 \otimes GY_2 \\
{\scriptstyle F^2_{X,Z}}\downarrow & {\scriptstyle \Downarrow F^2_{M,N}} \quad {\scriptstyle F^2_{Y_1,Y_2}}\downarrow & {\scriptstyle \Downarrow \beta^2_{Y_1 Y_2}} & & \downarrow{\scriptstyle G^2_{Y_1,Y_2}} \\
F(X_1 \otimes X_2) & \xrightarrow{F(M \otimes N)} & F(Y_1 \otimes Y_2) & \xrightarrow{\beta_{Y_1 \otimes Y_2}} & G(Y_1 \otimes Y_2) \\
\| & {\scriptstyle \Downarrow \beta_{M \otimes N}} & & & \| \\
F(X_1 \otimes X_2) & \xrightarrow[\beta_{X_1 \otimes X_2}]{} & G(X_1 \otimes X_2) & \xrightarrow[G(M \otimes N)]{} & G(Y_1 \otimes Y_2)
\end{array}
\; = \;
\begin{array}{ccccc}
FX_1 \otimes FX_2 & \xrightarrow{FM \otimes FN} & FY_1 \otimes FY_2 & \xrightarrow{\beta_{Y_1} \otimes \beta_{Y_2}} & GY_1 \otimes GY_2 \\
\| & & {\scriptstyle \Downarrow \beta_M \otimes \beta_N} & & \| \\
FX_1 \otimes FX_2 & \xrightarrow[\beta_{X_1} \otimes \beta_{X_2}]{} & GX_1 \otimes GX_2 & \xrightarrow[GM \otimes GN]{} & GY_1 \otimes GY_2 \\
{\scriptstyle F^2_{X_1,X_2}}\downarrow & {\scriptstyle \Downarrow \beta^2_{X_1,X_2}} & {\scriptstyle G^2_{X_1,X_2}}\downarrow & {\scriptstyle \Downarrow G^2_{M,N}} & \downarrow{\scriptstyle G^2_{Y_1,Y_2}} \\
F(X_1 \otimes X_2) & \xrightarrow[\beta_{X_1 \otimes X_2}]{} & G(X_1 \otimes X_2) & \xrightarrow[G(M \otimes N)]{} & G(Y_1 \otimes Y_2)
\end{array}
$$

expressing that the natural transformation giving the globular cell components of β is a monoidal natural transformation. (Note that the 'nullary' axiom corresponding to this 'binary' axiom holds automatically and need not be stated explicitly.)

We now consider monoidality of vertical transformations. Given the view of monoidal double categories and lax monoidal double functors as pseudomonoids and lax pseudomonoid maps, the obvious thing to consider would be the corresponding transformations of pseudomonoids, and this would yield the notion of monoidal vertical transformation considered in [HS19, Definition 2.15]. However, we will need something slightly more general for our applications (*cf.* Remark 5.4), which we will term a *pseudomonoidal* vertical transformation. The difference can be appreciated by noting that the monoidal vertical transformations of *loc. cit.* reduce to a Cat-enriched monoidal transformation on the vertical 2-category, while our transformations only yield a *monoidal pseudonatural transformation* in the sense of [DS97, Definition 3].

Definition 5.3 (Pseudomonoidal vertical transformation). Let $F, F' \colon \mathbb{C} \to \mathbb{D}$ be lax monoidal double functors. A *pseudomonoidal vertical transformation* $\sigma \colon F \Rightarrow F'$ is a vertical transformation equipped with squares

$$
\begin{array}{ccc}
FX_1 \otimes FX_2 \xrightarrow{\text{id}} FX_1 \otimes FX_2 & & I \xrightarrow{\text{id}} I \\
\Big\downarrow F^2_{X,Y} \qquad \Big\downarrow \sigma_{X_1} \otimes \sigma_{X_2} & & \Big\downarrow F^0 \qquad \Big\downarrow F'^0 \\
F(X_1 \otimes X_2) \quad \Downarrow \sigma^2_{X_1,X_2} \quad F'_{X_1} \otimes F'X_2 & \text{and} & FI \quad \Downarrow \sigma^0 \\
\Big\downarrow \sigma_{X_1 \otimes X_2} \qquad \Big\downarrow F'^2_{X_1,X_2} & & \Big\downarrow \sigma_I \\
F'(X_1 \otimes X_2) \xrightarrow[\text{id}]{} F'(X_1 \otimes X_2) & & F'I \xrightarrow[\text{id}]{} F'I,
\end{array} \tag{37}
$$

which are invertible in the vertical 2-category $\mathsf{V}(\mathbb{D})$ and satisfying the following five coherence axioms:

$$ \tag{38} $$

$$ $$

$$
\begin{array}{ccccc}
(FX_1 \otimes FX_2) \otimes FX_3 & \xrightarrow{\mathrm{id}} & (FX_1 \otimes FX_2) \otimes FX_3 & \xrightarrow{\mathrm{id}} & (FX_1 \otimes FX_2) \otimes FX_3 \\
\downarrow{\alpha} & & \downarrow{\alpha} & & \downarrow{\alpha} \\
FX_1 \otimes (FX_2 \otimes FX_3) & & FX_1 \otimes (FX_2 \otimes FX_3) \xrightarrow{\mathrm{id}} FX_1 \otimes (FX_2 \otimes FX_3) & & \\
\downarrow{1 \otimes F^2_{X_2,X_3}} & & \downarrow{1 \otimes F^2_{X_2,X_3}} & & \downarrow{1 \otimes \sigma_{X_2} \otimes \sigma_{X_3}} \\
FX_1 \otimes F(X_2 \otimes X_3) & \xrightarrow{\mathrm{id}} & FX_1 \otimes F(X_2 \otimes X_3) \quad \Downarrow \mathrm{id} \otimes \sigma^2 \quad FX_1 \otimes (F'X_2 \otimes F'X_3) \\
\end{array}
$$

(The first large commutative diagram, relating associativity coherence with σ^2, F^2, F'^2, σ_X, α, $F(\alpha)$.)

The second large commutative diagram likewise expresses the associativity coherence axiom.

$$
\begin{array}{ccc}
FX \otimes I & \xrightarrow{\mathrm{id}} & FX \otimes I \\
\downarrow{1 \otimes F^0} & & \downarrow{1 \otimes F^0} \\
FX \otimes FI & \xrightarrow{\mathrm{id}} & FX \otimes FI \\
\downarrow{F^2_{X,I}} & & \downarrow{\sigma_X \otimes \sigma_I} \\
F(X \otimes I) & \Downarrow \sigma^2 & F'X \otimes F'I \\
\downarrow{\sigma_{X \otimes I}} & & \downarrow{F'^2_{X,I}} \\
F'(X \otimes I) & \xrightarrow{\mathrm{id}} & F'(X \otimes I)
\end{array}
\quad = \quad
\begin{array}{ccc}
FX \otimes I & \xrightarrow{\mathrm{id}} & FX \otimes I \\
\downarrow{1 \otimes F^0} & & \downarrow \\
FX \otimes FI & \Downarrow \mathrm{id} \otimes \sigma^0 & \downarrow{1 \otimes F'^0} \\
\downarrow{1 \otimes \sigma_I} & & \\
FX \otimes F'I & \xrightarrow{\mathrm{id}} & FX \otimes F'I \\
\downarrow{\sigma_X \otimes 1} & & \downarrow{\sigma_X \otimes 1} \\
F'X \otimes F'I & & F'X \otimes F'I \\
\downarrow{F'^2_{X,I}} & & \downarrow{F'^2_{X,I}} \\
F'(X \otimes I) & \xrightarrow{\mathrm{id}} & F'(X \otimes I),
\end{array}
$$

$$
\begin{array}{ccc}
I \otimes FX & \xrightarrow{\mathrm{id}} & I \otimes FX \\
\downarrow{F^0 \otimes 1} & & \downarrow{F^0 \otimes 1} \\
FI \otimes FX & \xrightarrow{\mathrm{id}} & FI \otimes FX \\
\downarrow{F^2_{I,X}} & & \downarrow{\sigma_I \otimes \sigma_X} \\
F(I \otimes X) & \Downarrow \sigma^2 & F'I \otimes F'X \\
\downarrow{\sigma_{I \otimes X}} & & \downarrow{F'^2_{I,X}} \\
F'(I \otimes X) & \xrightarrow{\mathrm{id}} & F'X
\end{array}
\quad = \quad
\begin{array}{ccc}
I \otimes FX & \xrightarrow{\mathrm{id}} & I \otimes FX \\
\downarrow{F^0 \otimes 1} & & \downarrow{F'^0 \otimes 1} \\
FI \otimes FX & \Downarrow \sigma^0 \otimes \mathrm{id} & \downarrow \\
\downarrow{\sigma_I \otimes 1} & & \\
F'I \otimes FX & \xrightarrow{\mathrm{id}} & F'I \otimes FX \\
\downarrow{1 \otimes \sigma_X} & & \downarrow{1 \otimes \sigma_X} \\
F'I \otimes F'X & & F'I \otimes F'X \\
\downarrow{F'^2_{I,X}} & & \downarrow{F'^2_{I,X}} \\
F'(I \otimes X) & \xrightarrow{\mathrm{id}} & F'(I \otimes X).
\end{array}
$$

Note that the final four of these axioms only involve structure in the vertical 2-category $\mathsf{V}(\mathbb{D})$; and in fact, they correspond exactly to the axioms for a monoidal pseudonatural transformation from [DS97, Definition 3]. More explicitly, the second axiom expresses that the 2-cells $\sigma^2_{X_1,X_2}$ are components of a modification, while the third through fifth axioms are precisely the three coherence axioms of *loc. cit.*

If σ^0 and the components of σ^2 are identity 2-cells, then σ becomes a *monoidal* vertical transformation in the sense of [HS19, Definition 2.15]. In that case, $\sigma_0 \colon F_0 \Rightarrow F'_0$ and $\sigma_1 \colon F_1 \Rightarrow F'_1$ are monoidal transformations in the usual sense between lax monoidal functors.

Remark 5.4. The notion of a monoidal (rather than pseudomonoidal) vertical transformation is insufficiently general for the situation we are interested in: the monoidality of the free symmetric monoidal category double monad on the double category of small categories, functors and profunctors, as considered in Section 10. The underlying double functor of this double monad can be equipped with lax monoidal structure, with respect to which the monad unit is a monoidal vertical transformation; however, the monad multiplication is not a monoidal as a vertical transformation, but only *pseudomonoidal*. This can be seen as a consequence of the fact that the free symmetric monoidal category monad is not commutative, but only *pseudocommutative* in the sense of [HP02].

We now describe the final piece of structure needed for a double category of monoidal double functors.

Definition 5.5 (Monoidal modification). Let β, β' be monoidal horizontal transformations and let σ, τ be pseudomonoidal vertical transformations, as displayed on the boundary of:

$$
\begin{array}{ccc}
F & \xRightarrow{\beta} & G \\
\sigma\Vert & \Downarrow\gamma & \Vert\tau \\
F' & \xRightarrow[\beta']{} & G'.
\end{array}
$$

A *monoidal modification* γ filling this boundary is a modification of the displayed shape satisfying the axioms:

$$
\begin{array}{ccccc}
FX_1 \otimes FX_2 \xrightarrow{\beta_{X_1} \otimes \beta_{X_2}} GX_1 \otimes GX_2 \xrightarrow{\mathrm{id}} GX_1 \otimes GX_2 & \qquad & FX_1 \otimes FX_2 \xrightarrow{\mathrm{id}} FX_1 \otimes FX_2 \xrightarrow{\beta_{X_1} \otimes \beta_{X_2}} GX_1 \otimes GX_2
\end{array}
$$

with the equalities shown as diagram (39):

$$
\begin{array}{cc}
\begin{array}{ccc}
I & \xrightarrow{\mathrm{id}} I \xrightarrow{\mathrm{id}} I \\
F^0\downarrow & \Downarrow\beta^0 \quad \downarrow G^0 \\
FI & \xrightarrow[\beta_I]{} GI \quad \Downarrow\tau^0 \quad \Big\downarrow G'^0 \\
\sigma_I\downarrow & \Downarrow\gamma_I \quad \downarrow\tau_I \\
F'I & \xrightarrow[\beta'_I]{} G'I \xrightarrow[\mathrm{id}]{} G'I
\end{array}
& =
\begin{array}{ccc}
I & \xrightarrow{\mathrm{id}} I \xrightarrow{\mathrm{id}} I \\
F^0\downarrow & & \\
FI & \Downarrow\sigma^0 \quad F'^0\Downarrow\beta'^0 \quad \Big\downarrow G'^0 \\
\sigma_I\downarrow & & \\
F'I & \xrightarrow[\mathrm{id}]{} F'I \xrightarrow[\beta'_I]{} G'I.
\end{array}
\end{array}
\tag{39}
$$

We now provide an analogue of Proposition 3.9 in the monoidal setting, by constructing a double category of monoidal double functors between two monoidal double categories $\mathbb{C}$ and $\mathbb{D}$. It would be routine to construct a double category of lax monoidal double functors, monoidal vertical transformations, monoidal horizontal transformations, and monoidal modifications; however, because we wish to involve *pseudomonoidal* vertical transformations, a little more care is needed in checking the details.

Proposition 5.6 (Functor double categories, monoidal case). *Let $\mathbb{C}$, $\mathbb{D}$ be monoidal double categories. There is a double category $\mathsf{MonDblCat}[\mathbb{C}, \mathbb{D}]$ of lax monoidal (pseudo) double functors, pseudomonoidal vertical transformations, monoidal horizontal transformations, and monoidal modifications.*

Note that in Proposition 3.9, we considered *oplax* double functors; here we consider only (pseudo) double functors, but endowed with *lax* monoidal structure. While it certainly would be possible to consider "lax monoidal oplax double functors", this is not needed for our applications.

Proof. We first show that lax monoidal double functors and pseudomonoidal vertical transformations form a category. Given pseudomonoidal vertical transformations $\sigma\colon F \Rightarrow F'$ and $\tau\colon F' \Rightarrow F''$, we endow the composite vertical transformation $\tau\cdot\sigma\colon F \Rightarrow F''$ with pseudomonoidal structure via the pasting composites:

$$
\tag{40}
$$

It is now routine to verify the pseudomonoidal vertical transformation axioms for $\tau \cdot \sigma$, and to check that this composition law is associative and unital, so yielding the desired category.

We next show that monoidal horizontal transformations and monoidal modifications form a category; for which it suffices to verify that, given a pair of composable monoidal modifications, their composite *qua* modification, as in Definition 5.5, is again monoidal. This is straightforward.

We now provide the horizontal composition law for $\mathsf{MonDblCat}[\mathbb{C}, \mathbb{D}]$. Given monoidal horizontal transformations $\beta\colon F \Rightarrow G$ and $\gamma\colon G \Rightarrow H$, we endow the composite horizontal transformation $\gamma \circ \beta$ with monoidal structure via the pastings:

$$
\begin{array}{cc}
\begin{array}{c}
FX_1 \otimes FX_2 \xrightarrow{(\gamma_{X_1} \circ \beta_{X_1}) \otimes (\gamma_{X_2} \circ \beta_{X_2})} GX_1 \otimes GX_2
\end{array}
&
\begin{array}{c}
I \xrightarrow{\mathrm{id}_I} I
\end{array}
\\
\end{array}
$$

Direct verification yields the horizontal transformation axioms. To make the assignment $\beta, \gamma \mapsto \gamma \circ \beta$ into a functor, it now suffices to observe that that the horizontal composition of two monoidal modifications *qua* modification is again monoidal; this is again a matter of direct verification. Finally, the globular constraints a, ℓ, r of $\mathsf{MonDblCat}[\mathbb{C}, \mathbb{D}]$ are inherited from $\mathsf{DblCat}[\mathbb{C}, \mathbb{D}]$, and it is simply a matter of checking that these are indeed monoidal modifications. $\qquad\square$

The next result builds on Proposition 3.10.

Proposition 5.7. *A pseudomonoidal vertical transformation $\sigma\colon F \Rightarrow F'$ has a companion as a vertical 1-cell of $\mathsf{MonDblCat}[\mathbb{C}, \mathbb{D}]$ if and only if the underlying vertical transformation of σ has a companion as a vertical 1-cell of $\mathsf{DblCat}[\mathbb{C}, \mathbb{D}]$, i.e. it is special.*

Proof. The 'only if' direction is trivial: if σ has a companion in $\mathsf{MonDblCat}[\mathbb{C}, \mathbb{D}]$, then applying the forgetful double functor $\mathsf{MonDblCat}[\mathbb{C}, \mathbb{D}] \to \mathsf{DblCat}[\mathbb{C}, \mathbb{D}]$ shows it has a companion in $\mathsf{DblCat}[\mathbb{C}, \mathbb{D}]$. For the 'if' direction, given a pseudomonoidal transformation σ as in Definition 5.3, the additional necessary data for the induced horizontal transformation $\widehat{\sigma}\colon F \Rightarrow F'$ as described in the proof of Proposition 3.10 to be monoidal are the cells of (36). We obtain these as companion transposes of the structure data (37) of the pseudomonoidal vertical transformation σ, as in:

$$
(\widehat{\sigma})^2_{X_1, X_2} :=
\begin{array}{c}
FX_1 \otimes FX_2 \xrightarrow{\widehat{\sigma}_{X_1} \otimes \widehat{\sigma}_{X_2}} F'X_1 \otimes F'X_2
\end{array}
\qquad
(\widehat{\sigma})^0 :=
\begin{array}{c}
I \xrightarrow{\mathrm{id}_I} I
\end{array}
$$

where the top-left isomorphism arises due to the double functor $\otimes$ preserving companions. That this makes $\widehat{\sigma}$ into a monoidal horizontal transformation can now be checked by lengthy, but straightforward, calculations. Similarly, it is straightforward to verify that with respect to this structure, the companion 2-morphisms p_1 and p_2 in $\mathsf{DblCat}[\mathbb{C}, \mathbb{D}]$ are monoidal, and so lift to $\mathsf{MonDblCat}[\mathbb{C}, \mathbb{D}]$ as required. $\qquad\square$

6. Monoids in monoidal double categories

In this section, we consider horizontal and vertical monoids in a monoidal double category. When instantiated in the monoidal double categories $\mathsf{DblCat}[\mathbb{C}, \mathbb{C}]$ and $\mathsf{MonDblCat}[\mathbb{C}, \mathbb{C}]$ of Propositions 3.9 and 5.6, these will give us the notions of horizontal and vertical double monad, and of monoidal horizontal and vertical double monad respectively, to be considered in Sections 7 and 8.

We begin with the notion of a horizontal monoid in a monoidal double category. This is analogous to a pseudomonoid in a monoidal bicategory, in that the associativity and unit axioms do not hold on the nose, but rather up to invertible squares.

Definition 6.1. Let $\mathbb{C}$ be a monoidal double category. A *horizontal monoid* in $\mathbb{C}$ consists of:

- an object A;

- horizontal 1-cells $m\colon A \otimes A \rightarrowtail A$ and $e\colon I \rightarrowtail A$;
- invertible cells

$$
\begin{array}{ccc}
(A \otimes A) \otimes A \xrightarrow{m \otimes \mathrm{id}} A \otimes A \xrightarrow{m} A & \quad A \otimes I \xrightarrow{\mathrm{id} \otimes e} A \otimes A \xrightarrow{m} A & \quad I \otimes A \xrightarrow{e \otimes \mathrm{id}} A \otimes A \xrightarrow{m} A \\
\alpha \downarrow \quad \Downarrow \mathfrak{a} \quad \| & \rho \downarrow \quad \Downarrow \mathfrak{r} \quad \| \quad \text{and} & \lambda \downarrow \quad \Downarrow \mathfrak{l} \quad \| \\
A \otimes (A \otimes A) \xrightarrow[\mathrm{id} \otimes m]{} A \otimes A \xrightarrow[m]{} A & A \xrightarrow[\mathrm{id}]{} A & A \xrightarrow[\mathrm{id}]{} A.
\end{array}
\tag{41}
$$

These data are required to satisfy the coherence axioms that:

[coherence diagrams]

Remark 6.2. As discussed in Section 4, under fairly mild conditions the horizontal bicategory $\mathcal{H}(\mathbb{C})$ of a monoidal double category $\mathbb{C}$ will have the structure of a monoidal double category, whose monoidal associativity and unit constraint 1-cells are the companions of the corresponding constraints for $\mathbb{C}$. In this situation, horizontal monoids in $\mathbb{C}$ correspond to pseudomonoids in $\mathcal{H}(\mathbb{C})$ by taking the companion transposes of the coherence data (41).

Definition 6.3 (Vertical monoid). Let $\mathbb{C}$ be a monoidal double category. A *vertical monoid* in $\mathbb{C}$ is a monoid in the monoidal category $\mathbb{C}_0$. Explicitly, it is an object A endowed with vertical 1-cells $m\colon A \otimes A \to A$ and $e\colon I \to A$ satisfying the usual associativity and unit laws.

The next result shows how we may induce horizontal monoids from vertical ones, and will be applied in Theorem 7.4, relating horizontal and vertical double monads, and Theorem 8.4, relating monoidal horizontal and vertical double monads.

Theorem 6.4. *Let $\mathbb{C}$ be a monoidal double category and (A, m, e) be a vertical monoid in $\mathbb{C}$, such that m and e have companions. The companion transposes*

$$
\begin{array}{ccc}
(A \otimes A) \otimes A \xrightarrow{\widehat{m} \otimes \mathrm{id}} A \otimes A \xrightarrow{\widehat{m}} A & A \otimes I \xrightarrow{\mathrm{id} \otimes \widehat{e}} A \otimes A \xrightarrow{\widehat{m}} A & I \otimes A \xrightarrow{\widehat{e} \otimes \mathrm{id}} A \otimes A \xrightarrow{\widehat{m}} A \\
\alpha \downarrow \quad \Downarrow \mathfrak{a} \quad \| & \rho \downarrow \quad \Downarrow \mathfrak{r} \quad \| & \lambda \downarrow \quad \Downarrow \mathfrak{l} \quad \| \\
A \otimes (A \otimes A) \xrightarrow[\mathrm{id} \otimes \widehat{m}]{} A \otimes A \xrightarrow[\widehat{m}]{} A & A \xrightarrow[\mathrm{id}]{} A & A \xrightarrow[\mathrm{id}]{} A
\end{array}
$$

of the monoid identities endow $(A, \widehat{m}, \widehat{e})$ with the structure of a horizontal pseudomonoid.

Proof. The displayed 2-morphisms are constructed using transpose operations like (12) from the vertical associativity and unitality monoid axioms for A. They are vertically invertible since α, ρ and λ are, so that $\mathfrak{a}^{-1}$ may be constructed as companion transposes of the identities $m \circ (1 \otimes m) \circ \alpha^{-1} = m \circ (m \otimes 1)$, and similarly for $\mathfrak{l}^{-1}$ and $\mathfrak{r}^{-1}$.

The coherence axioms of Definition 6.1 for a horizontal pseudomonoid can now be checked by computing appropriate transposes of the required diagrams and making use of Lemma 3.12. $\qquad\square$

7. Double monads

For an ordinary category C, the category of endofunctors of C has a monoidal structure given by composition, and a monoid therein is precisely a monad on C. In the case of double categories, we can do something similar by exploiting our work in Sections 3 and 6, so leading to a notion of double monad: or rather, *two* notions of double monad, horizontal and vertical.

To begin with, observe that Proposition 3.9 states in particular that for any double category $\mathbb{C}$, there is a double category $\mathsf{DblCat}[\mathbb{C}, \mathbb{C}]$ of double endofunctors, vertical transformations, horizontal transformations and modifications (Definitions 3.1, 3.4, 3.5 and 3.8). In fact, as is well-known, this double category is monoidal:

Proposition 7.1 (Composition monoidal structure)**.** *Let* $\mathbb{C}$ *be a double category. The double category* $\mathsf{DblCat}[\mathbb{C}, \mathbb{C}]$ *admits a monoidal structure given by composition.*

Proof. We only sketch the proof; for a full treatment see, for example, [Gar06a, Proposition 39].

Given double endofunctors $F_1, F_2 \colon \mathbb{C} \to \mathbb{C}$, we define $F_1 \otimes F_2$ to be the double endofunctor $F_2 F_1$ with underlying ordinary functors $(F_2)_0 \circ (F_1)_0 \colon \mathbb{C}_0 \to \mathbb{C}_0$ and $(F_2)_1 \circ (F_1)_1 \colon \mathbb{C}_1 \to \mathbb{C}_1$, and with coherence data obtained by vertically pasting those for F and G. Given vertical transformations $\sigma_1 \colon F_1 \Rightarrow F_1'$ and $\sigma_2 \colon F_2 \Rightarrow F_2'$, we define $\sigma_1 \otimes \sigma_2$ to be the vertical transformation $\sigma_2 \sigma_1 \colon F_2 F_1 \Rightarrow F_2' F_1'$ with underlying ordinary natural transformations given by the horizontal composites $(\sigma_2)_0 * (\sigma_1)_0$ and $(\sigma_2)_0 * (\sigma_1)_0$. With the identity double functor as unit, this yields a strict monoidal structure on the category of double endofunctors and vertical transformations.

Next, given horizontal transformations $\beta_1 \colon F_1 \Rightarrow G_1$ and $\beta_2 \colon F_2 \Rightarrow G_2$ we define $\beta_1 \otimes \beta_2$ to be the horizontal transformation $\beta_2 \beta_1 \colon F_2 F_1 \Rightarrow G_2 G_1$ with horizontal 1-cell components[4]

$$(\beta_2 \beta_1)_X = F_2 F_1 X \xrightarrow{(\beta_2)_{F_1 X}} G_2 F_1 X \xrightarrow{G_2(\beta_1)_X} G_2 G_1 X \tag{42}$$

and remaining data obtained in an analogous way; whereas for modifications γ_1, γ_2 as in

$$
\begin{array}{ccc}
F_1 \overset{\beta_1}{\Longrightarrow} G_1 & \qquad & F_2 \overset{\beta_2}{\Longrightarrow} G_2 \\
\sigma_1 \big\Downarrow \quad \Downarrow\gamma_1 \quad \big\Downarrow \tau_1 & \qquad & \sigma_2 \big\Downarrow \quad \Downarrow\gamma_2 \quad \big\Downarrow \tau_2 \\
F_1' \underset{\beta_1'}{\Longrightarrow} G_1' & \qquad & F_2' \underset{\beta_2'}{\Longrightarrow} G_2'
\end{array}
\tag{43}
$$

we define $\gamma_1 \otimes \gamma_2$ to be the modification $\gamma_2 \gamma_1 \colon \beta_2 \beta_1 \Rightarrow \beta_2' \beta_1'$ with vertical source and target $\sigma_2 \sigma_1$ and $\tau_2 \tau_1$, and 2-morphism components:

$$
\begin{array}{ccc}
F_2 F_1 X \xrightarrow{(\beta_2)_{F_1 X}} G_2 F_1 X \xrightarrow{G_2(\beta_1)_X} G_2 G_1 X \\
(\sigma_2)_{F_1 X} \Big\downarrow \quad \Downarrow(\gamma_2)_{F_1 X} \quad (\sigma_2)_{F_1 X}\Big\downarrow \Downarrow(\sigma_2)_{(\beta_1)_X} \quad \Big\downarrow (\sigma_2)_{G_1 X} \\
F_2' F_1 X \xrightarrow[(\beta_2')_{F_1 X}]{} G_2' F_1 X \xrightarrow{G_2'(\beta_1)_X} G_2' G_1 X \\
F_2'(\sigma_1)_X \Big\downarrow \quad \Downarrow(\beta_2')_{(\sigma_1)_X} \quad G_2'(\sigma_1)_X\Big\downarrow \Downarrow G_2'(\gamma_1)_X \quad \Big\downarrow G_2'(\tau_1)_X \\
F_2' F_1' X \xrightarrow[(\beta_2')_{F_1' X}]{} G_2' F_1' X \xrightarrow[G_2'(\beta_1')_X]{} G_2' G_1' X.
\end{array}
\tag{44}
$$

These data endow the category of horizontal 1-cells and 2-morphisms with a *non*-strict monoidal structure; for the monoidal constraints, given horizontal transformations $\beta_1 \colon F_1 \Rightarrow G_1$, $\beta_2 \colon F_2 \Rightarrow G_2$ and $\beta_3 \colon F_3 \Rightarrow G_3$, the composites $\beta_3(\beta_2 \beta_1)$ and $(\beta_3 \beta_2)\beta_1$ have respective horizontal 1-cell components

$$G_3(G_2(\beta_1)_X \circ (\beta_2)_{F_1 X}) \circ (\beta_3)_{F_2 F_1 X} \qquad \text{and} \qquad G_3 G_2(\beta_1)_X \circ (G_3(\beta_2)_{F_1 X} \circ (\beta_3)_{F_2 F_1 X}) \tag{45}$$

and the desired globular associativity modification $(\beta_1 \otimes \beta_2) \otimes \beta_3 \Rightarrow \beta_1 \otimes (\beta_2 \otimes \beta_3)$ has components given by the evident globular 2-isomorphisms between these composites, built from associativity and functoriality of G_3. The unit constraints are handled similarly.

---

[4]In the provided reference, the alternate choice $(\beta_2 \beta_1)_X = (\beta_2)_{G_1 X} \circ F_2(\beta_1)_X$ is used; this results in a different but equivalent monoidal structure.

Finally, we must provide the globular 2-isomorphisms τ and η of (23). We describe only the case of τ; for which, consider horizontal transformations $\beta_1 \colon F_1 \Rightarrow G_1$, $\delta_1 \colon G_1 \Rightarrow H_1$, $\beta_2 \colon F_2 \Rightarrow G_2$ and $\delta_2 \colon G_2 \Rightarrow H_2$. The two composite horizontal transformations $(\delta_1 \circ \beta_1) \otimes (\delta_2 \circ \beta_2)$ and $(\delta_1 \otimes \delta_2) \circ (\beta_1 \otimes \beta_2)$ have respective horizontal 1-cell components

$$H_2((\delta_1)_X \circ (\beta_1)_X) \circ ((\delta_2)_{F_1 X} \circ (\beta_2)_{F_1 X}) \qquad \text{and} \qquad (H_2(\delta_1)_X \circ (\delta_2)_{G_1 X}) \circ (G_2(\beta_1)_X \circ (\beta_2)_{F_1 X}), \quad (46)$$

which are related by globular 2-isomorphisms built from functoriality constraints of H_2, associativity constraints of $\mathbb{C}$ and the coherence 2-isomorphism $(\delta_2)_{(\beta_1)_X}$ of the horizontal transformation δ_2. $\qquad \square$

By looking at horizontal and vertical monoids (as introduced in Definition 6.1 and Definition 6.3) in the endofunctor double category, we obtaine notions of *horizontal* and *vertical double monad*. These notions differ by the direction of the transformations for the the multiplication and unit and by their strictness: a horizontal monad induces a pseudomonad on the horizontal bicategory, while a vertical monad induces a 2-monad on the vertical 2-category. We shall relate these notions in Theorem 7.4.

Definition 7.2 (Horizontal double monad). Let $\mathbb{C}$ be a double category. A *horizontal double monad* on $\mathbb{C}$ is a horizontal monoid in the monoidal double category $\mathsf{DblCat}[\mathbb{C}, \mathbb{C}]$. Explicitly, it consists of:

- a double functor $T \colon \mathbb{C} \to \mathbb{C}$;
- a horizontal transformation $m \colon TT \Rrightarrow T$, with components $m_X \colon TTX \rightarrowtail TX$,

$$
\begin{array}{ccc}
TTX \xrightarrow{m_X} TX & & TTX \xrightarrow{TTM} TTY \xrightarrow{m_Y} TY \\
{\scriptstyle TTf}\downarrow \quad {\scriptstyle \Downarrow m_f} \quad \downarrow {\scriptstyle Tf} & \text{and} & \| \qquad {\scriptstyle \Downarrow m_M} \qquad \| \\
TTX' \xrightarrow{m_{X'}} TX' & & TTX \xrightarrow{m_X} TX \xrightarrow{TM} TY
\end{array}
\qquad (47)
$$

  for each object X, vertical 1-cell $f \colon X \to X'$ and horizontal 1-cell $M \colon X \rightarrowtail Y$;
- a horizontal transformation $e \colon 1 \Rrightarrow T$, with components $e_X \colon X \rightarrowtail TX$,

$$
\begin{array}{ccc}
X \xrightarrow{e_X} TX & & X \xrightarrow{M} Y \xrightarrow{e_Y} TY \\
{\scriptstyle f}\downarrow \quad {\scriptstyle \Downarrow e_f} \quad \downarrow {\scriptstyle Tf} & \text{and} & \| \qquad {\scriptstyle \Downarrow e_M} \qquad \| \\
X' \xrightarrow{e_{X'}} TX' & & X \xrightarrow{e_X} TX \xrightarrow{TM} TY
\end{array}
\qquad (48)
$$

  for each object X, vertical 1-cell $f \colon X \to X'$ and horizontal 1-cell $M \colon X \rightarrowtail Y$;
- invertible modifications $\mathfrak{a}$, $\mathfrak{l}$ and $\mathfrak{r}$ with respective components at $X \in \mathbb{C}$ given by:

$$
\begin{array}{ccccccc}
TTTX \xrightarrow{Tm_X} TTX \xrightarrow{m_X} TX & & TX \xrightarrow{e_{TX}} TTX \xrightarrow{m_X} TX & & TX \xrightarrow{Te_X} TTX \xrightarrow{m_X} TX \\
\| \quad {\scriptstyle \Downarrow \mathfrak{a}_X} \quad \| & & \| \quad {\scriptstyle \Downarrow \mathfrak{r}_X} \quad \| & & \| \quad {\scriptstyle \Downarrow \mathfrak{l}_X} \quad \| \\
TTTX \xrightarrow{m_{TX}} TTX \xrightarrow{m_X} TX & & TX \xrightarrow{\mathrm{id}_{TX}} TX & & TX \xrightarrow{\mathrm{id}_{TX}} TX
\end{array}
\qquad (49)
$$

These data are subject to the axioms of Definition 6.1, noting carefully the order-reversal stemming from the fact that $F_1 \otimes F_2 = F_2 F_1$.

A horizontal double monad is exactly the structure we need to define a horizontal Kleisli double category, as we shall do in Theorem 9.1 below. However, horizontal double monads involves non-trivial coherence axioms for associativity and unit; it is therefore useful in practice to have some ways of constructing them from simpler kinds of data. For this purpose, we recall from [GP04, §7] the following definition:

Definition 7.3 (Vertical double monad). Let $\mathbb{C}$ be a double category. A *vertical double monad* on $\mathbb{C}$ is a vertical monoid in $\mathsf{DblCat}[\mathbb{C}, \mathbb{C}]$. Explicitly, it consists of the following data:

- a double endofunctor $T \colon \mathbb{C} \to \mathbb{C}$;
- a vertical transformation $m \colon TT \Rightarrow T$, with components $m_X \colon TTX \to TX$ and

$$
\begin{array}{ccc}
TTX & \xrightarrow{TTM} & TTY \\
{\scriptstyle m_X}\downarrow & {\scriptstyle \Downarrow m_M} & \downarrow {\scriptstyle m_Y} \\
TX & \xrightarrow{TM} & TY
\end{array}
\qquad (50)
$$

  for each object X and horizontal 1-cell $M \colon X \rightarrowtail Y$;

- a vertical transformation $e \colon 1 \Rightarrow T$, with components $e_X \colon X \to TX$ and

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;M\;\;} & Y \\
{\scriptstyle e_X}\downarrow & \Downarrow e_M & \downarrow{\scriptstyle e_Y} \\
TX & \xrightarrow[\;\;TM\;\;]{} & TY
\end{array}
\tag{51}
$$

for each object X and horizontal 1-cell $M \colon X \longrightarrow Y$.

These data are required to satisfy associativity and unitality conditions, as in Definition 6.3.

The notion of a vertical monad is stricter than that of a horizontal monad and thus easier to exhibit in examples. Once we have a vertical monad, the following result allows us to enhance it to a horizontal one.

Theorem 7.4. *Let $\mathbb{C}$ be a double category and $T \colon \mathbb{C} \to \mathbb{C}$ be a vertical double monad. Assume that its multiplication $m \colon TT \Rightarrow T$ and unit $e \colon 1_{\mathbb{C}} \Rightarrow T$ are special vertical transformations. Then T induces a horizontal double monad $(T, \widehat{m}, \widehat{e})$ on $\mathbb{C}$.*

Proof. If we consider T as a vertical monoid in $\mathsf{DblCat}[\mathbb{C}, \mathbb{C}]$, Theorem 6.4 ensures that it induces a horizontal monoid therein (namely a horizontal double monad) whenever the unit e and multiplication m have companions as vertical transformations. By Proposition 3.10, this will happen if and only if they are special. $\qquad\square$

While a direct proof of Theorem 7.4 would certainly be possible, the more abstract approach we take has the advantage of being equally applicable to the case of *monoidal* double monads (Theorem 8.4), for which a direct approach seems less practicable. It is to this that we now turn.

8. Monoidal double monads

In this section, we retread the material of the previous section in the context of *monoidal* double categories, leading to the notions of a monoidal horizontal and monoidal vertical double monad, and results relating the two. In Section 9, we will exploit these notions in order to impose monoidal structure on the Kleisli double category of a horizontal double monad.

As a first step, we show that when $\mathbb{C}$ is a monoidal double category, we can extend the composition monoidal structure on the endofunctor double category $\mathsf{DblCat}[\mathbb{C}, \mathbb{C}]$ as recalled in Proposition 7.1, to a monoidal structure on the *monoidal* endofunctor double category $\mathsf{MonDblCat}[\mathbb{C}, \mathbb{C}]$ of Proposition 5.6.

Proposition 8.1 (Composition monoidal structure on $\mathsf{MonDblCat}[\mathbb{C}, \mathbb{C}]$)**.** *Let $\mathbb{C}$ be a monoidal double category. The composition monoidal structure of the double category $\mathsf{DblCat}[\mathbb{C}, \mathbb{C}]$ lifts to a monoidal structure on the double category $\mathsf{MonDblCat}[\mathbb{C}, \mathbb{C}]$ of monoidal endofunctors, pseudomonoidal vertical transformations, monoidal horizontal transformations and monoidal modifications.*

Proof. We must lift each of the pieces of data exhibited in the proof of Proposition 7.1 to the monoidal context. We first lift the strict monoidal structure on the category of 0-cells and vertical 1-cells. If $F_1, F_2 \colon \mathbb{C} \to \mathbb{C}$ are lax monoidal double endofunctors of $\mathbb{C}$, then their composite $F_2 F_1$ bears lax monoidal structure with vertical 1-cell components

$$
(F_2 F_1)^2_{X_1, X_2} = F_2 F_1 X_1 \otimes F_2 F_1 X_2 \xrightarrow{F_2^2} F_2(F_1 X_1 \otimes F_1 X_2) \xrightarrow{F_2 F_1^2} F_2 F_1(X_1 \otimes X_2),
$$

$$
(F_2 F_1)^0 = I \xrightarrow{F_2^0} F_2 I \xrightarrow{F_2 F_1^0} F_2 F_1 I
$$

and 2-morphism components given similarly by $(F_2 F_1)^2_{M, N} = F_2(F_1^2)_{M, N} \cdot (F_2^2)_{F_1 M, F_1 N}$. Next, if $\sigma_1 \colon F_1 \Rightarrow F_1'$ and $\sigma_2 \colon F_2 \Rightarrow F_2'$ are pseudomonoidal vertical transformations, then the composite vertical

transformation $\sigma_2\sigma_1\colon F_2F_1 \Rightarrow F_2'F_1'$ bears pseudomonoidal structure witnessed by the 2-morphisms

$$(\sigma_2\sigma_1)^2 =$$

$$
\begin{array}{ccccccc}
F_2F_1X_1 \otimes F_2F_1X_2 & \xrightarrow{\text{id}} & F_2F_1X_1 \otimes F_2F_1X_2 & \xrightarrow{\text{id}} & F_2F_1X_1 \otimes F_2F_1X_2 & \xrightarrow{\text{id}} & F_2F_1X_1 \otimes F_2F_1X_2 \\
\downarrow{F_2^2} & & \downarrow{F_2^2} & & \downarrow{\sigma_2\otimes\sigma_2} & & \downarrow{\sigma_2\otimes\sigma_2} \\
F_2(F_1X_1 \otimes F_1X_2) & \xrightarrow{\text{id}} & F_2(F_1X_1 \otimes F_1X_2) & \Downarrow\sigma_2{}^2 & F_2'F_1X_1 \otimes F_2'F_1X_2 & \xrightarrow{\text{id}} & F_2'F_1X_1 \otimes F_2'F_1X_2 \\
\downarrow{F_2F_1^2} & & \downarrow{\sigma_2} & & \downarrow{F_2'^2} & & \downarrow{F_2'\sigma_1\otimes F_2'\sigma_1} \\
F_2F_1(X_1 \otimes X_2) & & F_2'(F_1X_1 \otimes F_1X_2) & \xrightarrow{\text{id}} & F_2'(F_1X_1 \otimes F_1X_2) & & F_2'F_1'X_1 \otimes F_2'F_1'X_2 \\
\downarrow{\sigma_2} & & \downarrow{F_2'F_1^2} & & \downarrow{F_2'(\sigma_1\otimes\sigma_1)} & & \downarrow{F_2'^2} \\
F_2'F_1(X_1 \otimes X_2) & \xrightarrow{\text{id}} & F_2'F_1(X_1 \otimes X_2) & \Downarrow F_2'\sigma_1{}^2 & F_2'(F_1'X_1 \otimes F_1'X_2) & \xrightarrow{\text{id}} & F_2'(F_1'X_1 \otimes F_1'X_2) \\
\downarrow{F_2'\sigma_1} & & \downarrow{F_2'\sigma_1} & & \downarrow{F_2'F_1'^2} & & \downarrow{F_2'F_1'^2} \\
F_2'F_1'(X_1 \otimes X_2) & \xrightarrow[\text{id}]{} & F_2'F_1'(X_1 \otimes X_2) & \xrightarrow[\text{id}]{} & F_2'F_1'(X_1 \otimes X_2) & \xrightarrow[\text{id}]{} & F_2'F_1'(X_1 \otimes X_2)
\end{array}
$$

$$(\sigma_2\sigma_1)^0 =$$

$$
\begin{array}{ccccc}
I & \xrightarrow{\text{id}} & I & \xrightarrow{\text{id}} & I \\
\downarrow{F_2^0} & & \downarrow{F_2^0} & & \\
F_2I & \xrightarrow{\text{id}} & F_2I & \Downarrow\sigma_2^0 & \downarrow{F_2'^0} \\
\downarrow{F_2F_1^0} & & \downarrow{\sigma_2} & & \\
F_2F_1I & & F_2'I & \xrightarrow{\text{id}} & F_2F_1I \\
\downarrow{\sigma_2} & & \downarrow{F_2'F_1^0} & & \\
F_2'F_1I & \xrightarrow{\text{id}} & F_2'F_1I & \Downarrow F_2'\sigma_1^0 & \downarrow{F_2'F_1'^0} \\
\downarrow{F_2'\sigma_1} & & \downarrow{F_2'\sigma_1} & & \\
F_2'F_1'I & \xrightarrow[\text{id}]{} & F_2'F_1'I & \xrightarrow[\text{id}]{} & F_2'F_1'I
\end{array}
$$

where the empty squares are horizontal identities existing due to naturality of σ_2 and $F_2'^2$. It is direct to check that these pseudomonoidal constraint cells are stable under vertical composition, so that we have a functorial tensor product on the category of lax monoidal functors and pseudomonoidal vertical transformations. Taking this tensor product together with the (strict) monoidal $1_{\mathbb{C}}$ as unit, we obtain a strict monoidal category: indeed, for any lax monoidal double endofunctors F_3, F_2, F_1 of $\mathbb{C}$, $(F_3F_2)F_1 = F_3(F_2F_1)$ as lax monoidal double functors, since both have coherence vertical 1-cells given by

$$F_3F_2F_1X_1 \otimes F_3F_2F_1X_2 \xrightarrow{F_3^2} F_3(F_2F_1X_1 \otimes F_2F_1X_2) \xrightarrow{F_3F_2^2} F_3F_2(F_1X_1 \otimes F_1X_2) \xrightarrow{F_3F_2F_1^2} F_3F_2F_1(X_1 \otimes X_2),$$

using the fact that double functors strictly preserve vertical composition; and correspondingly for the coherence 2-morphisms.

We now show that the category of horizontal monoidal transformations and monoidal modifications is monoidal. If $\beta_1\colon F_1 \Rightarrow G_1$ and $\beta_2\colon F_2 \Rightarrow G_2$ are two monoidal horizontal transformations, then the horizontal transformation $\beta_2\beta_1\colon F_2F_1 \Rightarrow G_2G_1$ given as in (42) is monoidal, via the structure 2-morphisms

$$(\beta_2\beta_1)^2 =$$

$$
\begin{array}{ccccc}
F_2F_1X_1 \otimes F_2F_1X_2 & \xrightarrow{(G_2\beta_1\circ\beta_2)\otimes(G_2\beta_1\circ\beta_2)} & & & G_2G_1X_1 \otimes G_2G_1X_2 \\
\| & & \Downarrow\tau & & \| \\
F_2F_1X_1 \otimes F_2F_1X_2 & \xrightarrow{\beta_2\otimes\beta_2} G_2F_1X_1 \otimes G_2F_1X_2 & \xrightarrow{G_2\beta_1\otimes G_2\beta_1} & & G_2G_1X_1 \otimes G_2G_1X_2 \\
\downarrow{F_2^2} & \Downarrow\beta_2^2 \quad \downarrow{G_2^2} & \Downarrow(G_2^2)_{\beta_1,\beta_1} & & \downarrow{G_2^2} \\
F_2(F_1X_1 \otimes F_1X_2) & \xrightarrow{\beta_2} G_2(F_1X_1 \otimes F_1X_2) & \xrightarrow{G_2(\beta_1\otimes\beta_1)} & & G_2(G_1X_1 \otimes G_1X_2) \\
\downarrow{F_2(F_1^2)} & \Downarrow(\beta_2)_{F_1^2} \quad \downarrow{G_2(F_1^2)} & \Downarrow G_2(\beta_1^2) & & \downarrow{G_2(G_1^2)} \\
F_2F_1(X_1 \otimes X_2) & \xrightarrow{\beta_2} G_2F_1(X_1 \otimes X_2) & \xrightarrow{G_2\beta_1} & & G_2G_1(X_1 \otimes X_2)
\end{array}
$$

$$(\beta_2\beta_1)^0 = \quad \begin{array}{ccc}
I \xrightarrow{\ \mathrm{id}\ } I \xrightarrow{\ \mathrm{id}\ } I \\
F_2^0\downarrow \quad \Downarrow\beta_2^0 \quad G_2^0\downarrow \qquad\qquad \downarrow G_2^0 \\
F_2 I \xrightarrow{\ \beta_2\ } G_2 I \xrightarrow{\ \mathrm{id}\ } G_2 I \\
F_2 F_1^0\downarrow \quad \Downarrow(\beta_2)_{F_1^0} \quad G_2 F_1^0\downarrow \quad \Downarrow G_2(\beta_1^0) \quad \downarrow G_2 G_1^0 \\
F_2 F_1 I \xrightarrow[\ \beta_2\ ]{} G_2 F_1 I \xrightarrow[\ G_2\beta_1\ ]{} G_2 G_1 I.
\end{array}$$

Moreover, given monoidal modifications γ_1, γ_2 as in (43), their composite $\gamma_2\gamma_1\colon \beta_2\beta_1 \Rightarrow \beta_2'\beta_1'$ given by (44) can be verified to satisfy the axioms (39) that render it monoidal, using, among other things, the monoidality of γ_1 and γ_2. The functoriality of this tensor product is now inherited from $\mathsf{DblCat}[\mathbb{C},\mathbb{C}]_1$, given that monoidality is a mere condition on a modification. It is moreover easy to check that the associativity and unitality modifications in $\mathsf{DblCat}[\mathbb{C},\mathbb{C}]_1$ become monoidal on lifting them to $\mathsf{MonDblCat}[\mathbb{C},\mathbb{C}]_1$, so providing the last pieces of data for the desired monoidal structure.

It remains to lift τ and η from $\mathsf{DblCat}[\mathbb{C},\mathbb{C}]$ to $\mathsf{MonDblCat}[\mathbb{C},\mathbb{C}]$: and this is again simply a matter of checking that the modifications obtained from $\mathsf{DblCat}[\mathbb{C},\mathbb{C}]$ do indeed become monoidal modifications. $\quad\square$

Using this result, and paralleling the developments of Section 7, we can now give succinct definitions of the notions of monoidal horizontal and vertical double monad.

Definition 8.2 (Monoidal horizontal double monad)**.** Let $\mathbb{C}$ be a monoidal double category. A *monoidal horizontal double monad* on $\mathbb{C}$ is a horizontal monoid in the monoidal double category $\mathsf{MonDblCat}[\mathbb{C},\mathbb{C}]$. Explicitly, it is a horizontal double monad (T, m, e) on $\mathbb{C}$ in the sense of Definition 7.2 such that:

- the double functor $T\colon \mathbb{C} \to \mathbb{C}$ is lax monoidal, *i.e.*, it comes equipped with structure vertical 1-cells $T^2_{X_1,X_2}\colon TX_1 \otimes TX_2 \to T(X_1 \otimes X_2)$, $T^0\colon I \to TI$ and 2-morphisms

$$\begin{array}{ccc}
TX_1 \otimes TX_2 & \xrightarrow{TM \otimes TN} & TY_1 \otimes TY_2 \\
T^2_{X_1,X_2}\downarrow & \Downarrow T^2_{M,N} & \downarrow T^2_{Y_1,Y_2} \\
T(X_1 \otimes X_2) & \xrightarrow[T(M\otimes N)]{} & T(Y_1 \otimes Y_2)
\end{array}$$

  satisfying the axioms of Definition 5.1;
- the horizontal transformation $m\colon TT \Rightarrow T$ is monoidal, *i.e.*, it comes equipped with structure 2-morphisms:

$$\begin{array}{ccc}
TTX_1 \otimes TTX_2 & \xrightarrow{m_X \otimes m_Y} & TX_1 \otimes TX_2 \\
T^2_{TX_1,TX_2}\downarrow & & \downarrow \\
T(TX_1 \otimes TX_2) \quad \Downarrow m^2 & & T^2_{X_1,X_2} \\
T(T^2_{X_1,X_2})\downarrow & & \downarrow \\
TT(X_1 \otimes X_2) & \xrightarrow[m_{X_1 \otimes X_2}]{} & T(X_1 \otimes X_2)
\end{array} \quad\text{and}\quad \begin{array}{ccc}
I & \xrightarrow{\mathrm{id}_I} & I \\
T^0\downarrow & & \downarrow T^0 \\
TI & \Downarrow m^0 & T^0 \\
T(T^0)\downarrow & & \downarrow \\
TTI & \xrightarrow[m_I]{} & TI
\end{array} \tag{52}$$

  satisfying the axioms of Definition 5.2;
- the horizontal transformation $e\colon 1 \Rightarrow T$ is monoidal, *i.e.* comes with structure 2-morphisms

$$\begin{array}{ccc}
X_1 \otimes X_2 & \xrightarrow{e_{X_1} \otimes e_{X_2}} & TX_1 \otimes TX_2 \\
\| & \Downarrow e^2 & \downarrow T^2_{X_1,X_2} \\
X_1 \otimes X_2 & \xrightarrow[e_{X_1 \otimes X_2}]{} & T(X_1 \otimes X_2)
\end{array} \quad\text{and}\quad \begin{array}{ccc}
I & \xrightarrow{\mathrm{id}_I} & I \\
\| & \Downarrow e^0 & \downarrow T^0 \\
I & \xrightarrow[e_I]{} & TI
\end{array} \tag{53}$$

  satisfying the axioms of Definition 5.2;
- the modifications $\mathfrak{a}$, $\mathfrak{l}$, $\mathfrak{r}$ of (49) are monoidal as in Definition 5.5.

Definition 8.3 (Pseudomonoidal vertical double monad)**.** Let $\mathbb{C}$ be a monoidal double category. A *pseudomonoidal vertical double monad* on $\mathbb{C}$ a vertical monoid in $\mathsf{MonDblCat}[\mathbb{C},\mathbb{C}]$. Explicitly, it is a vertical double monad (T, m, e) on $\mathbb{C}$ in the sense of Definition 7.3, such that:

- the double functor $T\colon \mathbb{C} \to \mathbb{C}$ is lax monoidal, as in Definition 5.1;

- the vertical transformation $m\colon TT \Rightarrow T$ is pseudomonoidal, *i.e.*, it comes equipped with 2-morphisms

$$
\begin{array}{ccc}
TTX_1 \otimes TTX_2 & \xrightarrow{\ \mathrm{id}\ } & TTX_1 \otimes TTX_2 \\
\downarrow{\scriptstyle T^2_{TX,TY}} & & \downarrow{\scriptstyle m_{X_1} \otimes m_{X_2}} \\
T(TX_1 \otimes TX_2) & & TX_1 \otimes TX_2 \\
\downarrow{\scriptstyle T(T^2_{X_1,X_2})} \quad \Downarrow m^2 & & \downarrow{\scriptstyle T^2_{X_1,X_2}} \\
TT(X_1 \otimes X_2) & & \\
\downarrow{\scriptstyle m_{X_1 \otimes X_2}} & & \\
T(X_1 \otimes X_2) & \xrightarrow{\ \mathrm{id}\ } & T(X_1 \otimes X_2)
\end{array}
\qquad
\begin{array}{ccc}
I & \xrightarrow{\ \mathrm{id}\ } & I \\
\downarrow{\scriptstyle T^0} & & \downarrow{\scriptstyle T^0} \\
TI & & \\
\downarrow{\scriptstyle T(T^0)} \quad \Downarrow m^0 & & \downarrow{\scriptstyle T^0} \\
TTI & & \\
\downarrow{\scriptstyle m_I} & & \\
TI & \xrightarrow{\ \mathrm{id}\ } & TI
\end{array}
\tag{54}
$$

satisfying the axioms of Definition 5.3;

- the vertical transformation $e\colon 1 \Rightarrow T$ is pseudomonoidal , *i.e.*, it comes equipped with 2-morphisms

$$
\begin{array}{ccc}
X_1 \otimes X_2 & \xrightarrow{\ \mathrm{id}\ } & X_1 \otimes X_2 \\
\downarrow{\scriptstyle e_{X_1 \otimes X_2}} & & \downarrow{\scriptstyle e_{X_1} \otimes e_{X_2}} \\
& \Downarrow e^2 \quad TX_1 \otimes TX_2 & \\
& & \downarrow{\scriptstyle T^2_{X_1,X_2}} \\
T(X_1 \otimes X_2) & \xrightarrow{\ \mathrm{id}\ } & T(X_1 \otimes X_2)
\end{array}
\qquad
\begin{array}{ccc}
I & \xrightarrow{\ \mathrm{id}\ } & I \\
\downarrow{\scriptstyle e_I} \quad \Downarrow e^0 & & \downarrow{\scriptstyle T^0} \\
TI & \xrightarrow{\ \mathrm{id}\ } & TI
\end{array}
\tag{55}
$$

satisfying the axioms of Definition 5.3;

- the pseudomonoidal structures of the composites $m \circ Tm$ and $m \circ mT\colon TTT \Rightarrow T$ are equal, while the pseudomonoidal structures of $m \circ Te$ and $m \circ eT\colon T \Rightarrow T$ are both trivial.

For a horizontal monad that arises from a vertical one via Theorem 7.4, we are naturally interested in conditions on the vertical monad such that the induced horizontal monad is monoidal. Thankfully, the conditions under which a pseudomonoidal vertical double monad induces a monoidal horizontal double monad are the same as for the non-monoidal case of Theorem 7.4, as the next theorem shows.

Theorem 8.4. *Let $\mathbb{C}$ be a monoidal double category and (T, m, e) be a pseudomonoidal vertical double monad on $\mathbb{C}$. Assume that the underlying vertical transformations $m\colon TT \Rightarrow T$ and $e\colon 1_{\mathbb{C}} \Rightarrow T$ are special. Then (T, m, e) induces a monoidal horizontal double monad $(T, \widehat{m}, \widehat{e})$ on $\mathbb{C}$.*

Proof. If we consider T as a vertical monoid in $\mathsf{MonDblCat}[\mathbb{C}, \mathbb{C}]$, Theorem 6.4 ensures that it induces a horizontal pseudomonoid therein (namely a monoidal horizontal monad) when m and e have companions as pseudomonoidal vertical transformations. By Proposition 5.7, this is true if and only if m and e are special (Definition 3.6). For example, the unit of the induced monoidal horizontal double monad structure on T is the horizontal transformation $\widehat{e}\colon 1 \Rrightarrow T$ which becomes monoidal with structure cells

$$
\begin{array}{ccc}
X_1 \otimes X_2 & \xrightarrow{\ \widehat{e}_{X_1} \otimes \widehat{e}_{Y_2}\ } & TX_1 \otimes TX_2 \\
\| & \Downarrow & \downarrow{\scriptstyle T^2_{X_1,X_2}} \\
X_1 \otimes X_2 & \xrightarrow[\widehat{e}_{X_1 \otimes X_2}]{} & T(X_1 \otimes X_2)
\end{array}
\qquad
\begin{array}{ccc}
I & \xrightarrow{\ \mathrm{id}_I\ } & I \\
\| & \Downarrow & \downarrow{\scriptstyle T^0} \\
I & \xrightarrow[\widehat{e}_I]{} & TI
\end{array}
\tag{56}
$$

that bijectively correspond, under transpose operations, to those of (55). $\qquad\square$

While a direct proof of Theorem 8.4 should be possible, the construction of all the data for a monoidal horizontal double monad from that of a pseudomonoidal vertical double monad using companions, let alone the verification of the coherence axioms, would be a daunting task. It is at this point that the advantage of our abstract view becomes clear; as an added bonus, the proofs of Theorems 7.4 and 8.4 become essentially the same.

9. Monoidal Kleisli double categories

In this section, we first introduce the (horizontal) Kleisli double category $\mathbb{Kl}(T)$ for a horizontal double monad T (Definition 7.2) on a double category $\mathbb{C}$, with a particularly important case being where T is induced from a vertical double monad (Definition 7.3) as in Theorem 7.4.

We next consider what happens when the double category $\mathbb{C}$ is monoidal and the double monad T is also monoidal. We would naturally expect the monoidal structure of $\mathbb{C}$ to extend to $\mathbb{Kl}(T)$, just as happens with an ordinary monoidal monad on an ordinary monoidal category. However, because the monoidal constraint data for a horizontal double monad does not point exclusively in the horizontal direction, things are slightly more subtle. To even obtain monoidal structure we must assume certain companions exist, and even then, this structure is only *oplax* monoidal in general (Theorem 9.4). Again, the situation where T is induced from a vertical double monad will be important, and in this special case, we describe sufficient conditions for this oplax monoidal structure on $\mathbb{Kl}(T)$ to be normal oplax (Proposition 9.7) or (pseudo) monoidal (Corollary 9.8).

We begin with the construction of the horizontal Kleisli double category of a horizontal double monad. This construction is essentially contained in [CS10]; there, the authors start from a *vertical* double monad (T, m, e), and define from it a *horizontal* Kleisli double category (Definition 4.1 of *op. cit.*) which in general is only a so-called *virtual* double category. These are weaker structures than double categories, in which horizontal 1-cells do not compose, but instead are formed into a structure of "multi-2-morphisms"; however, [CS10, Theorem A.8] shows that, when the vertical transformations $e: 1 \Rightarrow T$ and $m: TT \Rightarrow T$ are *special*, this virtual double category is in fact a double category. In this case, the horizontal Kleisli double category of *loc. cit.* can be obtained as follows: first apply Theorem 7.4 to form the horizontal double monad $(T, \widehat{m}, \widehat{e})$ associated to (T, m, e); and then apply the following result.

Theorem 9.1. *Let $\mathbb{C}$ be a double category and (T, m, e) be a horizontal double monad on it. There is a double category $\mathbb{Kl}(T)$, called the* horizontal Kleisli double category *of $\mathbb{C}$, wherein:*

- **objects** *are objects of $\mathbb{C}$;*
- **vertical 1-cells** *are vertical 1-cells of $\mathbb{C}$;*
- **horizontal 1-cells** $M: X \rightsquigarrow Y$ *are horizontal 1-cells $X \rightarrow TY$ of $\mathbb{C}$;*
- **2-morphisms** *as to the left below, are the 2-morphisms of $\mathbb{C}$ as to the right:*

$$
\begin{array}{ccc}
X \overset{M}{\rightsquigarrow} Y & \qquad & X \overset{M}{\longrightarrow} TY \\
f \downarrow \quad \Downarrow\phi \quad \downarrow g & & f \downarrow \quad \Downarrow\phi \quad \downarrow Tg \\
X' \underset{M'}{\rightsquigarrow} Y' & & X' \underset{M'}{\longrightarrow} TY'.
\end{array}
$$

Proof. Vertical composition in $\mathbb{Kl}(T)$ is the same as in $\mathbb{C}$; horizontal composition of Kleisli 1-cells $M: X \rightsquigarrow Y$ and $N: Y \rightsquigarrow Z$ is given by

$$N \circ_{\mathbb{Kl}} M := X \xrightarrow{M} TY \xrightarrow{TN} TTZ \xrightarrow{m_Z} TZ; \tag{57}$$

while horizontal pasting of Kleisli 2-morphisms ϕ and ψ is given by

$$
\begin{array}{ccccccc}
X & \xrightarrow{M} & TY & \xrightarrow{TN} & TTZ & \xrightarrow{m_Z} & TZ \\
f\downarrow & \Downarrow\phi & Tg\downarrow & \Downarrow T\psi & TTh\downarrow & \Downarrow m_h & \downarrow Th \\
X' & \xrightarrow[M']{} & TY' & \xrightarrow[TN']{} & TTZ' & \xrightarrow[m_{Z'}]{} & TZ' \ .
\end{array}
$$

The horizontal identity 1-cell on X is

$$\mathrm{id}_X^{\mathbb{Kl}} := X \xrightarrow{e_X} TX \tag{58}$$

and the horizontal identity 2-morphism on $f: X \to X'$ is

$$
\begin{array}{ccc}
X & \xrightarrow{e_X} & TX \\
f\downarrow & \Downarrow e_f & \downarrow Tf \\
X' & \xrightarrow[e_{X'}]{} & TX'.
\end{array}
$$

It is easy to see that horizontal composition of 2-morphisms is vertically functorial, and so it remains to give the coherence constraints. Given Kleisli 1-cells $M\colon X \rightsquigarrow Y$ and $N\colon Y \rightsquigarrow Z$ and $P\colon Z \rightsquigarrow W$, the associativity constraint is given by the following pasting, in which the horizontal 1-cell at the top is $(P \circ_{\mathbb{Kl}} N) \circ_{\mathbb{Kl}} M$ and the one at the bottom is $P \circ_{\mathbb{Kl}} (N \circ_{\mathbb{Kl}} M)$:

$$
\begin{array}{c}
X \xrightarrow{M} TY \xrightarrow{\quad T(m_W \circ TP \circ N)\quad} TTW \xrightarrow{m_W} TW \\
\Downarrow \xi \\
TY \xrightarrow{TN} TTZ \xrightarrow{TTP} TTTW \xrightarrow{Tm_W} TTW \xrightarrow{m_W} TW \\
\Downarrow \mathfrak{a} \\
TTZ \xrightarrow{TTP} TTTW \xrightarrow{m_{TW}} TTW \xrightarrow{m_W} TW \\
\Downarrow m_P \\
X \xrightarrow{M} TY \xrightarrow{TN} TTZ \xrightarrow{m_Z} TZ \xrightarrow{TP} TTW \xrightarrow{m_W} TW.
\end{array}
$$

The unit constraints are as follows, where the horizontal top 1-cells are $\mathrm{id}_Y^{\mathbb{Kl}} \circ_{\mathbb{Kl}} M$ and $M \circ_{\mathbb{Kl}} \mathrm{id}_X^{\mathbb{Kl}}$:

$$
\begin{array}{c}
X \xrightarrow{M} TY \xrightarrow{Te_Y} TTY \xrightarrow{m_Y} TY \\
\| \qquad \| \qquad \Downarrow \mathfrak{l} \qquad \| \\
X \xrightarrow{M} TY \xrightarrow{\quad \mathrm{id}_{TY}\quad} TY
\end{array}
\qquad
\begin{array}{c}
X \xrightarrow{e_X} TX \xrightarrow{TM} TTY \xrightarrow{m_Y} TY \\
\| \quad \Downarrow (e_M)^{-1} \quad \| \qquad \| \\
X \xrightarrow{M} TY \xrightarrow{e_{TY}} TTY \xrightarrow{m_Y} TY \\
\| \qquad \| \qquad \Downarrow \mathfrak{r} \qquad \| \\
X \xrightarrow{M} TY \xrightarrow{\quad \mathrm{id}_{TY}\quad} TY.
\end{array}
$$

Above, the 2-cells labelled $\mathfrak{a}, \mathfrak{l}, \mathfrak{r}$ are as in (49) and the components of m, e are as in (47, 48). The coherence axioms follow by the usual argument for a Kleisli bicategory, *cf.* [CS10, FGHW18]. $\qquad\square$

We have not ascribed any kind of universal property to the construction of the Kleisli double category, and for our purposes we do not need to; however, if we were to do so, then, following [Str72], we would express it in terms of *universal opalgebra* structure on the canonical embedding of $\mathbb{C}$ into $\mathbb{Kl}(T)$:

Definition 9.2. Let $\mathbb{C}$ be a double category and (T, m, e) be a horizontal double monad on it. The *canonical embedding* $F_T\colon \mathbb{C} \to \mathbb{Kl}(T)$ is the double functor which is the identity on objects and vertical 1-cells; sends a horizontal 1-cell $M\colon X \rightarrow Y$ to $e_Y \circ M\colon X \rightsquigarrow Y$, and correspondingly for 2-morphisms between horizontal 1-cells.

Applying Lemma 3.3 to this canonical embedding, we immediately obtain the following result concerning companions in Kleisli double categories (*cf.* [CS10, Proposition 7.5]):

Proposition 9.3. *Let $\mathbb{C}$ be a double category, T a horizontal double monad on $\mathbb{C}$, and $f\colon X \to X'$ a vertical 1-cell of $\mathbb{C}$. If f has a companion as a vertical 1-cell of $\mathbb{C}$, then it has a companion also as a vertical 1-cell of $\mathbb{Kl}(T)$.* $\qquad\square$

We now consider the Kleisli double category when $\mathbb{C}$ is a monoidal double category and T is a monoidal horizontal double monad (Definition 8.2). As discussed above, it does not seem to be true in general that the monoidal structure of $\mathbb{C}$ will extend to $\mathbb{Kl}(T)$; however, under mild assumptions which are satisfied in our applications, we do obtain at least an oplax monoidal structure (Definition 4.1) on $\mathbb{Kl}(T)$:

Theorem 9.4. *Let $\mathbb{C}$ be a monoidal double category and T a monoidal horizontal double monad on $\mathbb{C}$. If the vertical 1-cell $T^0\colon I \to TI$ and each vertical 1-cell $T^2_{X_1, X_2}\colon TX_1 \otimes TX_2 \to T(X_1 \otimes X_2)$ has a companion, then the monoidal structure of $\mathbb{C}$ induces an oplax monoidal structure on $\mathbb{Kl}(T)$.*

Proof. The monoidal structure on the category of objects and vertical 1-cells $\mathbb{Kl}(T)_0 = \mathbb{C}_0$ is inherited from $\mathbb{C}$. For the monoidal structure on the category $\mathbb{Kl}(T)_1$ of horizontal 1-cells and 2-morphisms, we define the tensor product of $M_1\colon X_1 \rightsquigarrow Y_1$ and $M_2\colon X_2 \rightsquigarrow Y_2$ and the monoidal unit J to be:

$$
M_1 \boxtimes M_2 := X_1 \otimes X_2 \xrightarrow{M_1 \otimes M_2} TY_1 \otimes TY_2 \xrightarrow{\widehat{T^2_{Y_1, Y_2}}} T(Y_1 \otimes Y_2), \qquad J := I \xrightarrow{\widehat{T^0}} TI;
$$

while the binary tensor product of 2-morphisms is given by:

$$
\begin{array}{ccc}
X_1 \xrightarrow{M_1} TY_1, & X_2 \xrightarrow{M_2} TY_2 \\
f_1 \downarrow \quad \Downarrow\phi \quad \downarrow Tg_1 & f_2 \downarrow \quad \Downarrow\psi \quad \downarrow Tg_2 \\
X_1' \xrightarrow{M_1'} TY_1', & X_2' \xrightarrow{M_2'} TY_2'
\end{array}
\quad \mapsto \quad
\begin{array}{c}
X_1 \otimes X_2 \xrightarrow{M_1 \otimes M_2} TY_1 \otimes TY_2 \xrightarrow{\widehat{T^2_{Y_1,Y_2}}} T(Y_1 \otimes Y_2) \\
f_1 \otimes f_2 \downarrow \quad \Downarrow\phi\otimes\psi \quad Tg_1\otimes Tg_2 \downarrow \quad \Downarrow \quad \downarrow T(g_1 \otimes g_2) \\
X_1' \otimes X_2' \xrightarrow[M_1'\otimes M_2']{} TY_1' \otimes TY_2' \xrightarrow[\widehat{T^2_{Y_1',Y_2'}}]{} T(Y_1' \otimes Y_2')
\end{array}
$$

where the right-hand 2-morphism is a companion transpose of the equality $T(g_1 \otimes g_2) \circ T^2_{Y_1,Y_2} = T^2_{Y_1',Y_2'} \circ Tg_1 \otimes Tg_2$ of vertical 1-cells expressing naturality of T^2. The associativity constraint is given by the following pasting, where the horizontal composite at the top is $(M_1 \boxtimes M_2) \boxtimes M_3$ and the one at the bottom is $M_1 \boxtimes (M_2 \boxtimes M_3)$:

$$
\begin{array}{c}
(X_1 \otimes X_2) \otimes X_3 \xrightarrow{(\widehat{T^2}\circ(M_1\otimes M_2))\otimes M_3} T(Y_1 \otimes Y_2) \otimes TY_3 \xrightarrow{\widehat{T^2}} T((Y_1 \otimes Y_2) \otimes Y_3) \\
\| \qquad \Downarrow\tau\,(23) \qquad \| \qquad \| \\
(X_1 \otimes X_2) \otimes X_3 \xrightarrow{(M_1\otimes M_2)\otimes X_3} (TY_1 \otimes TY_2) \otimes TY_3 \xrightarrow{\widehat{T^2}\otimes \mathrm{id}} T(Y_1 \otimes Y_2) \otimes TY_3 \xrightarrow{\widehat{T^2}} T((Y_1 \otimes Y_2) \otimes Y_3) \\
\alpha_{X_1,X_2,X_3} \downarrow \quad \Downarrow\alpha_{M_1,M_2,M_3}\,(22) \quad \downarrow\alpha_{TY_1,TY_2,TY_3} \qquad \Downarrow(*) \qquad \downarrow T\alpha_{Y_1,Y_2,Y_3} \\
X_1 \otimes (X_2 \otimes X_3) \xrightarrow{M_1\otimes(M_2\otimes M_3)} TY_1 \otimes (TY_2 \otimes TY_3) \xrightarrow{\mathrm{id}\otimes\widehat{T^2}} TY_1 \otimes T(Y_2 \otimes Y_3) \xrightarrow{\widehat{T^2}} T(Y_1 \otimes (Y_2 \otimes Y_3)) \\
\| \qquad \Downarrow\tau^{-1}\,(23) \qquad \| \qquad \| \\
X_1 \otimes (X_2 \otimes X_3) \xrightarrow{M_1\otimes(\widehat{T^2}\circ(M_2\otimes M_3))} TY_1 \otimes T(Y_2 \otimes Y_3) \xrightarrow{\widehat{T^2}} T(Y_1 \otimes (Y_2 \otimes Y_3)).
\end{array}
$$

Here, the 2-cell $(*)$ is the transpose of the equality $(T\alpha)T^2(T^2 \otimes 1) = T^2(1 \otimes T^2)\alpha$ of vertical 1-cells expressing the associativity axiom for the vertical part of the monoidal double functor T; note that $(*)$ is invertible (as all other 2-cells in the above composite) by Proposition 2.8(iii) and (vi). The unit constraints are formed similarly as follows, where the top horizontal 1-cells are $J \boxtimes M$ and $M \boxtimes J$, respectively:

$$
\begin{array}{cc}
\begin{array}{c}
I \otimes X \xrightarrow{\widehat{T^0}\otimes M} TI \otimes TY \xrightarrow{\widehat{T^2_{I,Y}}} T(I \otimes Y) \\
\| \quad \Downarrow\tau \quad \| \quad \| \\
I \otimes X \xrightarrow{\mathrm{id}\otimes M} I \otimes TY \xrightarrow{\widehat{T^0}\otimes\mathrm{id}} TI \otimes TY \xrightarrow{\widehat{T^2_{I,Y}}} T(I \otimes Y) \\
\lambda\downarrow \quad \Downarrow\lambda_M \quad \downarrow\lambda \quad \Downarrow(*) \quad \downarrow T\lambda \\
X \xrightarrow{M} TY \xrightarrow{\mathrm{id}} TY,
\end{array}
&
\begin{array}{c}
X \otimes I \xrightarrow{M\otimes\widehat{T^0}} TY \otimes TI \xrightarrow{\widehat{T^2_{Y,I}}} T(Y \otimes I) \\
\| \quad \Downarrow\tau \quad \| \quad \| \\
X \otimes I \xrightarrow{M\otimes\mathrm{id}} TY \otimes I \xrightarrow{\mathrm{id}\otimes\widehat{T^0}} TY \otimes TI \xrightarrow{\widehat{T^2_{Y,I}}} T(Y \otimes I) \\
\rho\downarrow \quad \Downarrow\rho_M \quad \downarrow\rho \quad \Downarrow(*) \quad \downarrow T\rho \\
X \xrightarrow{M} TY \xrightarrow{\mathrm{id}} TY.
\end{array}
\end{array}
\tag{59}
$$

Here, the 2-cells $(*)$ are the transposes of the unit axioms for the vertical part of the monoidal double functor T, and are invertible by the same reasoning as before.

The oplax monoidal structure map to the left of (23) has the form:

$$
\begin{array}{c}
X_1 \otimes X_2 \xrightarrow{(N_1\circ_{\mathbb{KI}}M_1)\boxtimes(N_2\circ_{\mathbb{KI}}M_2)} Z_1 \otimes Z_2 \\
\| \qquad \Downarrow\tau \qquad \| \\
X_1 \otimes X_2 \xrightarrow[M_1\boxtimes M_2]{} Y_1 \otimes Y_2 \xrightarrow[N_1\boxtimes N_2]{} Z_1 \otimes Z_2
\end{array}
$$

where $\circ_{\mathbb{Kl}}$ is defined as in (57); we obtain it as the pasting composite

$$(60)$$

where the top left isomorphism is the monoidal interchange (23) applied twice, the 2-morphism labelled $\widehat{m^2}$ is a companion transpose of the structure 2-morphism m^2 of the monoidal horizontal transformation m as in (52), and the 2-morphism $\widehat{T^2_{N_1,N_2}}$ is a companion transpose of the component $T^2_{N_1,N_2}$ of the lax monoidal structure on the double functor T as in (34).

The globular 2-morphism η to the right of (23) is obtained as follows, where the horizontal 1-cell at the top is $\mathrm{id}^{\mathbb{Kl}}_{X_1} \boxtimes \mathrm{id}^{\mathbb{Kl}}_{X_2}$ and the one at the bottom is $\mathrm{id}^{\mathbb{Kl}}_{X_1 \otimes X_2}$:

$$
\begin{array}{ccc}
X_1 \otimes X_2 \xrightarrow{e_{X_1} \otimes e_{X_2}} TX_1 \otimes TX_2 \xrightarrow{\widehat{T^2_{X_1,X_2}}} T(X_1 \otimes X_2) \\
\| \qquad\qquad \Downarrow \widehat{e^2} \qquad\qquad \| \\
X_1 \otimes X_2 \xrightarrow[\quad e_{X_1 \otimes X_2} \quad]{} T(X_1 \otimes X_2).
\end{array}
\qquad (61)
$$

where $\mathrm{id}^{\mathbb{Kl}}$ is defined as in (58). Here, the 2-morphism filling the square is a companion transpose of the structure 2-morphism e^2 of the monoidal horizontal transformation e as in (53). Finally, the globular structure 2-morphisms δ and ι of (24) are defined to be

$$
\delta = \begin{array}{ccc}
I \xrightarrow{\quad \widehat{T^0} \quad} TI \\
\| \quad \Downarrow \widehat{m^0} \quad \| \\
I \xrightarrow[\widehat{T^0}]{} TI \xrightarrow[TT\widehat{T^0}]{} TTI \xrightarrow[m_I]{} TI
\end{array}
\qquad
\iota = \begin{array}{ccc}
I \xrightarrow{\widehat{T^0}} TI \\
\| \quad \Downarrow \widehat{e^0} \quad \| \\
I \xrightarrow[e_I]{} TI
\end{array}
\qquad (62)
$$

obtained as the companion transpose of the structure 2-morphism m^0 from (52) (using that $TT\widehat{T^0}$ is a companion of TT^0 by Lemma 3.3); and the companion transpose of the structure 2-morphism e^0 from (53) respectively.

With some effort, one may show that with these structure cells, the horizontal double Kleisli category $\mathbb{Kl}(T)$ is an oplax monoidal double category in the sense of Definition 4.1. We do not provide the details here, but in Appendix A.2 we give some sample verifications, along with a number of technical lemmas used repeatedly in the calculations. $\square$

It is very natural to ask when the oplax monoidal structure of the preceding definition is in fact a genuine (pseudo) monoidal structure, or at least a *normal* oplax monoidal structure. For our purposes, we will only answer this question in the case of primary interest, where our monoidal horizontal monad is induced from a pseudomonoidal vertical monad (Definition 8.3). To start with, putting together Theorem 8.4 and Theorem 9.4 gives us:

Corollary 9.5. *Let $\mathbb{C}$ be a monoidal double category and T be a pseudomonoidal vertical double monad. If it is true that:*

    (i) the multiplication and unit of T are special; and

    (ii) all vertical 1-cells $T^2_{X_1,X_2} : TX_1 \otimes TX_2 \to T(X_1 \otimes X_2)$ and $T^0 : I \to TI$ have companions,

then the Kleisli double category $\mathbb{Kl}(T)$ of the induced monoidal horizontal monad $(T, \widehat{m}, \widehat{e})$ admits an oplax monoidal structure found as in Theorem 9.4. $\square$

Remark 9.6. Notice that in the situation of the above corollary, it is only the interchange 2-morphisms τ of the oplax monoidal structure which may not be invertible. Indeed, each of the other structure 2-morphisms η, δ, ι, as displayed in (61) and (62), must be invertible; for example, in this case η is the

transpose of the 2-morphism (56) which is, in turn, the transpose of the $\mathsf{V}(\mathbb{C})$-invertible 2-morphism e^2 (55) of the pseudomonoidal vertical transformation e, and as such, is invertible by Proposition 2.8(vi).

As explained above, we will now investigate when, in the situation of Corollary 9.5, the oplax monoidal structure on $\mathbb{K}\mathsf{l}(T)$ is in fact normal in the sense of Definition 4.3. To this end, motivated by the theory of pseudo-commutative monads [HP02], we define for a pseudomonoidal vertical monad (T, m, e) as in Definition 8.3 a vertical double transformation $\kappa\colon (-) \otimes T(?) \Rightarrow T \circ (- \otimes ?)$ called the *strength*; this is given as the vertical composite $T^2 \circ (e \otimes 1)$ with components

$$
\begin{array}{ccc}
\begin{array}{ccc}
X_1 \otimes TX_2 & \xrightarrow{M_1 \otimes TM_2} & Y_1 \otimes TY_2 \\
\kappa_{X_1, X_2} \downarrow & \Downarrow \kappa_{M_1, M_2} & \downarrow \kappa_{Y_1, Y_2} \\
T(X_1 \otimes X_2) & \xrightarrow[T(M_1 \otimes M_2)]{} & T(Y_1 \otimes Y_2)
\end{array}
& := &
\begin{array}{ccc}
X_1 \otimes TX_2 & \xrightarrow{M_1 \otimes TM_2} & Y_1 \otimes TY_2 \\
e_{X_1} \otimes 1 \downarrow & \Downarrow e_{M_1} \otimes 1_{TM_2} & \downarrow e_{Y_1} \otimes 1 \\
TX_1 \otimes TX_2 & \xrightarrow{TM_1 \otimes TM_2} & TY_1 \otimes TY_2 \\
T^2_{X_1, X_2} \downarrow & \Downarrow T^2_{M_1, M_2} & \downarrow T^2_{Y_1, Y_2} \\
T(X_1 \otimes X_2) & \xrightarrow[T(M_1 \otimes M_2)]{} & T(Y_1 \otimes Y_2) \ .
\end{array}
\end{array}
\tag{63}
$$

In an analogous way, we can also define the *costrength* as a vertical transformation $T(-) \otimes (?) \Rightarrow T \circ (- \otimes ?)$. It turns out that requiring these two vertical transformations to be special, as in Definition 3.6, is sufficient to make the oplax monoidal structure on $\mathbb{K}\mathsf{l}(T)$ normal:

Proposition 9.7. *Let $\mathbb{C}$ be a monoidal double category and T be a pseudomonoidal vertical double monad. If it is true that:*

    (i) the multiplication and unit of T are special;
    (ii) all vertical 1-cells $T^2_{X_1, X_2}\colon TX_1 \otimes TX_2 \to T(X_1 \otimes X_2)$ and $T^0\colon I \to TI$ have companions; and
    (iii) the strength and costrength of T are special vertical transformations,

then the oplax monoidal double structure on $\mathbb{K}\mathsf{l}(T)$ found as in Corollary 9.5 is normal.

Proof. We need to verify that each oplax double functor $X_1 \boxtimes (-)\colon \mathbb{K}\mathsf{l}(T) \to \mathbb{K}\mathsf{l}(T)$ and $(-) \boxtimes X_2\colon \mathbb{K}\mathsf{l}(T) \to \mathbb{K}\mathsf{l}(T)$ is in fact a (pseudo) double functor. By Remark 9.6, we already know that the square η in (61) is invertible, which expresses the invertibity of the identity constraints for these double functors. As for the binary functoriality constraints, it suffices by symmetry to consider the case of $X_1 \boxtimes (-)$. To say that its binary constraints are invertible is to say that the 2-morphism in (60) is invertible when $X = X_1 = Y_1 = Z_1$ and $M_1 = N_1 = \widehat{e}_{X_1}$. The 2-morphism in question is given by:

$$\tag{64}$$

Clearly the first and final rows of this diagram are invertible. On the second row, the 2-morphism $\widehat{m^2}$ corresponds under transpose to the $\mathsf{V}(\mathbb{C})$-invertible cell m^2 of the pseudomonoidal vertical transformation m in (54), and as such is invertible by Proposition 2.8(vi). Thus, we will be done if we can also prove the invertibility of the third row of the diagram.

Now, since the strength (63) is by assumption a special vertical transformation, it is in particular true that the companion transpose of the component $\kappa_{\widehat{e}, N_2}$ is invertible. This transpose is equally well the

composite:

$$
\begin{array}{ccccccc}
X_1 \otimes TY_2 & \xrightarrow{\hat{e}\otimes TN_2} & TX_1 \otimes TTZ_2 & \xrightarrow{T\hat{e}\otimes 1} & TTX_1 \otimes TTZ_2 & \xrightarrow{\widehat{T^2}} & T(TX_1 \otimes TZ_2) \\
\| & & \Downarrow (*) & & \| & & \| \\
X_1 \otimes TY_2 & \xrightarrow{\hat{e}\otimes 1} & TX_1 \otimes TY_2 & \xrightarrow{T\hat{e}\otimes TN_2} & TTX_1 \otimes TTZ_2 & \xrightarrow{\widehat{T^2}} & T(TX_1 \otimes TZ_2) \\
\| & & \| & & \Downarrow \widehat{T^2} & & \| \\
X \otimes TY_2 & \xrightarrow[\hat{e}\otimes 1]{} & TX \otimes TY_2 & \xrightarrow[\widehat{T^2}]{} & T(X_1 \otimes Y_2) & \xrightarrow[T(\hat{e}\otimes N_2)]{} & T(TX_2 \otimes TZ_2)
\end{array}
$$

where the 2-morphism $(*)$ on the top row is the transpose of the 2-cell $e_{\hat{e}_{X_1}} \otimes 1_{N_2}$. But by Lemma 3.12, this $(*)$ is itself the transpose of a vertical identity, and as such, is invertible by Proposition 2.8(vi). It follows that the composite 2-morphism comprising the bottom row of this diagram is invertible, which now implies the invertibility of the third row of (64) as desired. $\square$

Although this will not be the case in our applications, we note in particular the following sufficient conditions for the induced oplax monoidal structure on $\mathbb{Kl}(T)$ to be not just normal oplax, but in fact a genuine (pseudo) monoidal structure.

Corollary 9.8. *Let $\mathbb{C}$ be a monoidal double category and T be a pseudomonoidal vertical double monad. If it is true that:*

    (i) the multiplication and unit of T are special;

    (ii) all vertical 1-cells $T^2_{X_1,X_2} \colon TX_1 \otimes TX_2 \to T(X_1 \otimes X_2)$ and $T^0 \colon I \to TI$ have companions; and

    (iii) the monoidality constraint T^2 of T is a special vertical transformation,

then the oplax monoidal double structure on $\mathbb{Kl}(T)$ found as in Corollary 9.5 is genuinely monoidal.

Proof. η, δ and ι are already known to be invertible, and, arguing as before, the assumption that T^2 is special ensures that *every* component (60) of the oplax monoidal interchange τ is invertible. $\square$

In the situation of Proposition 9.7, the fact that $\mathbb{Kl}(T)$ is a normal oplax monoidal double category implies that its horizontal bicategory inherits the monoidal structure, in the sense specified in Definition 4.8. We thus obtain the following result as the culmination of the abstract development of the paper thus far. This will be the result we use to obtain the monoidal structure on the bicategory of coloured symmetric sequences in Section 10.

Corollary 9.9. *Let $\mathbb{C}$ be a monoidal double category and T be a pseudomonoidal vertical double monad. Under the assumptions of Proposition 9.7, the horizontal bicategory of $\mathbb{Kl}(T)$ admits a normal oplax monoidal structure.*

Proof. This follows from Theorem 4.10 and Propositions 9.3 and 9.7. $\square$

10. The arithmetic product of coloured symmetric sequences

In this section, we apply the theory developed in the previous sections to our intended application, namely coloured symmetric sequences. Throughout this section, we fix a cocomplete *cartesian closed* category $\mathcal{V}$, considered as a symmetric monoidal closed category with respect to its cartesian closed structure. The restriction to a cartesian monoidal structure was already made in [DH14, GL16] and indeed it is essential for some of our results, as we explain further below.

In order to help readers follow our development, let us display the main double categories to be considered in this section in a commutative diagram of inclusions:

$$
\begin{array}{ccc}
\mathbb{Mat}_{\mathcal{V}} & \longrightarrow & \mathbb{Sym}_{\mathcal{V}} \\
\downarrow & & \downarrow \\
\mathbb{Prof}_{\mathcal{V}} & \longrightarrow & \mathbb{CatSym}_{\mathcal{V}}.
\end{array}
$$

On the left-hand side of the diagram, $\mathbb{Mat}_{\mathcal{V}}$ is the double category of matrices of Example 2.3 and $\mathbb{Prof}_{\mathcal{V}}$ is the double category of profunctors of Example 2.4. On the right-hand side of the diagram, $\mathbb{CatSym}_{\mathcal{V}}$ is the double category of categorical symmetric sequences which arises from $\mathbb{Prof}_{\mathcal{V}}$ as a Kleisli double category, and $\mathbb{Sym}_{\mathcal{V}}$ is its full double subcategory spanned by discrete $\mathcal{V}$-categories—much like the double category $\mathbb{Mat}_{\mathcal{V}}$ is a full double subcategory of $\mathbb{Prof}_{\mathcal{V}}$. We will define the double categories $\mathbb{CatSym}_{\mathcal{V}}$

and $\mathbb{Sym}_{\mathcal{V}}$ explicitly in Theorem 10.4, but in order to do so, we must first introduce the relevant double monad for the Kleisli construction.

Let X be a small $\mathcal{V}$-category. For $n \in \mathbb{N}$, let us define the $\mathcal{V}$-category $S_n(X)$ as follows. The objects of $S_n(X)$ are n-tuples $\vec{x} = (x_1, \ldots, x_n)$ of objects of X. Given two such n-tuples $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{x}' = (x_1', \ldots, x_n')$, the hom-object of maps between them is defined by

$$S_n(X)[\vec{x}, \vec{x}'] := \bigsqcup_{\sigma \in \mathfrak{S}_n} \prod_{1 \leqslant i \leqslant n} X[x_{\sigma(i)}, x_i']$$

where $\mathfrak{S}_n$ is the n-th symmetric group, and where $\bigsqcup$ and $\prod$ denote coproduct and product respectively. We then let SX be the following coproduct in $\mathsf{Cat}_{\mathcal{V}}$:

$$SX = \bigsqcup_{n \in \mathbb{N}} S_n(X).$$

The $\mathcal{V}$-category SX admits a symmetric strict monoidal structure in which the tensor product, written as $\vec{x}, \vec{y} \mapsto \vec{x} \otimes \vec{y}$, is given by concatenation of sequences; the tensor unit is given by the empty sequence, written ( ); and the symmetry is given by the evident permutations. The operation mapping X to SX extends to a 2-functor $S \colon \mathsf{Cat}_{\mathcal{V}} \to \mathsf{Cat}_{\mathcal{V}}$, which is part of a 2-monad whose strict algebras are symmetric strict monoidal $\mathcal{V}$-categories. The multiplication of this 2-monad has components $m_X \colon SSX \to SX$, for $X \in \mathsf{Cat}_{\mathcal{V}}$, defined by taking a list of lists to its flattening:

$$m_X(\vec{x}^1, \ldots, \vec{x}^k) := \vec{x}^1 \otimes \ldots \otimes \vec{x}^k.$$

The unit of the 2-monad has components $e_X \colon X \to SX$, for $X \in \mathsf{Cat}_{\mathcal{V}}$, defined by taking an object $x \in X$ to the singleton list $(x) \in SX$.

We now show that, firstly, S extends to a vertical double monad on $\mathbb{P}\mathsf{rof}_{\mathcal{V}}$, and secondly, that this vertical double monad can be turned into a horizontal double monad. To say that S extends to a vertical double monad is equivalently to say that the underlying 2-functor of S extends along the inclusion of bicategories $\mathsf{Cat}_{\mathcal{V}} \to \mathbb{P}\mathsf{rof}_{\mathcal{V}}$; while to say that this vertical monad can be turned into a horizontal one amounts to saying that the whole 2-monad S extends from $\mathsf{Cat}_{\mathcal{V}}$ to $\mathbb{P}\mathsf{rof}_{\mathcal{V}}$. This is a known result, and there are two approaches in the literature to proving it. The first uses the theory of pseudo-distributive laws; see, for example [FGHW18]. The second, which we follow here, is essentially a categorification of the approach of [Bar70].

Proposition 10.1. *The free symmetric strict monoidal category 2-monad S on $\mathsf{Cat}_{\mathcal{V}}$ extends in an essentially unique way to a vertical double monad on $\mathbb{P}\mathsf{rof}_{\mathcal{V}}$. The multiplication and unit vertical transformations of this double monad are special.*

Here, we say that a vertical double monad T on a double category $\mathbb{C}$ *extends* a 2-monad R on the vertical 2-category $\mathsf{V}(\mathbb{C})$, if R is isomorphic (as a 2-monad) to the 2-monad on $\mathsf{V}(\mathbb{C})$ induced by T.

Proof. Because any double functor preserves companions (Lemma 3.3), any extension of S to $\mathbb{P}\mathsf{rof}_{\mathcal{V}}$ must satisfy $S(\widehat{F}) \cong \widehat{SF}$ for a $\mathcal{V}$-functor F. Because $\widehat{F} \dashv \check{F}$ in $\mathcal{H}(\mathbb{P}\mathsf{rof}_{\mathcal{V}})$, and any double functor preserves adjunctions in the horizontal bicategory, we must also have $S(\check{F}) \cong \check{SF}$. Since by [Str80, §6], every $\mathcal{V}$-profunctor $M \colon X \longrightarrow Y$ admits a globular isomorphism to one of the form $\check{G} \circ \widehat{F}$ for a suitable cospan of $\mathcal{V}$-functors $F \colon X \to Z \leftarrow Y \colon G$, the preceding conditions determine the action of S on horizontal 1-cells of $\mathbb{P}\mathsf{rof}_{\mathcal{V}}$ to within unique isomorphism. Using this idea, one obtains the following explicit definition: given $M \colon X \longrightarrow Y$, $SM \colon SX \longrightarrow SY$ is defined as the $\mathbb{N}$-indexed coproduct in $(\mathbb{P}\mathsf{rof}_{\mathcal{V}})_1$ of $S_n(M) \colon S_n(X) \longrightarrow S_n(Y)$, where

$$S_n(M)(\vec{y}, \vec{x}) := \bigsqcup_{\sigma \in \mathfrak{S}_n} \prod_{1 \leqslant i \leqslant n} M(y_{\sigma(i)}, x_i). \tag{65}$$

The action of S on 2-morphisms of $\mathbb{P}\mathsf{rof}_{\mathcal{V}}$ is now forced by Proposition 2.8(iii) and the fact that any double functor preserves globularity; the reader will easily guess an explicit formula, and this guess is the correct one. This completes the definition of S *qua* double functor on $\mathbb{P}\mathsf{rof}_{\mathcal{V}}$; that these data are indeed double functorial is verified, for example, in [Gar06b, Proposition 55], to which we refer for further details.

We now extend the unit e and multiplication m of the 2-monad S; the missing data are the 2-morphism components e_M and m_M (50, 51) associated to a horizontal 1-cell $M \colon X \longrightarrow Y$. Writing $M \cong \check{G} \circ \widehat{F}$ as

before, we see that e_M and m_M are determined by $e_{\widehat{F}}$, $e_{\check{G}}$, $m_{\widehat{F}}$ and $m_{\check{G}}$:

$$
e_M =
\begin{array}{c}
\begin{array}{ccccc}
X & \xrightarrow{\quad M \quad} & & & Z \\
\| & & \cong & & \| \\
X & \xrightarrow{\widehat{F}} & Z & \xrightarrow{\check{G}} & Y \\
e_X\downarrow & \Downarrow e_{\widehat{F}} & e_Z\downarrow\ \Downarrow e_{\check{G}} & & \downarrow e_Y \\
SX & \xrightarrow{S\widehat{F}} & SZ & \xrightarrow{S\check{G}} & SY \\
\| & & \cong & & \| \\
SX & \xrightarrow{\quad SM \quad} & & & SY
\end{array}
\end{array}
\quad\text{and}\quad
m_M =
\begin{array}{c}
\begin{array}{ccccc}
SSX & \xrightarrow{\quad SSM \quad} & & & SSZ \\
\| & & \cong & & \| \\
SSX & \xrightarrow{SS\widehat{F}} & SSZ & \xrightarrow{SS\check{G}} & SSY \\
m_X\downarrow & \Downarrow m_{\widehat{F}} & m_Z\downarrow\ \Downarrow m_{\check{G}} & & \downarrow m_Y \\
SX & \xrightarrow{S\widehat{F}} & SZ & \xrightarrow{S\check{G}} & SY \\
\| & & \cong & & \| \\
SX & \xrightarrow{\quad SM \quad} & & & SY.
\end{array}
\end{array}
\tag{66}
$$

To the left, Lemma 3.12 implies that $e_{\widehat{F}}$ is the companion transpose of the identity of vertical 1-cells $e_Z \circ F = SF \circ e_X$; while a suitable dual of Lemma 3.12 implies that $e_{\check{F}}$ is the *conjoint* transpose of the identity $e_Z \circ G = SG \circ e_Y$. In a similar way, the 2-morphisms $m_{\widehat{F}}$ and $m_{\check{G}}$ are forced. An explicit verification that these data satisfy the vertical double monad axioms is, again, given in [Gar06b, Proposition 55].

Before continuing, we note a fact which will be used repeatedly in what follows. Suppose given profunctors $N\colon X \nrightarrow Y$ and $M\colon SY \nrightarrow Z$. Then for any $z \in Z$ and $\vec{x} = (x_1, \ldots, x_n) \in SX$, the value at $(z, \vec{x})$ of the composite $M \circ SN\colon SX \nrightarrow SY \nrightarrow Z$, as to the left below, is equally given as to the right:

$$
\int^{\vec{y} \in SY} M(z, \vec{y}) \times SN(\vec{y}, \vec{x}) \cong \int^{\vec{y} \in Y^n} M(z, \vec{y}) \times \prod_{1 \leqslant i \leqslant n} N(y_i, x_i) .
\tag{67}
$$

Indeed, we can immediately reduce the left-hand coend to one over $\vec{y} \in S_n Y$; and for such a $\vec{y}$, we have $M(z, \vec{y}) \times SN(\vec{y}, \vec{x}) \cong \bigsqcup_{\sigma \in \mathfrak{S}_n} M(z, \vec{y}) \times \prod_{1 \leqslant i \leqslant n} N(y_{\sigma(i)}, x_i)$. On the σ-summand of this coproduct, we define the component of the desired isomorphism (67) to be

$$
M(z, \vec{y}) \times \prod_{1 \leqslant i \leqslant n} N(y_{\sigma(i)}, x_i) \xrightarrow{M(1, \sigma^{-1}) \times 1} M(z, \sigma^* \vec{y}) \times \prod_{1 \leqslant i \leqslant n} N\big((\sigma^* \vec{y})_i, x_i\big) \hookrightarrow \int^{\vec{y} \in Y^n} M(z, \vec{y}) \times \prod_{1 \leqslant i \leqslant n} N(y_i, x_i)
$$

where $(\sigma^* \vec{y})_i = y_{\sigma(i)}$ and where $\sigma^{-1}\colon \vec{y} \to \sigma^* \vec{y}$ is the evident symmetry isomorphism in SY.

It remains only to check that the extended vertical transformations e and m are special, *i.e.* that the following companion transpose 2-morphisms are invertible:

$$
\begin{array}{c}
\begin{array}{ccccc}
Z & \xrightarrow{\quad M \quad} & Y & \xrightarrow{\quad \widehat{e}_Y \quad} & SY \\
\| & & \Downarrow \widehat{e}_M & & \| \\
Z & \xrightarrow{\widehat{e}_Z} & SZ & \xrightarrow{SM} & SY
\end{array}
\end{array}
\quad\text{and}\quad
\begin{array}{c}
\begin{array}{ccccc}
SSZ & \xrightarrow{\quad SSM \quad} & SSY & \xrightarrow{\quad \widehat{m}_Y \quad} & SY \\
\| & & \Downarrow \widehat{m}_M & & \| \\
SSZ & \xrightarrow{\widehat{m}_Z} & SZ & \xrightarrow{SM} & SY.
\end{array}
\end{array}
$$

Starting to the left, let $z \in Z$ and $\vec{y} = (y_1, \ldots, y_n) \in SY$. To within isomorphism, using the formulas for composition and companions for profunctors (8, 11) as well as (65), the profunctor at the bottom of the square sends $(\vec{y}, z)$ to

$$
SM(\vec{y}, (z)) = \begin{cases} M(y_1, z) & \text{if } n = 1; \\ 0 & \text{otherwise.} \end{cases}
\tag{68}
$$

On the other hand, the profunctor around the top sends $(\vec{y}, z)$ to:

$$
\int^{y' \in Y} SY[\vec{y}, (y')] \times M(y', z) = \begin{cases} \int^{y' \in Y} Y[y_1, y'] \times M(y', z) & \text{if } n = 1; \\ 0 & \text{otherwise.} \end{cases}
\tag{69}
$$

In the only non-trivial case where $n = 1$, the comparison 2-cell $\widehat{e}_M$ from (69) to (68) is given by composition: and this is invertible by the Yoneda lemma.

We proceed similarly for $\widehat{m}_M$. Let $\vec{z} = (\vec{z}^1, \ldots, \vec{z}^n) \in SSZ$ with $\vec{z}^i = (z_{m_{i-1}+1}, \ldots, z_{m_i})$ for some $0 = m_0 \leqslant m_1 \leqslant \cdots \leqslant m_n$, and let $\vec{y} = (y_1, \ldots, y_m) \in SY$. The only non-trivial case is when $m_n = m$ so we immediately restrict to that. This time, the profunctor in the bottom row sends $(\vec{y}, \vec{z})$ to

$$
SM\Big(\vec{y}, \bigotimes_{1 \leqslant i \leqslant n} \vec{z}^i\Big) = \bigsqcup_{\sigma \in \mathfrak{S}_m} \prod_{1 \leqslant i \leqslant m} M(y_{\sigma(m)}, z_m).
\tag{70}
$$

On the other hand, the profunctor around the top sends $(\vec{y}, \vec{z})$ to

$$
\int^{\vec{w} \in SSY} SY[\vec{y}, \bigotimes_{1 \leqslant i \leqslant m} \vec{w}^i] \times SSM(\vec{w}, \vec{z}) \cong \int^{\vec{w}^1, \ldots, \vec{w}^n \in SY} SY[\vec{y}, \bigotimes_{1 \leqslant i \leqslant m} \vec{w}^i] \times \prod_{1 \leqslant j \leqslant n} SM(\vec{w}^j, \vec{z}^j)
$$

$$
\cong \int^{w_1, \ldots, w_m \in Y} SY[\vec{y}, \vec{w}] \times \prod_{\substack{1 \leqslant j \leqslant n \\ m_{i-1}+1 \leqslant i \leqslant m_i}} M(w_i, z_i) \tag{71}
$$

$$
\cong \bigsqcup_{\sigma \in \mathfrak{S}_m} \int^{w_1, \ldots, w_m \in Y} \prod_{1 \leqslant i \leqslant m} Y[y_{\sigma(i)}, w_i] \times M(w_i, z_i),
$$

using (67) once at the first step, and n times at the second step. The comparison 2-morphism $\widehat{m}_M$ from (71) to (70) is again given by composition, and this is again invertible by the Yoneda lemma. $\square$

Corollary 10.2. *The 2-monad $S \colon \mathsf{Cat}_{\mathcal{V}} \to \mathsf{Cat}_{\mathcal{V}}$ induces a horizontal double monad $S \colon \mathbb{P}\mathsf{rof}_{\mathcal{V}} \to \mathbb{P}\mathsf{rof}_{\mathcal{V}}$.*

Proof. Apply Theorem 7.4 to the vertical double monad of Proposition 10.1 to obtain the horizontal double monad $(S, \widehat{m}, \widehat{e})$. $\square$

We are now ready to recall the definition of categorical and coloured symmetric sequences.

Definition 10.3 (Categorical and coloured symmetric sequences).
- Let X and Y be small $\mathcal{V}$-categories. A *categorical symmetric sequence* $M \colon X \rightsquigarrow Y$ is a profunctor $M \colon X \rightarrow SY$, i.e. a $\mathcal{V}$-functor $M \colon SY^{\mathrm{op}} \times X \to \mathcal{V}$.
- Let X and Y be sets. A *coloured symmetric sequence* $M \colon X \rightsquigarrow Y$ is a categorical symmetric sequence from X to Y, considered as discrete $\mathcal{V}$-categories.

Categorical symmetric sequences and coloured symmetric sequences are the horizontal 1-cells of double categories that we denote $\mathbb{C}\mathsf{atSym}_{\mathcal{V}}$ and $\mathbb{S}\mathsf{ym}_{\mathcal{V}}$, which we may now obtain by forming the horizontal Kleisli double category (Theorem 9.1) of the horizontal double monad $S \colon \mathbb{P}\mathsf{rof}_{\mathcal{V}} \to \mathbb{P}\mathsf{rof}_{\mathcal{V}}$.

Theorem 10.4.
  (i) *There exists a double category $\mathbb{C}\mathsf{atSym}_{\mathcal{V}}$ having small $\mathcal{V}$-categories as objects, $\mathcal{V}$-functors as vertical 1-cells and categorical symmetric sequences as horizontal 1-cells.*
  (ii) *There exists a double subcategory $\mathbb{S}\mathsf{ym}_{\mathcal{V}}$ having sets as objects, functions as vertical 1-cells and coloured symmetric sequences as horizontal 1-cells.*

Proof. For Theorem 10.4(i), it suffices that we apply Theorem 9.1 to the horizontal double monad $(S, \widehat{m}, \widehat{e}) \colon \mathbb{P}\mathsf{rof}_{\mathcal{V}} \to \mathbb{P}\mathsf{rof}_{\mathcal{V}}$ of Corollary 10.2, where m and e are as in (66). Indeed, a categorical symmetric sequence $M \colon X \rightsquigarrow Y$ is precisely a horizontal Kleisli 1-cell, and so by (57), the composition of categorical symmetric sequences $M \colon X \rightarrow SY$ and $N \colon Y \rightarrow SZ$ is the profunctor $N \circ_{\mathbb{Kl}} M \colon X \rightarrow SZ$ found as the composite

$$
X \xrightarrow{\;M\;} SY \xrightarrow{\;SN\;} SSZ \xrightarrow{\;\widehat{m}_Z\;} SZ.
$$

Using (8), (11) and (65), this composite has value at $\vec{z} = (z_1, \ldots, z_m) \in SZ$ and $x \in X$ given by

$$
(N \circ_{\mathbb{Kl}} M)(\vec{z}, x) = \int^{\vec{w} \in SSZ, \vec{y} \in SY} SZ[\vec{z}, \bigotimes_i \vec{w}^i] \times SN(\vec{w}, \vec{y}) \times M(\vec{y}, x)
$$

which by applying (67) simplifies to the following well-known formula (c.f. [FGHW08, eq. (10)]):

$$
(N \circ_{\mathbb{Kl}} M)(\vec{z}, x) = \int^{\vec{w}^1, \ldots, \vec{w}^m \in SZ, \vec{y} \in SY} SZ[\vec{z}, \bigotimes_i \vec{w}^i] \times \prod_{1 \leqslant i \leqslant m} N(\vec{w}^i, y_i) \times M(\vec{y}, x)
$$

which is a generalisation of the substitution monoidal structure for symmetric sequences [Kel05]. Theorem 10.4(ii) follows immediately and the formula for composition does not actually simplify significantly, since SY and SZ are genuine categories even when Y and Z are sets. $\square$

Remark 10.5. Even if the primary focus of our interest is the double category of coloured symmetric sequences $\mathbb{S}\mathsf{ym}_{\mathcal{V}}$, it is useful to consider the larger double category of categorical symmetric sequences $\mathbb{C}\mathsf{atSym}_{\mathcal{V}}$. The reason is that the latter arises naturally from the double category of profunctors as a Kleisli double category and enjoys better closure properties than the former, since the free symmetric strict monoidal category on a discrete $\mathcal{V}$-category is not discrete.

We now wish to apply the theory developed in the previous sections in order to obtain the desired oplax monoidal structures on $\mathbb{CatSym}_{\mathcal{V}}$ and $\mathbb{Sym}_{\mathcal{V}}$. First note that, by Example 4.6, the double category $\mathbb{Prof}_{\mathcal{V}}$ has a monoidal structure induced from that on $\mathcal{V}$. Thus, by Proposition 9.7, it suffices to show that the vertical double monad $S\colon \mathbb{Prof}_{\mathcal{V}} \to \mathbb{Prof}_{\mathcal{V}}$ has well-behaved pseudomonoidal structure.

The key fact which allows us to do this is that, as a 2-monad on $\mathsf{Cat}_{\mathcal{V}}$, S is pseudomonoidal [HP02, Kel74]. Indeed, [HP02, Section 3.3] shows that S can be equipped with the structure of a pseudo-commutative 2-monad, while [HP02, Theorem 7] states that every pseudo-commutative 2-monad is pseudomonoidal, *cf.* also [Koc72, Theorem 2.3]. For our purposes, it will be convenient to describe the relevant structure explicitly. First of all, S admits a *strength* [Koc72] given by:

$$\begin{aligned} \kappa\colon X \times SY &\to S(X \times Y) \\ (x, \vec{y}) &\mapsto \big((x, y_1), \ldots, (x, y_n)\big), \end{aligned} \tag{72}$$

as well as a *costrength* $\kappa'\colon SX \times Y \to S(X \times Y)$ given dually. Note that, because the formula for $\kappa(x, \vec{y})$ repeats the variable x, we can only make the assignment of (72) $\mathcal{V}$-functorial when $\mathcal{V}$ is *cartesian* monoidal— and this explains the reason for us making this restriction in the first place. In this situation, the 2-functor S acquires *two* canonical lax monoidal structures built from the strength, the costrength and the monad multiplication as in [Koc72, eqs. (2.1) and (2.2)]. In our case, one of these lax monoidal structures has $S^0\colon 1 \to S1$ given by the monad unit, and $S^2_{X,Y}\colon SX \times SY \to S(X \times Y)$ defined by lexicographic product:

$$\big((x_1, \ldots, x_m), (y_1, \ldots, y_n)\big) \quad \mapsto \quad \big((x_1, y_1), (x_1, y_2), \ldots, (x_1, y_n), (x_2, y_1), \ldots, (x_m, y_n)\big), \tag{73}$$

which we sometimes also denote by $\vec{x} \boxtimes \vec{y}$ as in (5). The other lax monoidal structure has the same S^0 and binary constraints $SX \times SY \to S(X \times Y)$ given by *colexicographic* product. Evidently, these two lax monoidal structures are isomorphic, and this is the key aspect of S being pseudo-commutative in the sense of [HP02].

In this situation, by [HP02, Theorem 7], which is a higher-dimensional adaptation of [Koc72, Theorem 2.3], S becomes a pseudomonoidal 2-monad with respect to the lax monoidal structure S^2. It is not hard to see that the monad unit $e\colon 1 \Rightarrow S$ is in fact a genuine monoidal transformation; however, the multiplication $m\colon SS \Rightarrow S$ is not monoidal, but only a *pseudomonoidal* transformation; which is to say that the two sides of the diagram

$$\begin{array}{ccc} SSX \times SSY \xrightarrow{\ S^2\ } S(SX \times SY) \xrightarrow{\ S(S^2)\ } SS(X \times Y) \\ {\scriptstyle m_X \times m_Y}\downarrow \qquad\qquad \Downarrow m^2_{X,Y} \qquad\qquad \downarrow{\scriptstyle m_{X \times Y}} \\ SX \times SY \xrightarrow{\qquad\qquad\qquad S^2 \qquad\qquad\qquad} S(X \times Y). \end{array} \tag{74}$$

are not equal, but only coherently isomorphic via a 2-cell as displayed. We now describe this 2-cell $m^2_{X,Y}$ explicitly. To this end, let us take a typical element of $SSX \times SSY$, say $\big((\vec{x}^1, \ldots, \vec{x}^k), (\vec{y}^1, \ldots \vec{y}^\ell)\big)$ where $\vec{x}^i = (x^i_1, \ldots, x^i_{m_i})$ for $1 \leqslant i \leqslant k$ and $\vec{y}^j = (y^j_1, \ldots, y^j_{n_j})$ for $1 \leqslant j \leqslant \ell$. On the one hand, around the lower side of (74), this element is sent first to $\big((x^1_1, \ldots, x^k_{m_k}), (y^1_1, \ldots, y^\ell_{n_\ell})\big)$ and then to

$$\big((x^1_1, y^1_1), (x^1_1, y^1_2), \ldots, (x^1_1, y^\ell_{n_\ell}), (x^1_2, y^1_1), \ldots, (x^1_2, y^\ell_{n_\ell}), \ldots, (x^k_{m_k}, y^\ell_{n_\ell})\big).$$

This is the lexicographic order on four indices. On the other hand, around the upper side of (74) we obtain first $\big((\vec{x}^1, \vec{y}^1), (\vec{x}^1, \vec{y}^2), \ldots, (\vec{x}^k, \vec{y}^\ell)\big)$ and then, applying (73) to each pair, we get

$$\Big(\big((x^1_1, y^1_1), (x^1_1, y^1_2), \ldots, (x^1_{m_1}, y^1_{n_1})\big), \big((x^1_1, y^2_1), (x^1_1, y^2_2), \ldots, (x^1_{m_1}, y^2_{n_2})\big), \ldots, \big((x^k_1, y^\ell_1), (x^k_1, y^\ell_2), \ldots, (x^k_{m_k}, y^\ell_{n_\ell})\big)\Big)$$

and so finally

$$\big((x^1_1, y^1_1), (x^1_1, y^1_2), \ldots, (x^1_{m_1}, y^1_{n_1}), (x^1_1, y^2_1), \ldots, (x^1_{m_1}, y^2_{n_2}), \ldots, (x^k_{m_k}, y^\ell_{n_\ell})\big) \ .$$

This is a twisted lexicographic order on the indices, with the significance order being 1–3–2–4 rather than 1–2–3–4. Clearly, there is a unique bijection θ which exchanges these orderings, giving the components of the desired natural isomorphism $m^2_{X,Y}$ filling (74). This establishes the binary pseudomonoidality of m; the corresponding nullary pseudomonoidality constraint m^0 is in fact the *identity*. That these data satisfy the necessary coherences to form a pseudomonoidal monad is now asserted in [HP02, Theorem 7], but one could also establish this directly, following a reasoning similar to that used in [HP02, Section 3.3] to establish pseudocommutativity.

The next lemma extends the pseudomonoidal structure of the 2-monad $S\colon \mathsf{Cat}_\mathcal{V} \to \mathsf{Cat}_\mathcal{V}$ to a pseudomonoidal structure in the sense of Definition 8.3 on the vertical double monad $S\colon \mathbb{P}\mathsf{rof}_\mathcal{V} \to \mathbb{P}\mathsf{rof}_\mathcal{V}$ of Proposition 10.1. Before doing this, let us note that under our assumption that $\mathcal{V}$ is cartesian monoidal, the induced monoidal structure on $\mathbb{P}\mathsf{rof}_\mathcal{V}$ is also cartesian, in the sense that the ordinary monoidal structures on $(\mathbb{P}\mathsf{rof}_\mathcal{V})_0$ and $(\mathbb{P}\mathsf{rof}_\mathcal{V})_1$ that underlie it are both cartesian monoidal; as such, we will continue to write $\times$ rather than $\otimes$ for this tensor product, in particular, for its action on horizontal 1-cells of $\mathbb{P}\mathsf{rof}_\mathcal{V}$.

Lemma 10.6. *The vertical double monad* $S\colon \mathbb{P}\mathsf{rof}_\mathcal{V} \to \mathbb{P}\mathsf{rof}_\mathcal{V}$ *is pseudomonoidal.*

Proof. We need to check that S is a lax monoidal double functor as in Definition 5.1 and that m and e are pseudomonoidal vertical transformations as in Definition 5.3. Let us begin by showing that S admits a lax monoidal structure. For $X_1, X_2 \in \mathbb{P}\mathsf{rof}_\mathcal{V}$, the vertical 1-cells $S^2_{X_1,X_2}\colon SX_1 \times SX_2 \to S(X_1 \times X_2)$ and $S^0\colon 1 \to S1$ are given in (73). For $M_1\colon X_1 \rightarrow Y_1$ and $M_2\colon X_2 \rightarrow Y_2$, the squares

$$
\begin{array}{ccc}
SX_1 \times SX_2 & \xrightarrow{SM_1 \times SM_2} & SY_1 \times SY_2 \\
{\scriptstyle S^2_{X_1,X_2}}\Big\downarrow & \Downarrow S^2_{M_1,M_2} & \Big\downarrow{\scriptstyle S^2_{Y_1,Y_2}} \\
S(X_1 \times X_2) & \xrightarrow[S(M_1 \times M_2)]{} & S(Y_1 \times Y_2)
\end{array}
\tag{75}
$$

can be constructed following the same reasoning as in the proof of Proposition 10.1, *i.e.* reducing to the cases where $M = \widehat{F}$, $N = \widehat{G}$ and $M = \check{F}$, $N = \check{G}$, and using that S is lax monoidal on $\mathsf{Cat}_\mathcal{V}$.

We already observed that the unit e is genuinely monoidal at the 2-monad level, and the same is true for e *qua* vertical transformation. As for the vertical transformation m, the axioms for a pseudomonoidal vertical transformation that concern only the vertical fragment are exactly those expressing that m is a pseudomonoidal natural transformation in $\mathsf{Cat}_\mathcal{V}$, which we have discussed above. The only axiom not of this form is (38), and this can be verified using the construction of the squares in (75) via a reduction to companions and conjoints and the modification axiom for the 2-cells filling (74). $\qquad\square$

So S extends to a pseudomonoidal vertical double monad $S\colon \mathbb{P}\mathsf{rof}_\mathcal{V} \to \mathbb{P}\mathsf{rof}_\mathcal{V}$; and the last step required to establish the normal oplax monoidal structure of coloured symmetric sequences is to verify that this vertical double monad satisfies the additional conditions of Proposition 9.7.

Lemma 10.7. *The pseudomonoidal vertical double monad* $S\colon \mathbb{P}\mathsf{rof}_\mathcal{V} \to \mathbb{P}\mathsf{rof}_\mathcal{V}$ *has the properties that:*

  (i) *The multiplication* $m\colon SS \Rightarrow S$ *and unit* $e\colon 1 \Rightarrow S$ *are special vertical transformations.*
  (ii) *The vertical 1-cells* $S^2_{X,Y}\colon SX \times SY \to S(X \times Y)$ *and* $S^0\colon 1 \to S1$ *have companions.*
  (iii) *The strength and costrength of* S *are special vertical transformations.*

Proof. Item (i) was already shown as part of Proposition 10.1. Item (ii) is immediate since every vertical 1-cell in $\mathbb{P}\mathsf{rof}_\mathcal{V}$ has a companion, see Example 2.7. For Item (iii), the two cases are dual, so we only provide details for one. According to Proposition 9.7, the strength in question is defined from the lax monoidal structure S^2 of S via (63); but because we originally obtained S^2 using the strength κ of (72) and the dual costrength, it follows as in [Koc72, Theorem 2.3] that the strength of (63) is this same κ, and similarly for the costrength. Thus, the condition we must prove *e.g.* for the costrength is that, for any $M\colon X \rightarrow Y$ and $N\colon W \rightarrow Z$, the companion transpose 2-morphism

$$
\begin{array}{ccccc}
SX \times W & \xrightarrow{SM \times N} & SY \times Z & \xrightarrow{\widehat{\kappa}_{Y,Z}} & S(Y \times Z) \\
\Big\| & & \Downarrow \widehat{\kappa}_{M,N} & & \Big\| \\
SX \times W & \xrightarrow[\widehat{\kappa}_{X,Z}]{} & S(X \times W) & \xrightarrow[S(M \times N)]{} & S(Y \times Z)
\end{array}
\tag{76}
$$

is invertible. To this end, let $\vec{u} = \big((y_1, z_1), \ldots, (y_m, z_m)\big)$ in $S(Y \times Z)$ and let $(\vec{x}, w) \in SX \times W$ where $\vec{x} = (x_1, \ldots, x_m)$. Note we assume that $\vec{u}$ and $\vec{x}$ have the same length; that we may do so without loss of generality will be clear from the formulae which follow. Now, the profunctor along the bottom of (76) has value at $(\vec{u}, (\vec{x}, w))$ given by

$$
S(M \times N)\big(\vec{u}, \kappa(\vec{x}, w)\big) = \bigsqcup_{\sigma \in \mathfrak{S}_m} \prod_{1 \leqslant i \leqslant m} (M \times N)\big((y_{\sigma(i)}, z_{\sigma(i)}), (x_i, w)\big)
$$

$$
= \bigsqcup_{\sigma \in \mathfrak{S}_m} \prod_{1 \leqslant i \leqslant m} M(y_{\sigma(i)}, x_i) \times N(z_{\sigma(i)}, w).
$$

On the other hand, the profunctor across the top of (76) has value at $(\vec{u}, (\vec{x}, w))$ given by

$$\int^{\vec{v} \in SY, z' \in Z} S(Y \times Z)[\vec{u}, \kappa(\vec{v}, z')] \times (SM \times N)((\vec{v}, z'), (\vec{x}, w))$$

$$= \int^{\vec{v} \in SY, z' \in Z} S(Y \times Z)[\vec{u}, \kappa(\vec{v}, z')] \times SM(\vec{v}, \vec{x}) \times N(z', w)$$

$$\cong \int^{v_1, \ldots, v_m \in Y, z' \in Z} S(Y \times Z)[\vec{u}, \kappa(\vec{v}, z')] \times \prod_{1 \leqslant i \leqslant m} M(v_i, x_i) \times N(z', w)$$

$$\cong \bigsqcup_{\sigma \in \mathfrak{S}_m} \int^{v_1, \ldots, v_m \in Y, z' \in Z} \prod_{1 \leqslant i \leqslant m} (Y \times Z)[(y_{\sigma(i)}, z_{\sigma(i)}), (v_i, z')] \times \prod_{1 \leqslant i \leqslant m} M(v_i, x_i) \times N(z', w)$$

$$\cong \bigsqcup_{\sigma \in \mathfrak{S}_m} \int^{v_1, \ldots, v_m \in Y, z' \in Z} \prod_{1 \leqslant i \leqslant m} Y[y_{\sigma(i)}, v_i] \times Z[z_{\sigma(i)}, z'] \times \prod_{1 \leqslant i \leqslant m} M(v_i, x_i) \times N(z', w)$$

$$\cong \bigsqcup_{\sigma \in \mathfrak{S}_m} \prod_{1 \leqslant i \leqslant m} M(y_{\sigma(i)}, x_i) \times N(z_{\sigma(i)}, w) \,,$$

where at the first isomorphism we use (67) and at the final one we use the Yoneda lemma. By tracing it through we may see that the isomorphism constructed in this way is exactly $\widehat{\kappa}_{M,N}$, which is thus invertible. As noted above, the specialness of the costrength follows by an identical dual argument. $\qquad \square$

Theorem 10.8. *The double category* $\mathbb{C}\mathsf{atSym}_{\mathcal{V}}$ *of categorical symmetric sequences admits a normal oplax monoidal structure, given by arithmetic product of categorical symmetric sequences. Moreover, this restricts to a normal oplax monoidal structure on the double category* $\mathbb{S}\mathsf{ym}_{\mathcal{V}}$ *of coloured symmetric sequences.*

Proof. $\mathbb{C}\mathsf{atSym}_{\mathcal{V}}$ is the horizontal Kleisli double category of the horizontal double monad induced by the vertical double monad $S \colon \mathbb{P}\mathsf{rof}_{\mathcal{V}} \to \mathbb{P}\mathsf{rof}_{\mathcal{V}}$, as seen in the proof of Theorem 10.4. Moreover, by Lemma 10.6, the vertical double monad S is pseudomonoidal, and by Lemma 10.7, it satisfies the further hypotheses of Corollary 9.5 and Proposition 9.7. Applying these results, we see that the monoidal structure of $\mathbb{P}\mathsf{rof}_{\mathcal{V}}$ extends to a normal oplax monoidal structure on $\mathbb{C}\mathsf{atSym}_{\mathcal{V}}$, which clearly restricts back to the full sub-double-category $\mathbb{S}\mathsf{ym}_{\mathcal{V}}$.

It remains to show that the tensor product of horizontal 1-cells computes the arithmetic product of categorical symmetric sequences. For categorical symmetric sequences $M_1 \colon X_1 \rightsquigarrow Y_1$ and $M_2 \colon X_2 \rightsquigarrow Y_2$, the tensor product $M_1 \boxtimes M_2 \colon X_1 \times X_2 \rightsquigarrow Y_1 \times Y_2$ is defined as the profunctor:

$$X_1 \times X_2 \xrightarrow{M_1 \times M_2} SY_1 \times SY_2 \xrightarrow{\widehat{S^2_{Y_1, Y_2}}} S(Y_1 \times Y_2).$$

We now unfold this expression explicitly. First, by the definition of a companion in (11) applied to $S^2_{Y_1, Y_2}$, the second of these profunctors is given by

$$\widehat{S^2_{Y_1, Y_2}}(\vec{y}, (\vec{y}_1, \vec{y}_2)) = S(Y_1 \times Y_2)[\vec{y}, \vec{y}_1 \boxtimes \vec{y}_2],$$

where $\vec{y}_1 \boxtimes \vec{y}_2$ is given by the lexicographic ordering (73). Thus, using the definition of tensor product of profunctors (26) and of composition of profunctors in (8), we obtain

$$(M_1 \boxtimes M_2)(\vec{y}, (x_1, x_2)) = \int^{\vec{y}_1, \vec{y}_2} S(Y_1 \times Y_2)[\vec{y}, \vec{y}_1 \boxtimes \vec{y}_2] \times M_1(\vec{y}_1, x_1) \times M_2(\vec{y}_2, x_2), \qquad (77)$$

which is the formula for the arithmetic product of categorical and coloured symmetric sequences and in particular gives the formula in (4) for coloured symmetric sequences. $\qquad \square$

The final step is to obtain the desired oplax monoidal structures at the level of bicategories rather than double categories. Indeed, the horizontal bicategories of $\mathbb{C}\mathsf{atSym}_{\mathcal{V}}$ and $\mathbb{S}\mathsf{ym}_{\mathcal{V}}$ are the bicategories of categorical and coloured symmetric sequences introduced in [FGHW18] for $\mathcal{V} = \mathsf{Set}$ and in [GJ17] for a general $\mathcal{V}$. Thus, we can apply Corollary 9.9 to obtain:

Theorem 10.9. *The bicategory of categorical symmetric sequences* $\mathsf{CatSym}_{\mathcal{V}}$ *admits a normal oplax monoidal structure, given by the arithmetic product of categorical symmetric sequences. Furthermore, this normal oplax monoidal structure restricts to the bicategory of coloured symmetric sequences* $\mathsf{Sym}_{\mathcal{V}}$.

Apart from its intrinsic interest, and the construction of an example of a sophisticated kind of low-dimensional categorical structure by purely algebraic means, without any appeal to homotopy theory, Theorem 10.8 and Theorem 10.9 will be essential for subsequent work on the Boardman–Vogt tensor product of bimodules between symmetric coloured operads, extending that in [DH14] for bimodules between symmetric operads.

Appendix A.

A.1. **Coherence axioms for an oplax monoidal double category.** In this appendix, we spell out in detail the axioms for an oplax monoidal category as in Definition 4.1. In [HS19], the explicit axioms for a *monoidal* double category can be found, which are analogous but differ in that we use the opposite orientation for our (non-invertible) structure maps, and also provide additional non-invertible structure cells δ and ι as in (24), which in the pseudo case can be chosen to be identities and so omitted.

The 2-cells τ and η as in (23) satisfy the following axioms, for $M_i \colon X_i \longrightarrow Y_i$, $N_i \colon Y_i \longrightarrow Z_i$ and $P_i \colon Z_i \longrightarrow U_i$:

These axioms make $\otimes$ into an oplax double functor. Notice that at the bottom of the left diagram of the first axiom there is a composition associativity constraint implied.

The 2-cells δ and ι as in (24) satisfy the following axioms

$$(78)$$

which make $I \colon \mathbf{1} \to \mathbb{C}$ into an oplax double functor.

Next, for horizontal 1-cells $M_i\colon X_i \rightarrow Y_i$ and $N_i\colon Y_i \rightarrow Z_i$, the following axioms hold

$$
\begin{array}{cc}
\text{(left diagram)} & \text{(right diagram)}
\end{array}
$$

which make associativity into a vertical transformation of oplax double functors (together with the naturality of components which comes from $(\mathbb{D}_1, \otimes_1, I_1)$ being a monoidal category). Moreover, the following axioms hold

$$ \tag{79} $$

which make the left unit constraint $\lambda\colon \otimes \circ (I \times 1) \Rightarrow 1$ and the right unit constraint $\rho\colon \otimes \circ (1 \times I) \Rightarrow 1$ into vertical transformations of oplax double functors (together with the naturality of components coming from $(\mathbb{D}_1, \otimes_1, I_1)$ being a monoidal category).

A.2. **Proof of Theorem 9.4.** In this section, we illustrate the main ideas in the proof of Theorem 9.4. We begin by stating a couple of technical lemmas which, along with Lemma 3.12, are used repeatedly in the proof.

Lemma A.1. Let $F\colon \mathbb{C} \to \mathbb{D}$ be a double functor. For any horizontal 1-cell $M\colon X \rightarrow Y$ and vertical 1-cell $f\colon Y \to Y'$ in $\mathbb{C}$, a 2-morphism in $\mathbb{D}$ of the form on the left corresponds, under transpose operations,

to a 2-morphism of the form on the right

$$
\begin{array}{ccc}
FX \xrightarrow{F(\widehat{f}\circ M)} FY' & & FX \xrightarrow{FM} FY \xrightarrow{\mathrm{id}_{FY}} FY \\
\| \quad\Downarrow\xi\quad \| & & \\
FX \xrightarrow{FM} FY \xrightarrow{F\widehat{f}} FY' & & \Downarrow\phi \quad \Downarrow\psi \quad Ff \\
\Downarrow\phi \quad \Downarrow\widehat{\psi} & & \\
\bullet \longrightarrow \bullet \longrightarrow \bullet & & \bullet \longrightarrow \bullet \longrightarrow \bullet
\end{array}
$$

for any 2-morphisms ϕ, ψ of the right shape.

Proof. By pasting the 2-morphism $F(p_2 \circ 1_M)$ on the top of the left diagram, we obtain

$$
\begin{array}{ccc}
(\text{left diagram}) & = & (\text{middle diagram}) & = & (\text{right diagram})
\end{array}
$$

where the first equality is due to naturality of components of ξ, and the second one (up to pasting with appropriate coherence isomorphisms) follows from the definition of the transpose $\widehat{\psi}$ and the unitality axiom for F. $\qquad\square$

Lemma A.2. *Let $\beta\colon F \Rightarrow G$ be a horizontal transformation and $f\colon X \to X'$ a vertical 1-cell in $\mathbb{C}$. If f has a companion $\widehat{f}$, then the globular coherence 2-isomorphism $\beta_{\widehat{f}}$ is vertically inverse to the transpose of the 2-morphism component β_f, i.e. :*

$$
\begin{array}{ccc}
FX \xrightarrow{\mathrm{id}_{FX}} FX \xrightarrow{\beta_X} GX \xrightarrow{G\widehat{f}} GX' & & FX \xrightarrow{\beta_X} GX \xrightarrow{G\widehat{f}} GX' \\
\| \; \Downarrow Fp_2 \; Ff \quad \Downarrow\beta_f \quad Gf \Downarrow Gp_1 \; \| & = & \| \quad \Downarrow\beta_{\widehat{f}}^{-1} \quad \| \\
FX \xrightarrow{F\widehat{f}} FX' \xrightarrow{\beta_{X'}} GX' \xrightarrow{\mathrm{id}_{GX'}} GX' & & FX \xrightarrow{F\widehat{f}} FX' \xrightarrow{\beta_{X'}} GX'
\end{array}
$$

Proof. We will show that if we vertically compose the transpose of β_f with $\beta_{\widehat{f}}$ on both sides, it produces a vertical identity (up to coherence isomorphisms). Indeed,

$$
\begin{array}{cccc}
(\text{diagram}) & \overset{(15)}{=} & (\text{diagram}) & \overset{\substack{(14)\\(10)}}{=} 1_{\beta_{X'}\circ F\widehat{f}}
\end{array}
$$

$$
\begin{array}{cccc}
(\text{diagram}) & \overset{(15)}{=} & (\text{diagram}) & \overset{\substack{(14)\\(10)}}{=} 1_{G\widehat{f}\circ \beta_X} \quad\square
\end{array}
$$

We now provide a sample verification of one of the axioms needed in the proof of Theorem 9.4. We will show that the left unit constraint of the monoidal structure of a Kleisli double category $\mathbb{K}l(T)$ for a monoidal horizontal double monad is compatible with horizontal composition, namely the top left axiom of (79).

First of all, the left unit constraint components $J \boxtimes M \Rightarrow M$ as in (59) bijectively correspond to cells

$$
\begin{array}{ccccc}
I \otimes X & \xrightarrow{\mathrm{id}\otimes M} & I \otimes TY & \xrightarrow{\mathrm{id}} & I \otimes TY \\
& & & & \downarrow T^0\otimes 1 \\
& & & & TI \otimes TY \\
\lambda\downarrow & \Downarrow\lambda_M \quad \lambda\downarrow & & \Downarrow\mathrm{id}_\lambda & \downarrow T^2 \\
& & & & T(I\otimes Y) \\
& & & & \downarrow T\lambda \\
X & \xrightarrow{M} & TY & \xrightarrow{\mathrm{id}} & TY
\end{array}
$$

using Lemma A.1, since τ is the composition comparison structure map for the double functor $\otimes$ of the monoidal double category $\mathbb{C}$. Using appropriate transpose operations like (12), we can therefore transform the axiom at question to one that does not involve companions of 1-cells and 2-morphisms as follows: the right-hand side of (79) becomes

$$
\begin{array}{ccccc}
I \otimes X & \xrightarrow{\mathrm{id}\otimes(m\circ TN\circ M)} & I \otimes TZ & \xrightarrow{\mathrm{id}} & I \otimes TZ \\
& & & & \downarrow T^0\otimes 1 \\
& & & & TI \otimes TZ \\
\lambda\downarrow & \Downarrow\lambda_{m\circ TN\circ M} & \lambda\downarrow & & \downarrow T^2 \\
& & & & T(I\otimes Z) \\
& & & & \downarrow T\lambda \\
X \xrightarrow{M} TY \xrightarrow{TN} TTZ \xrightarrow{m_Z} TZ & & & \xrightarrow{\mathrm{id}} & TZ
\end{array} \tag{80}
$$

whereas the left-hand side, using naturality of τ, becomes

$$
\tag{81}
$$

Now using the fact that (T^2, T^0) and (m^2, m^0) are the structure maps of lax monoidal functors, namely

$$
\begin{array}{ccc}
I \otimes TY & \xrightarrow{\mathrm{id} \otimes TN} & I \otimes TTZ \\
{\scriptstyle T^0 \otimes 1}\downarrow & {\scriptstyle \Downarrow \mathrm{id}_{T^0} \otimes 1} & \downarrow {\scriptstyle T^0 \otimes 1} \\
TI \otimes TY & \xrightarrow{\mathrm{id} \otimes TN} & TI \otimes TTZ \\
{\scriptstyle T^2}\downarrow & {\scriptstyle \Downarrow T^2_{1,N}} & \downarrow {\scriptstyle T^2} \\
T(I \otimes Y) & \xrightarrow[T(\mathrm{id} \otimes N)]{} & T(I \otimes TZ) \;= \\
\downarrow & & \downarrow \\
{\scriptstyle T\lambda}\downarrow & {\scriptstyle \Downarrow T\lambda_N} & \downarrow {\scriptstyle T\lambda} \\
TY & \xrightarrow[TN]{} & TTZ
\end{array}
\qquad
\begin{array}{ccc}
I \otimes TY & \xrightarrow{\mathrm{id} \otimes TN} & I \otimes TTZ \\
\lambda \downarrow & {\scriptstyle \Downarrow \lambda_{TN}} & \downarrow \lambda \\
TY & \xrightarrow[TN]{} & TTZ
\end{array}
$$

$$
\begin{array}{ccc}
I \otimes TTZ & \xrightarrow{\mathrm{id} \otimes m} & I \otimes TZ \\
{\scriptstyle T^0 \otimes 1}\downarrow & & \downarrow {\scriptstyle T^0 \otimes 1} \\
TI \otimes TTZ & {\scriptstyle \Downarrow m^0 \otimes 1} & \\
{\scriptstyle TT^0 \otimes 1}\downarrow & & \\
TTI \otimes TTZ & \xrightarrow{m \otimes m} & TI \otimes TZ \\
{\scriptstyle T^2}\downarrow & & \downarrow {\scriptstyle T^2} \\
T(TI \otimes TZ) & {\scriptstyle \Downarrow m^2} & \\
{\scriptstyle TT^2}\downarrow & & \\
TT(I \otimes Z) & \xrightarrow{m} & T(I \otimes Z) \\
{\scriptstyle TT\lambda}\downarrow & {\scriptstyle \Downarrow m_\lambda} & \downarrow {\scriptstyle T\lambda} \\
TTZ & \xrightarrow[m]{} & TZ
\end{array}
\qquad
\begin{array}{ccc}
I \otimes TTZ & \xrightarrow{\mathrm{id} \otimes m} & I \otimes TZ \\
\lambda \downarrow & {\scriptstyle \Downarrow \lambda_m} & \downarrow \lambda \\
TTZ & \xrightarrow[m]{} & TZ
\end{array}
$$

the diagram (81) reduces to

$$
\begin{array}{ccccccc}
I \otimes X & & \xrightarrow{\quad \mathrm{id} \otimes (m \circ TN \circ M) \quad} & & & & I \otimes TZ \\
\| & & & {\scriptstyle \Downarrow \tau} & & & \| \\
I \otimes X & \xrightarrow{\mathrm{id} \otimes M} & I \otimes Y & \xrightarrow{\mathrm{id} \otimes TN} & I \otimes TTZ & \xrightarrow{\mathrm{id} \otimes m} & I \otimes TZ \\
\lambda \downarrow & {\scriptstyle \Downarrow \lambda_M} & \lambda \downarrow & {\scriptstyle \Downarrow \lambda_{TN}} & \lambda \downarrow & {\scriptstyle \Downarrow \lambda_m} & \downarrow \lambda \\
X & \xrightarrow[M]{} & TY & \xrightarrow[TN]{} & TTZ & \xrightarrow[m]{} & TZ
\end{array}
$$

which is equal to (80), since λ is a vertical transformation of double functors (18).

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