AN EMBEDDING THEOREM FOR TANGENT CATEGORIES

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Abstract. *Tangent categories* were introduced by Rosický as a categorical setting for differential structures in algebra and geometry; in recent work of Cockett, Crutwell and others, they have also been applied to the study of differential structure in computer science. In this paper, we prove that every tangent category admits an embedding into a representable tangent category—one whose tangent structure is given by exponentiating by a free-standing tangent vector, as in, for example, any well-adapted model of Kock and Lawvere's synthetic differential geometry. The key step in our proof uses a coherence theorem for tangent categories due to Leung to exhibit tangent categories as a certain kind of enriched category.

1. Introduction

Tangent categories, introduced by Rosický in [23], provide an category-theoretic setting for differential structures in geometry, algebra and computer science. A tangent structure on a category $\mathcal{C}$ comprises a functor $T\colon \mathcal{C} \to \mathcal{C}$ together with associated natural transformations—for example a natural transformation $p\colon T \Rightarrow 1$ making each TM into an object over M—which capture just those properties of the "tangent bundle" functor on the category **Man** of smooth manifolds that are necessary to develop a reasonable abstract differential calculus. The canonical example is **Man** itself, but others include the category of schemes (using the Zariski tangent spaces), the category of convenient manifolds [2] and, in computer science, any model of Ehrhard and Regnier's *differential* λ*-calculus* [10].

A more powerful category-theoretic approach to differential structures is the *synthetic differential geometry* developed by Kock, Lawvere, Dubuc and others [9, 18, 22]. It is more powerful because it presupposes more: a so-called "well-adapted" model of synthetic differential geometry is a Grothendieck topos $\mathcal{E}$ equipped with a full embedding $\iota\colon \mathbf{Man} \to \mathcal{E}$ of the category of smooth manifolds, obeying axioms which, among other things, assert that the affine line $R = \iota(\mathbb{R})$ has enough nilpotent elements to detect the differential structure. In particular, in a well-adapted model, the tangent bundle of a smooth manifold M is determined by the cartesian closed structure of $\mathcal{E}$ through the equation $\iota(TM) = \iota(M)^D$; here, D is the "disembodied tangent vector", which in the internal logic of $\mathcal{E}$ comprises the elements of R which square to zero.

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Any model $\mathcal{E}$ of synthetic differential geometry gives rise to a tangent category, whose underlying category comprises the *microlinear* objects [22, Chapter V] of $\mathcal{E}$ (among which are found the embeddings $\iota(M)$ of manifolds) and whose "tangent bundle" functor is $(–)^D$; see [6, Section 5]. This raises the question of whether any tangent category can be embedded into the microlinear objects of a well-adapted model of synthetic differential geometry; and while this is probably too much to ask, it has been conjectured that any tangent category should at least be embeddable in a *representable* tangent category—one whose "tangent bundle" functor is of the form $(–)^D$. The goal of this article is to prove this conjecture.

Our approach uses ideas of enriched category theory [17]. By exploiting Leung's coherence result [20] for tangent categories, we are able to describe a cartesian closed category $\mathcal{E}$ such that tangent categories are the same thing as $\mathcal{E}$-enriched categories admitting certain *powers* [17, Section 3.7]—a kind of enriched-categorical limit. Standard enriched category theory then shows that, for any small $\mathcal{E}$-category $\mathcal{C}$, the $\mathcal{E}$-category of presheaves $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ is complete, cocomplete and cartesian closed as a $\mathcal{E}$-category. Completeness means that, in particular, $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ bears the powers necessary for tangent structure; but cocompleteness and cartesian closure allow these powers to be computed as internal homs $(–)^D$, so that any presheaf $\mathcal{E}$-category bears *representable* tangent structure. It follows that, for any (small) tangent category $\mathcal{C}$, the $\mathcal{E}$-categorical Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ is a full embedding of $\mathcal{C}$ into a representable tangent category.

Beyond allowing an outstanding conjecture to be settled, we believe that the enriched-categorical approach to tangent structure has independent value, which will be explored further in future work. In one direction, the category $\mathcal{E}$ over which our enrichment exists admits an abstract version of the *Campbell–Baker–Hausdorff* construction by which a Lie algebra can be formally integrated to a formal group law (i.e., encoding the purely algebraic part of Lie's theorems). Via enrichment, this construction can be transported to any suitable $\mathcal{E}$-enriched category, so allowing a version of Lie theory to be associated uniformly with any category with differential structure. Another direction we intend to explore in future research involves modifying the category $\mathcal{E}$ to capture generalised forms of differential structure. One possibility involves *non-linear* or *arithmetic* differential geometry in the sense of [5], which should involve enrichments over a suitable category of *k-k-birings* in the sense of [25, 3]. Another possibility would be to explore "two-dimensional Lie theory" by replacing the cartesian closed category $\mathcal{E}$ with a suitable cartesian closed bicategory of k-linear categories, and considering generalised enrichments over this in the sense of [13].

Besides this introduction, this paper comprises the following parts. Section 2 recalls the basic notions of tangent category and representable tangent category, along with the coherence result of Leung on which our constructions will rest. Section 3 extends Leung's result so as to exhibit an equivalence between the 2-category of tangent categories and a certain 2-category of *actegories* [21]—categories equipped with an action by a monoidal category. Section 4 then applies two results from enriched category theory, due to Wood and Day, to exhibit these actegories as categories enriched over a certain base $\mathcal{E}$. Then, in Section 5, we see that this base $\mathcal{E}$ is complete, cocomplete and cartesian closed, and using this, deduce that the desired embedding arises simply as the $\mathcal{E}$-enriched Yoneda embedding. Finally, in Section 6, we unfold the abstract constructions to give a concrete description of the embedding of any tangent category into a representable one.

2. Background

We begin by recalling the notion of tangent category and representable tangent category. Rosický's original definition in [23] requires *abelian group* structure on the fibres of the tangent bundle; with motivation from computer science, Cockett and Crutwell weaken this in [6] to involve only *commutative monoid* structure, and we adopt their more general formulation here, though our results are equally valid under the narrower definition.

Definition 1. A *tangent category* is a category $\mathcal{C}$ equipped with:

(i) A functor $T\colon \mathcal{C} \to \mathcal{C}$ and a natural transformation $p\colon T \Rightarrow \mathrm{id}_{\mathcal{C}}$ such that each n-fold fibre product $TX \times_{p_X} \cdots \times_{p_X} TX$ exists in $\mathcal{C}$ and is preserved by each functor T^m;

(ii) Natural transformations

$$e\colon \mathrm{id}_{\mathcal{C}} \Rightarrow T \qquad m\colon T \times_p T \Rightarrow T \qquad \ell\colon T \Rightarrow TT \quad \text{and} \quad c\colon TT \Rightarrow TT\ ,$$

subject to the following axioms:

(iii) The maps e_X and m_X endow each $p_X\colon TX \to X$ with the structure of a commutative monoid in the slice category $\mathcal{C}/X$;

(iv) The following squares commute:

$$\begin{array}{ccc} T & \xrightarrow{\ell} & T^2 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Tp} \\ \mathrm{id}_{\mathcal{C}} & \xrightarrow{e} & T \end{array} \qquad \begin{array}{ccc} \mathrm{id}_{\mathcal{C}} & \xrightarrow{e} & T \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle Te} \\ T & \xrightarrow{\ell} & T^2 \end{array} \qquad \begin{array}{ccc} T \times_p T & \xrightarrow{\ell \times_e \ell} & T^2 \times_{Tp} T^2 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle Tm} \\ T & \xrightarrow{\ell} & T^2 \end{array}\ ;$$

(v) The following squares commute:

$$\begin{array}{ccc} T^2 & \xrightarrow{c} & T^2 \\ {\scriptstyle Tp}\downarrow & & \downarrow{\scriptstyle pT} \\ T & \xrightarrow{\mathrm{id}} & T \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{\mathrm{id}} & T \\ {\scriptstyle Te}\downarrow & & \downarrow{\scriptstyle eT} \\ T^2 & \xrightarrow{c} & T^2 \end{array} \qquad \begin{array}{ccc} T^2 \times_{Tp} T^2 & \xrightarrow{c \times_T c} & T^2 \times_{pT} T^2 \\ {\scriptstyle Tm}\downarrow & & \downarrow{\scriptstyle mT} \\ T^2 & \xrightarrow{c} & T^2 \end{array}\ ;$$

(vi) $c^2 = \mathrm{id}$, $c\ell = \ell$, and the following diagrams commute:

$$\begin{array}{ccc} T & \xrightarrow{\ell} & T^2 \\ {\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle T\ell} \\ T^2 & \xrightarrow{\ell T} & T^3 \end{array} \qquad \begin{array}{ccccc} T^3 & \xrightarrow{Tc} & T^3 & \xrightarrow{cT} & T^3 \\ {\scriptstyle cT}\downarrow & & & & \downarrow{\scriptstyle Tc} \\ T^3 & \xrightarrow{Tc} & T^3 & \xrightarrow{cT} & T^3 \end{array} \qquad \begin{array}{ccccc} T^2 & \xrightarrow{\ell T} & T^3 & \xrightarrow{Tc} & T^3 \\ {\scriptstyle c}\downarrow & & & & \downarrow{\scriptstyle cT} \\ T^2 & & \xrightarrow{T\ell} & & T^3 \end{array}\ ;$$

(vii) Writing w for the composite $T \times_p T \xrightarrow{\ell \times_e eT} T^2 \times_{Tp} T^2 \xrightarrow{Tm} T^2$, each diagram of the following form is an equaliser:

$$TX \times_{p_X} TX \xrightarrow{\ w_X\ } T^2X \underset{e_X \cdot p_X \cdot p_{TX}}{\overset{Tp_X}{\rightrightarrows}} TX\ . \tag{2.1}$$

In the sequel we will, as in [6], write $T_nX := TX \times_{p_X} \cdots \times_{p_X} TX$ for the n-fold fibre product of TX over X in any tangent category. We refer to these fibre products and the equalisers in (2.1) collectively as *tangent limits*.

Examples 2.

(i) The category **Man** of smooth manifolds is a tangent category under the structure for which $p_X : TX \to X$ is the usual tangent bundle of X.

(ii) Let $\mathcal{E}$ be any model of synthetic differential geometry [18, 22] with embedding $\iota\colon \mathbf{Man} \to \mathcal{E}$ of the category of smooth manifolds. The full subcategory of $\mathcal{E}$ on the *microlinear* objects [22, Chapter V] is a tangent category under the structure which sends a microlinear $X \in \mathcal{E}$ to the exponential X^D by the object $D = \{x \in \iota(\mathbb{R}) : x^2 = 0\}$; see [6, Section 5].

(iii) The category **Sch** of schemes over Spec $\mathbb{Z}$ is a tangent category under the structure which sends a scheme X to its Zariski tangent space TX. One way to see this is via the full embedding of **Sch** into the local ring classifier $\mathbf{Zar} = \mathbf{Sh}(\mathbf{CRng}_f^{\mathrm{op}}, \mathcal{Z})$. The generic local ring $R \in \mathbf{Zar}$ is known to be a ring of line type [18], and so the category $\mathbf{Zar}_{ml}$ of R-microlinear objects in **Zar** is by [6, Section 5] a tangent category under the tangent functor $(-)^D$, where $D = \text{Spec } \mathbb{Z}[x]/x^2$. Now, every affine scheme is microlinear by the Zariski analogue of [22, Proposition 7.1]; whereupon it follows easily that a general scheme is microlinear since it is a patching of affine schemes along open immersions, and these are formally étale by [15, Proposition 17.1.3(i)]. Thus **Sch** is contained in $\mathbf{Zar}_{ml}$; finally, by [4, Lemma 5.12.1], **Sch** is closed in $\mathbf{Zar}_{ml}$ under the tangent functor described above, and moreover this coincides with the Zariski tangent space.[1]

(iv) The category **CRig** of commutative rigs (rings without negatives) has a tangent structure with $TA = A[x]/x^2$ and with $p_A : TA \to A$ defined by $p_A(a+bx) = a$. It follows that $T_2A \cong A[x,y]/x^2, y^2, xy$ and that $T^2A \cong A[x,y]/x^2, y^2$, in which terms the remaining structure is given by:

$$e_A(a) = a \qquad m_A(a + bx + cy) = a + (b + c)x$$
$$\ell_A(a + bx) = a + bxy \qquad c_A(a + bx + cy + dxy) = a + cx + by + dxy \ .$$

(v) The functor $T\colon \mathbf{CRig} \to \mathbf{CRig}$ of (iv) preserves limits and filtered colimits, and so has a left adjoint S. It is easy to see that this endows $\mathbf{CRig}^{\mathrm{op}}$ with tangent structure (see [6, Proposition 5.17]).

(vi) Consider the category $\mathcal{W}$ whose objects are formal tensor products of the form $W_{n_1} \otimes \cdots \otimes W_{n_k}$ and whose morphisms are rig homomorphisms between the corresponding actual tensor products of commutative rigs, where here

$$W_n := \mathbb{N}[x_1, \ldots, x_n]/(x_i x_j)_{1 \leqslant i \leqslant j \leqslant n} \ .$$

[1] Another way to obtain the tangent structure on **Sch** is via the join restriction category $\mathbf{Aff}_p$ of affine schemes and partial maps defined on Zariski open subschemes. Because the tangent functor on affine schemes preserves open immersions, it extends to a tangent structure on $\mathbf{Aff}_p$; whence **Sch** is a tangent category by [6, Corollary 6.26], since it is the total category of the manifold completion of $\mathbf{Aff}_p$.

(We may also write W for W_1). The replete image $\overline{\mathcal{W}}$ of $\mathcal{W}$ in **CRig** is closed under the tangent structure of (iv), since this structure satisfies $T_nA \cong W_n \otimes A$; transporting this restricted tangent structure across the equivalence $\mathcal{W} \simeq \overline{\mathcal{W}}$ yields one on $\mathcal{W}$ with $T_n(A) = W_n \otimes A$.

Definition 3. If $\mathcal{C}$ is a cartesian closed category, then a tangent structure on $\mathcal{C}$ is *representable* if each functor T_n is of the form $(\text{–})^{D_n}$ for some $D_n \in \mathcal{C}$; equivalently, if $T \cong (\text{–})^D$ for some $D \in \mathcal{C}$ and all finite fibre coproducts $D_n = D +_0 \cdots +_0 D$ exist. Here, $0\colon 1 \to D$ is the map determined by the requirement that $p_X\colon X^D \to X$ is given by evaluation at 0; by naturality of p and of exponential transpose, this map is equally the composite

$$1 \xrightarrow{\overline{\mathrm{id}_D}} D^D \xrightarrow{p_D} D\ .$$

Example 4. The tangent structure in Examples 2(ii) above is representable, with $D_n = \{\vec{x} \in \iota(\mathbb{R})^n : x_ix_j = 0 \text{ for all } 1 \leqslant i \leqslant j \leqslant n\}$. The tangent structure on schemes in (iii) is similarly representable, with $D_n = \mathrm{Spec}(\mathbb{Z}[x_1, \dots, x_n]/(x_ix_j))$. It is also the case that the tangent structure in (v) is representable. Indeed, for each n we have $S_n \dashv T_n\colon \mathbf{CRig} \to \mathbf{CRig}$; since $T_nA \cong W_n \otimes A$ and tensor product in **CRig** is also coproduct, the left adjoint S_n "co-exponentiates" by W_n in **CRig**, and so dually is the exponential $(\text{–})^{W_n}$ in $\mathbf{CRig}^{\mathrm{op}}$.

Definition 5. A *tangent functor* between tangent categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $H\colon \mathcal{C} \to \mathcal{D}$ that preserves tangent limits—which we reiterate means each n-fold pullback $TX \times_{p_X} \cdots \times_{p_X} TX$ and each equaliser (2.1)—together with a natural isomorphism $\varphi\colon HT \Rightarrow TH$ rendering commutative each diagram:

$$\begin{array}{ccc} H & \xrightarrow{eH} & TH \\ {\scriptstyle He}\downarrow & \nearrow{\scriptstyle \varphi} & \downarrow{\scriptstyle pH} \\ HT & \xrightarrow[Hp]{} & H \end{array} \qquad \begin{array}{ccc} HT \times_{Hp} HT & \xrightarrow{\varphi \times_H \varphi} & TH \times_{pH} TH \\ {\scriptstyle Hm}\downarrow & & \downarrow{\scriptstyle mH} \\ HT & \xrightarrow{\varphi} & TH \end{array}$$

(2.2)

$$\begin{array}{ccccc} HT & & \xrightarrow{\varphi} & & TH \\ {\scriptstyle H\ell}\downarrow & & & & \downarrow{\scriptstyle \ell H} \\ HTT & \xrightarrow{\varphi T} & THT & \xrightarrow{T\varphi T} & TTH \end{array} \qquad \begin{array}{ccccc} HTT & \xrightarrow{\varphi T} & THT & \xrightarrow{T\varphi} & TTH \\ {\scriptstyle Hc}\downarrow & & & & \downarrow{\scriptstyle cH} \\ HTT & \xrightarrow{\varphi T} & THT & \xrightarrow{T\varphi} & TTH\ . \end{array}$$

A *tangent transformation* between tangent functors $H, K\colon \mathcal{C} \to \mathcal{D}$ comprises a natural transformation $\alpha\colon H \Rightarrow K$ such that $\varphi.\alpha T = T\alpha.\varphi\colon HT \Rightarrow TK$. We write **TANG** for the 2-category of tangent categories.

Remark 6. If we drop from the definition of tangent functor the requirements that H preserve tangent limits and that φ be invertible, we obtain the notion of *lax tangent functor* $H\colon \mathcal{C} \to \mathcal{D}$; we will make brief use of this in Section 6 below.

The goal of this paper is to show that every small tangent category admits a full embedding into a representable tangent category. Our result relies heavily on the following coherence result, which is Theorem 14.1 of [20]:

Theorem 7. *The tangent category $\mathcal{W}$ of Examples 2(vi) classifies tangent structures; by this we mean that, to within isomorphism, tangent structures on a category*

$\mathcal{C}$ correspond with strong monoidal functors $\Phi\colon (\mathcal{W}, \otimes, \mathbb{N}) \to ([\mathcal{C}, \mathcal{C}], \circ, \mathrm{id})$ sending tangent limits in $\mathcal{W}$ to pointwise limits in $[\mathcal{C}, \mathcal{C}]$.

Proof (sketch). The monoidal category $\mathcal{W}$ as defined above is *strict* monoidal, and every object is, in a unique way, the tensor of objects of the form W_n. This "flexibility" of $\mathcal{W}$ means that any strong monoidal functor $\mathcal{W} \to [\mathcal{C}, \mathcal{C}]$ can be replaced by an isomorphic strict monoidal one, and so we may reduce to this case.

Suppose, then, that we have a strict monoidal, tangent-limit-preserving functor $\Phi\colon \mathcal{W} \to [\mathcal{C}, \mathcal{C}]$. Let us write $T = \Phi(W)\colon \mathcal{C} \to \mathcal{C}$, and write $p\colon T \Rightarrow \mathrm{id}$ for $\Phi(p_{\mathbb{N}})\colon \Phi(W) \to \Phi(\mathbb{N})$. Since $W_n \in \mathcal{W}$ is the tangent limit $W \times_{\mathbb{N}} \cdots \times_{\mathbb{N}} W$, we see that $\Phi(W_n)$ must be a (pointwise) pullback $T \times_p \cdots \times_p T$ in $[\mathcal{C}, \mathcal{C}]$, which, as previously, we will denote by T_n. The definition of Φ on a general object of $\mathcal{W}$ is now forced by strict monoidality to be:

$$\Phi(W_{n_1} \otimes \cdots \otimes W_{n_k}) = T_{n_1} \circ \cdots \circ T_{n_k}\ . \tag{2.3}$$

Given this equation, the images under Φ of the maps $e_{\mathbb{N}}, m_{\mathbb{N}}, \ell_{\mathbb{N}}$ and $c_{\mathbb{N}}$ of the tangent structure on $\mathcal{W}$ thereby provide the remaining data for a tangent structure on $\mathcal{C}$. The corresponding axioms are all immediate except for the requirement that T^m should preserve the n-fold pullback $T \times_p \cdots \times_p T$. To see that this holds, note that for any $A \in \mathcal{W}$ the square left below, being a tangent limit, is sent by Φ to a pointwise pullback in $[\mathcal{C}, \mathcal{C}]$; but in the category of squares in $\mathcal{W}$, it is isomorphic (via the symmetry maps) to the one on the right, which is thus also sent to a pointwise pullback; now taking $A = W^{\otimes m}$ gives the result.

$$\begin{array}{ccc} W_{n+k} \otimes A & \longrightarrow & W_n \otimes A \\ \downarrow & & \downarrow {\scriptstyle !\otimes A} \\ W_k \otimes A & \xrightarrow{!\otimes A} & A \end{array} \qquad \begin{array}{ccc} A \otimes W_{n+k} & \longrightarrow & A \otimes W_n \\ \downarrow & & \downarrow {\scriptstyle A\otimes !} \\ A \otimes W_k & \xrightarrow{A\otimes !} & A\ . \end{array}$$

It follows that each Φ as in the statement of the theorem gives rise to a tangent structure on $\mathcal{C}$. Conversely, given a tangent structure on $\mathcal{C}$, we define the corresponding Φ on objects by (2.3). On morphisms, we define the images of $p_{\mathbb{N}}$, $e_{\mathbb{N}}$, $m_{\mathbb{N}}$, $\ell_{\mathbb{N}}$ and $c_{\mathbb{N}}$ to be the p, e, m, ℓ and c of our given tangent structure; what is rather less trivial is defining Φ on the other morphisms of $\mathcal{W}$.

First one shows that every morphism of $\mathcal{W}$ can be constructed from $p_{\mathbb{N}}$, $e_{\mathbb{N}}$, $m_{\mathbb{N}}$, $\ell_{\mathbb{N}}$ and $c_{\mathbb{N}}$ using only the monoidal structure and tangent limits; this is done in [20, Proposition 9.1]. Since Φ is supposed to be strict monoidal and tangent-limit-preserving, this now provides a prospective definition of Φ on morphisms. It remains to prove that this definition makes Φ well-defined, functorial, strict monoidal, and tangent-limit-preserving; the (hard) proof of this is the content of Sections 12 and 13 of [20]. □

In what follows, we will find it convenient to make use of the following straightforward reformulation of Leung's result.

Corollary 8. *The tangent category $\mathcal{W}$ is the free tangent category on an object, in the sense that for any tangent category $\mathcal{C}$, the functor*

$$\mathbf{TANG}(\mathcal{W}, \mathcal{C}) \to \mathcal{C} \tag{2.4}$$

given by evaluation at $\mathbb{N} \in \mathcal{W}$ is an equivalence of categories.

Proof. Let $\mathcal{C}$ be any tangent category. By Theorem 7, there is an essentially-unique strict monoidal tangent-limit-preserving functor $\Phi\colon \mathcal{W} \to [\mathcal{C},\mathcal{C}]$, and it is easy to see that this functor is in fact a tangent functor, when $\mathcal{W}$ is endowed with its canonical tangent structure and $[\mathcal{C},\mathcal{C}]$ with the pointwise one coming from $\mathcal{C}$. Now for any $X \in \mathcal{C}$, the evaluation functor $\mathrm{ev}_X\colon [\mathcal{C},\mathcal{C}] \to \mathcal{C}$ is also a tangent functor, and so the composite

$$F = \mathcal{W} \xrightarrow{\Phi} [\mathcal{C},\mathcal{C}] \xrightarrow{\mathrm{ev}_X} \mathcal{C}$$

is a tangent functor with $F(\mathbb{N}) = X$. This shows that (2.4) is surjective on objects; it remains to prove that it is fully faithful. Given tangent functors $H, K\colon \mathcal{W} \rightrightarrows \mathcal{C}$, it is easy to see that any tangent transformation $\alpha\colon H \Rightarrow K$ must render commutative each square

$$\begin{array}{ccccc}
HT_{n_1}\cdots T_{n_k} & \xrightarrow{\varphi^H_{n_1}T_{n_2}\cdots T_{n_k}} & \cdots & \xrightarrow{T_{n_1}\cdots T_{n_{k-1}}\varphi^H_{n_k}} & T_{n_1}\cdots T_{n_k}H \\
{\scriptstyle \alpha T_{n_1}\cdots T_{n_k}}\downarrow & & & & \downarrow{\scriptstyle T_{n_1}\cdots T_{n_k}\alpha} \\
KT_{n_1}\cdots T_{n_k} & \xrightarrow{\varphi^K_{n_1}T_{n_2}\cdots T_{n_k}} & \cdots & \xrightarrow{T_{n_1}\cdots T_{n_{k-1}}\varphi^K_{n_k}} & T_{n_1}\cdots T_{n_k}K
\end{array}$$

where we write $\varphi^H_n = \varphi^H \times_H \cdots \times_H \varphi^H$ and similarly for φ^K_n. As both horizontal maps are invertible, we see on evaluating at $\mathbb{N}$ that the component of α at a general object $W_{n_1} \otimes \cdots \otimes W_{n_k} = T_{n_1}(T_{n_2}(\cdots T_{n_k}(\mathbb{N})\dots))$ of $\mathcal{W}$ is determined by that at $\mathbb{N}$; whence (2.4) is faithful. For fullness, we must check that defining components in this manner from *any* map $\alpha_{\mathbb{N}}\colon H(\mathbb{N}) \to K(\mathbb{N})$ yields a tangent transformation α. The key point is naturality, which will follow from the equalities in (2.2) *so long as* we can show that every map in $\mathcal{W}$ is the $\mathbb{N}$-component of some transformation derived from the tangent structure; which follows by [20, Proposition 9.1]. □

3. Tangent categories as actegories

We will need to extend Leung's result from tangent categories to the maps between them; for this it will be convenient to deploy the notion of actegory [21]. If $\mathcal{M}$ is a monoidal category, then an $\mathcal{M}$*-actegory* is a category $\mathcal{C}$ equipped with an "action" functor $*\colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ which is associative and unital to within coherent isomorphism. By this, we mean that it comes endowed with natural families of isomorphisms $\alpha\colon (M \otimes N) * X \to M * (N * X)$ and $\lambda\colon I * X \to X$ satisfying a pentagon and a triangle axiom, as given in [21, §3], for example. $\mathcal{M}$-actegories are the objects of a 2-category $\mathcal{M}$-**ACT**, wherein a 1-cell is a functor $F\colon \mathcal{C} \to \mathcal{D}$ equipped with natural isomorphisms $\mu\colon F(M * X) \to M * FX$ compatible with α and λ; and a 2-cell is a transformation $\alpha\colon F \Rightarrow G$ compatible with μ.

With $\mathcal{W} = (\mathcal{W}, \otimes, \mathbb{N})$ given as before, we now define a *tangent* $\mathcal{W}$*-actegory* to be a $\mathcal{W}$-actegory $(\mathcal{C}, *)$ for which each functor $(-) * X\colon \mathcal{W} \to \mathcal{C}$ preserves tangent limits; these span a full and locally full sub-2-category $\mathcal{W}$-$\mathbf{ACT}_{\mathrm{t}}$ of $\mathcal{W}$-**ACT**.

Theorem 9. *The* 2*-category* **TANG** *is equivalent to* $\mathcal{W}$-$\mathbf{ACT}_{\mathrm{t}}$.

Proof. We define a 2-functor $\Gamma\colon \mathbf{TANG} \to \mathcal{W}$-$\mathbf{ACT}_{\mathrm{t}}$ as follows. First, given a tangent category $\mathcal{C}$, the strong monoidal $\Phi\colon \mathcal{W} \to [\mathcal{C},\mathcal{C}]$ of Theorem 7 transposes to a $\mathcal{W}$-action $*\colon \mathcal{W} \times \mathcal{C} \to \mathcal{C}$ which preserves tangent limits in its first variable. Next, given a map of tangent categories $F\colon \mathcal{C} \to \mathcal{D}$, consider (following [16]) the category $\mathcal{K}$ whose objects are triples $(A \in [\mathcal{C},\mathcal{C}], B \in [\mathcal{D},\mathcal{D}], \alpha\colon FA \cong BF)$ and whose morphisms are compatible pairs of natural transformations. $\mathcal{K}$ bears a tangent structure with "tangent bundle" functor

$$(A, B, \alpha) \qquad \mapsto \qquad (TA, TB, FTA \xrightarrow{\varphi A} TFA \xrightarrow{T\alpha} TBF)\ ,$$

with remaining data inherited from the pointwise tangent structures on $[\mathcal{C},\mathcal{C}]$ and $[\mathcal{D},\mathcal{D}]$, and with axioms following from those for the tangent functor F. By initiality of $\mathcal{W}$, there is an essentially-unique tangent functor $H\colon \mathcal{W} \to \mathcal{K}$ sending $\mathbb{N}$ to $(\mathrm{id}_\mathcal{C}, \mathrm{id}_\mathcal{D}, \mathrm{id}_F)$. Since the projections from $\mathcal{K}$ to $[\mathcal{C},\mathcal{C}]$ and $[\mathcal{D},\mathcal{D}]$ are clearly tangent functors, this H sends each $V \in \mathcal{W}$ to a triple

$$(V * (-) \in [\mathcal{C},\mathcal{C}],\ V * (-) \in [\mathcal{D},\mathcal{D}],\ \mu_V\colon F(V * -) \to V * F(-)\,)$$

whose third component gives the maps necessary to make F into a morphism of $\mathcal{M}$-actegories $\mathcal{C} \to \mathcal{D}$. This defines Γ on morphisms; the definition on 2-cells now follows on replacing $\mathcal{D}$ by $\mathcal{D}^\mathbf{2}$ in the preceding construction.

It is immediate from Theorem 7 that Γ is essentially surjective on objects, and so we need only show that it is fully faithful on 1- and 2-cells. So let $\mathcal{C}$ and $\mathcal{D}$ be tangent categories and $(F, \mu)\colon \Gamma\mathcal{C} \to \Gamma\mathcal{D}$ a map of the corresponding $\mathcal{W}$-actegories. The maps $\mu_{W,-}$ constitute a natural isomorphism $F(W * -) \Rightarrow W * F(-)$ which, since $W * (-) \cong T$ in both domain and codomain, determines and is determined by one $\varphi\colon FT \Rightarrow TF$. The axioms for a map of $\mathcal{W}$-actegories now imply commutativity of the diagrams (2.2), and so (F, φ) will be a tangent functor so long as F preserves tangent limits. For the pullbacks, we consider the diagram

$$\begin{array}{ccc} F(W_n * X) & \longrightarrow & FTX \times_{FX} \cdots \times_{FX} FTX \\ {\scriptstyle \mu_{W_n}}\downarrow & & \downarrow{\scriptstyle \varphi \times_{FX} \cdots \times_{FX} \varphi} \\ W_n * FX & \longrightarrow & TFX \times_{FX} \cdots \times_{FX} TFX\ . \end{array}$$

with top edge induced by the maps $F(\pi_i * X)\colon F(W_n * X) \to F(W * X) \cong FTX$ and bottom induced by the maps $\pi_i * FX\colon W_n * FX \to W * FX \cong TFX$. The square commutes by naturality of μ, and our assumptions means that the top, left and right sides are isomorphisms; whence also the bottom. The argument for preservation of the equalisers (2.1) is similar, and so (F, φ) is a map of tangent categories. It is moreover easily unique such that $\Gamma(F, \varphi) = (F, \mu)$, so that Γ is fully faithful on 1-cells; the argument on 2-cells is similar on replacing $\mathcal{D}$ by $\mathcal{D}^\mathbf{2}$. □

4. Tangent categories as enriched categories

We now exploit Theorem 9 in order to exhibit tangent categories as particular kinds of *enriched category* in the sense of [17]; more precisely, we construct a base for enrichment $\mathcal{E}$ such that tangent categories are the same thing as $\mathcal{E}$-enriched categories admitting *powers* by a certain class of objects in $\mathcal{E}$; here, we recall that:

Definition 10. If $\mathcal{C}$ is a category enriched over the symmetric monoidal base $\mathcal{V}$, then a *power* (resp. *copower*) of $X \in \mathcal{C}$ by $V \in \mathcal{V}$ is an object $V \pitchfork X$ (resp. $V \cdot X$) of $\mathcal{C}$

together with a $\mathcal{V}$-natural family of isomorphisms in $\mathcal{V}$ as to the left or right in:

$$\mathcal{C}(Y, V \pitchfork X) \xrightarrow{\cong} \mathcal{V}(V, \mathcal{C}(Y, X)) \qquad\qquad \mathcal{C}(V \cdot X, Y) \xrightarrow{\cong} \mathcal{V}(V, \mathcal{C}(X, Y)) \ .$$

Note that, by $\mathcal{V}$-naturality, such isomorphisms are determined by a *unit* map $V \to \mathcal{C}(V \pitchfork X, X)$ or $V \to \mathcal{C}(X, V \cdot X)$ as appropriate.

The characterisation result in question is our Theorem 19 below; it will follow from two basic arguments in the theory of enriched categories. The first, due to Richard Wood, identifies actegories over a small symmetric $\mathcal{M}$ with $\mathcal{PM}$-enriched categories admitting powers by representables; here, $\mathcal{PM}$ is the category $[\mathcal{M}, \mathbf{Set}]$ of presheaves on $\mathcal{M}^{\mathrm{op}}$ under Day's *convolution* monoidal structure:

Definition 11. Let $\mathcal{M}$ be small symmetric monoidal. The *convolution* monoidal structure on $[\mathcal{M}, \mathbf{Set}]$ is the symmetric monoidal closed structure whose unit object is $yI = \mathcal{V}(I, –)$, whose binary tensor product and internal hom are as displayed below, and whose coherence data are given as in [7]:

$$(4.1) \qquad \begin{aligned} (F \otimes G)(X) &= \textstyle\int^{M,N \in \mathcal{M}} \mathcal{M}(M \otimes N, X) \times FM \times GN \\ [F, G](X) &= \textstyle\int_{M \in \mathcal{M}} [FM, G(X \otimes M)] \ . \end{aligned}$$

The first step in proving Wood's result uses his characterisation of general $\mathcal{PM}$-enriched categories.

Lemma 12 (Wood)**.** *Let $\mathcal{M}$ be a small symmetric monoidal category. To give a $\mathcal{PM}$-enriched category $\mathcal{C}$ is equally to give:*

- *A set $\mathrm{ob}\,\mathcal{C}$ of objects;*
- *For each $x, y \in \mathrm{ob}\,\mathcal{C}$, a functor $\mathcal{C}(x, y) \colon \mathcal{M} \to \mathbf{Set}$;*
- *For each $x \in \mathrm{ob}\,\mathcal{C}$, an identity element $\mathrm{id}_x \in \mathcal{C}(x, x)(I)$;*
- *For each $x, y, z \in \mathrm{ob}\,\mathcal{C}$, a family of composition functions*

$$\mathcal{C}(x, y)(M) \times \mathcal{C}(y, z)(N) \to \mathcal{C}(x, z)(M \otimes N)$$

  natural in $M, N \in \mathcal{M}$,

subject to three axioms expressing associativity and unitality of composition.

Proof. This is [27, Proposition 1]; the key point is to use the Yoneda lemma to deduce that maps $yI \to F$ out of the unit in $\mathcal{PM}$ are in natural bijection with elements of FI, and that maps $h \colon F \otimes G \to H$ out of a binary tensor product are in natural bijection with natural families of maps $\bar{h}_{AB} \colon FM \times GN \to H(M \otimes N)$. □

The following key result is essentially contained in Chapter 1, §7 of Wood's Ph.D. thesis [26]; the proof is simple enough for us to include here.

Proposition 13 (Wood)**.** *Let $\mathcal{M}$ be a small symmetric monoidal category. There is a correspondence, to within isomorphism, between $\mathcal{M}$-actegories and $\mathcal{PM}$-categories admitting powers by representables.*

Proof. First let $\mathcal{C}$ be a $\mathcal{PM}$-category admitting powers by representables. As usual, we write $\mathcal{C}_0$ for the underlying ordinary category of $\mathcal{C}$, whose objects are those of $\mathcal{C}$ and whose hom-sets are $\mathcal{C}_0(x, y) = \mathcal{C}(x, y)(I)$. We endow $\mathcal{C}_0$ with an $\mathcal{M}$-action by taking $M * X := y_M \pitchfork X$. Functoriality of $*$ follows by the functoriality of enriched limits; the associativity constraints are given by

$$y_{M\otimes N} \pitchfork X \cong (y_M \otimes y_N) \pitchfork X \cong y_V \pitchfork (y_W \pitchfork X)$$

where the first isomorphism comes from the definition of the convolution monoidal structure, and the second is the associativity of iterated powers [17, Equation 3.18]; and the unit constraints are analogous. This gives an assignation $\mathcal{C} \mapsto (\mathcal{C}_0, y_{(-)} \pitchfork (-))$ from $\mathcal{PM}$-categories admitting powers by representables to $\mathcal{M}$-actegories.

Conversely, if $(\mathcal{C}_0, *)$ is an $\mathcal{M}$-actegory, then we define a $\mathcal{PM}$-category $\mathcal{C}$ with objects those of $\mathcal{C}_0$, with hom-objects $\mathcal{C}(X,Y)\colon \mathcal{M} \to \mathbf{Set}$ given by $\mathcal{C}(X,Y)(M) = \mathcal{C}_0(X, M * Y)$, with unit elements $\lambda_X \in \mathcal{C}(X,X)(I) = \mathcal{C}(X, I \otimes X)$, and with composition maps $\mathcal{C}(X,Y)(M) \times \mathcal{C}(Y,Z)(N) \to \mathcal{C}(X,Z)(M \otimes N)$ given by sending $f\colon X \to M * Y$ and $g\colon Y \to N * Z$ to the composite

$$X \xrightarrow{f} M * Y \xrightarrow{M*g} M * (N * Z) \xrightarrow{\cong} (M \otimes N) * Z \ .$$

It is straightforward to check that this $\mathcal{C}$ has powers by representables given by taking $y_V \pitchfork X := V * X$. Finally, it is easy to see that the preceding two constructions are inverse to within an isomorphism. □

In fact, by using results of [14], this correspondence can be enhanced to an equivalence of 2-categories. Let us write $\mathcal{PM}\text{-}\mathbf{CAT}_{\pitchfork}$ for the locally full sub-2-category of $\mathcal{PM}\text{-}\mathbf{CAT}$ whose objects are $\mathcal{PM}$-categories admitting powers by representables, and whose 1-cells are $\mathcal{PM}$-functors preserving such powers.

Proposition 14. *Let $\mathcal{M}$ be small symmetric monoidal. The correspondence of Proposition 13 underlies an equivalence of 2-categories $\mathcal{M}\text{-}\mathbf{ACT} \simeq \mathcal{PM}\text{-}\mathbf{CAT}_{\pitchfork}$.*

Proof. In [14, §3], the assignation $\mathcal{C} \mapsto (\mathcal{C}_0, y_{(-)} \pitchfork (-))$ of the preceding proposition is made into the action on objects of a 2-functor $\mathcal{PM}\text{-}\mathbf{CAT}_{\pitchfork} \to \mathcal{M}\text{-}\mathbf{ACT}$. The preceding Proposition shows that this 2-functor is essentially surjective on objects, and it is 2-fully faithful by [14, Theorem 3.4]. □

In particular, with $\mathcal{W} = (\mathcal{W}, \otimes, \mathbb{N})$ given as in the preceding sections, this proposition identifies $\mathcal{W}$-actegories with $\mathcal{PW}$-categories admitting powers by representables. What it does not yet capture are the limit-preservation properties required of a *tangent* $\mathcal{W}$-actegory; for this, we require a second basic result of enriched category theory, concerning enrichment over a *monoidally reflective* subcategory.

Definition 15. A *symmetric monoidal reflection* is an adjunction

$$(\mathcal{V}', \otimes', I') \underset{J}{\overset{L}{\rightleftarrows}} (\mathcal{V}, \otimes, I) \tag{4.2}$$

in the 2-category **SMC** of symmetric monoidal categories, symmetric (lax) monoidal functors and monoidal transformations for which J is the inclusion of a full, replete subcategory $\mathcal{V}' \subseteq \mathcal{V}$. We may also say that $\mathcal{V}'$ is *monoidally reflective* in $\mathcal{V}$.

Any symmetric monoidal functor $F\colon \mathcal{V}_1 \to \mathcal{V}_2$ induces a "change of base" 2-functor $F_*\colon \mathcal{V}_1\text{-}\mathbf{CAT} \to \mathcal{V}_2\text{-}\mathbf{CAT}$ which sends a $\mathcal{V}_1$-category $\mathcal{A}$ to the $\mathcal{V}_2$-category $F_*\mathcal{A}$ with the same objects and with $(F_*\mathcal{A})(x,y) = F(\mathcal{A}(x,y))$. Similarly, any symmetric monoidal transformation $\alpha\colon F \Rightarrow G$ between monoidal symmetric functors induces a 2-natural transformation $\alpha_*\colon F_* \Rightarrow G_*$ between the corresponding change of base 2-functors. The assignations $F \mapsto F_*$ and $\alpha \mapsto \alpha_*$ are evidently 2-functorial, and so any monoidal reflection (4.2) gives rise to a reflection of 2-categories $J_*\colon \mathcal{V}'\text{-}\mathbf{CAT} \leftrightarrows \mathcal{V}\text{-}\mathbf{CAT}\colon L_*$. It follows that:

Lemma 16. *For any symmetric monoidal reflection as in* (4.2)*, the* 2*-functor* $J_*\colon \mathcal{V}'\text{-}\mathbf{CAT} \to \mathcal{V}\text{-}\mathbf{CAT}$ *induces a* 2*-equivalence between* $\mathcal{V}'\text{-}\mathbf{CAT}$ *and the full and locally full sub-*2*-category of* $\mathcal{V}\text{-}\mathbf{CAT}$ *on those* $\mathcal{V}$*-categories with hom-objects in* $\mathcal{V}'$*.*

To obtain symmetric monoidal reflections, we use Day's *reflection theorem*:

Proposition 17 (Day)**.** *Let* $(\mathcal{V}, \otimes, I)$ *be symmetric monoidal closed, let* $J\colon \mathcal{V}' \leftrightarrows \mathcal{V}\colon L$ *exhibit* $\mathcal{V}'$ *as a full, replete reflective subcategory of* $\mathcal{V}$*, and suppose that we have:*

$$A \in \mathcal{A} \;\text{ and }\; V \in \mathcal{V} \qquad \Longrightarrow \qquad [V, A] \in \mathcal{A}\ . \tag{4.3}$$

Then $\mathcal{V}'$ *is symmetric monoidal on taking* $I' = LI$ *and* $A \otimes' B = L(IA \otimes IB)$*, and this structures makes* $\mathcal{V}'$ *monoidally reflective in* $\mathcal{V}$*. Furthermore,* $\mathcal{V}'$ *is closed monoidal with internal hom inherited from* $\mathcal{V}$*.*

Proof. This is [8, Theorem 1.2], and a full proof is given there; we sketch an alternative approach via symmetric *multicategories* [19]. Let $\mathbf{V}$ be the underlying symmetric multicategory of $\mathcal{V}$: so we have $\operatorname{ob}\mathbf{V} = \operatorname{ob}\mathcal{V}$ and $\mathbf{V}(A_1, \dots, A_n; B) = \mathcal{V}(A_1 \otimes \cdots \otimes A_n, B)$. Write $I\colon \mathbf{V}' \to \mathbf{V}$ for the full sub-multicategory on those objects from $\mathcal{V}'$. Of course, we have natural isomorphisms $\mathcal{V}'(LA, B) \cong \mathcal{V}(A, IB)$, but by closedness and (4.3), there are more general natural isomorphisms:

$$\mathbf{V}'(LA_1, \dots, LA_n; B) \cong \mathbf{V}(A_1, \dots, A_n; IB)\ ,$$

giving an adjunction of symmetric multicategories $I\colon \mathbf{V}' \leftrightarrows \mathbf{V}\colon L$. We will now be done as long as we can show that $\mathbf{V}'$, like $\mathbf{V}$, is representable. Since any left adjoint multifunctor preserves universal multimorphisms, we have for any $A, B \in \mathcal{V}'$ a universal multimorphism

$$A, B \xrightarrow{\varepsilon_A^{-1}, \varepsilon_B^{-1}} LIA, LIB \xrightarrow{L(IA \otimes IB)} L(IA \otimes IB)$$

exhibiting $L(IA \otimes IB)$ as the binary tensor of A and B in $\mathbf{V}'$; the same argument shows that LI provides a unit object. □

We now use the Day reflection theorem to find a monoidally reflective subcategory of $\mathcal{PW}$ which encodes the preservation of limits required for a tangent $\mathcal{W}$-actegory.

Proposition 18. *The full subcategory* $\mathcal{E} \subset \mathcal{PW}$ *on those functors* $F\colon \mathcal{W} \to \mathbf{Set}$ *which preserve tangent limits (in the sense of sending them to limits in* $\mathbf{Set}$*) is monoidally reflective.*

Proof. Clearly $\mathcal{E}$ is a full, replete subcategory of $\mathcal{PW}$, and its reflectivity is quite standard; see [12], for example. To show it is monoidally reflective, it thus suffices to verify the closure condition (4.3). So given $F \in \mathcal{PW}$ and $G \in \mathcal{E}$, we must show that $[F, G] \in \mathcal{E}$; writing F as a colimit $\operatorname{colim} y_{A_i}$ of representables, we have $[F, G] \cong [\operatorname{colim}_i y_{A_i}, G] \cong \lim_i [y_{A_i}, G]$, and since $\mathcal{E}$ is closed under limits in $\mathcal{PW}$, it now suffices to show that $[y_A, G] \in \mathcal{E}$ whenever $G \in \mathcal{E}$. This follows because

$[y_A, G](-) \cong G(A \otimes -)$ is the composite of $G\colon \mathcal{W} \to \mathbf{Set}$ with the map of tangent categories $A \otimes (-)\colon \mathcal{W} \to \mathcal{W}$. □

Since each representable in $\mathcal{PW}$ clearly lies in $\mathcal{E}$, we may write $\mathcal{E}\text{-}\mathbf{CAT}_\pitchfork$ to denote the locally full sub-2-category of $\mathcal{E}\text{-}\mathbf{CAT}$ on the $\mathcal{E}$-categories and $\mathcal{E}$-functors which admit, respectively preserve, powers by representables. With this notation, we can now give our promised representation of tangent categories as enriched categories.

Theorem 19. *The* 2*-category* **TANG** *is equivalent to* $\mathcal{E}\text{-}\mathbf{CAT}_\pitchfork$.

Proof. By Lemma 16, we can identify $\mathcal{E}\text{-}\mathbf{CAT}$ with a full sub-2-category of $\mathcal{PW}\text{-}\mathbf{CAT}$; but since the inclusion $\mathcal{E} \to \mathcal{PW}$ preserves internal homs, an $\mathcal{E}$-category will admit powers by representables *qua* $\mathcal{E}$-category just when it does so *qua* $\mathcal{PW}$-category, and so we may identify $\mathcal{E}\text{-}\mathbf{CAT}_\pitchfork$ with the full sub-2-category of $\mathcal{PW}\text{-}\mathbf{CAT}_\pitchfork$ on those $\mathcal{C}$ for which each $\mathcal{C}(X, Y)\colon \mathcal{W} \to \mathbf{Set}$ preserves tangent limits. Transporting across the equivalence $\mathcal{PW}\text{-}\mathbf{CAT}_\pitchfork \simeq \mathcal{W}\text{-}\mathbf{ACT}$ of Proposition 14, we may thus identify $\mathcal{E}\text{-}\mathbf{CAT}_\pitchfork$ with the full sub-2-category of $\mathcal{W}\text{-}\mathbf{ACT}$ on those $(\mathcal{C}, *)$ for which each

$$\mathcal{C}(Y, (-) * X)\colon \mathcal{W} \to \mathbf{Set}$$

preserves tangent limits. By the Yoneda lemma, this is the same as asking that each functor $(-) * X\colon \mathcal{W} \to \mathcal{C}$ preserves tangent limits—which is to ask that $(\mathcal{C}, *)$ be a tangent $\mathcal{W}$-actegory. So $\mathcal{E}\text{-}\mathbf{CAT}_\pitchfork \simeq \mathcal{W}\text{-}\mathbf{ACT}_\mathrm{t}$, and now composing with the equivalence of Theorem 9 yields the result. □

Remark 20. It is not hard to show that, if $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{E}$-categories admitting powers by representables, then a general $\mathcal{E}$-functor (not necesssarily preserving such powers) corresponds to a lax tangent functor in the sense of Remark 6. We will use this fact in Section 6 below.

5. An embedding theorem for tangent categories

We now use the representation of tangent categories as enriched categories to show that any small tangent category $\mathcal{C}$ has a full tangent-preserving embedding into a representable tangent category. This embedding will simply be the Yoneda embedding $Y\colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ of $\mathcal{C}$ seen as an $\mathcal{E}$-enriched category; since a presheaf category always admits powers, and since the Yoneda embedding preserves any powers that exist, this is certainly an embedding of tangent categories, and so all we need to show is that the tangent structure on $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ is in fact representable. The reason that this is true is that the monoidal structure on $\mathcal{E}$ is in fact *cartesian.*

Lemma 21. *The category $\mathcal{E}$ of Proposition 18 is complete and cocomplete, and has its symmetric monoidal structure given by cartesian product.*

Proof. $\mathcal{E}$ is a small-orthogonality class in a presheaf category, so locally presentable, so complete and cocomplete; see [1], for example. To see that its monoidal structure is cartesian, note first that the monoidal structure $(\mathcal{W}, \otimes, \mathbb{N})$ is *cocartesian*, so that each $A \in \mathcal{W}$ bears a commutative monoid structure, naturally in A. Since the restricted Yoneda embedding $\mathcal{W}^{\mathrm{op}} \to \mathcal{E}$ is strong monoidal, each $y_A \in \mathcal{E}$ bears a cocommutative comonoid structure, naturally in A; since any colimit of commutative comonoids is again a commutative comonoid, and since the representables are dense in $\mathcal{E}$, it follows that each $X \in \mathcal{E}$ has a cocommutative comonoid structure, naturally in X: which implies [11] that the monoidal structure is in fact cartesian. □

Corollary 22. *For any small $\mathcal{E}$-category $\mathcal{C}$, the presheaf $\mathcal{E}$-category $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ is complete, cocomplete, and cartesian closed as an $\mathcal{E}$-category.*

Proof. Since $\mathcal{E}$ is complete and cocomplete as an ordinary category, the completeness and cocompleteness of $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ as an $\mathcal{E}$-category follows from [17, Proposition 3.75]. As for cartesian closedness, we must show that each $\mathcal{E}$-functor

$$(-) \times F \colon [\mathcal{C}^{\mathrm{op}}, \mathcal{E}] \to [\mathcal{C}^{\mathrm{op}}, \mathcal{E}] \tag{5.1}$$

admits a right adjoint. Now, for each $X \in \mathcal{E}$, $(-) \times X \colon \mathcal{E} \to \mathcal{E}$ is the $\mathcal{E}$-functor taking *copowers* by X and so is cocontinuous. As limits and colimits in functor $\mathcal{E}$-categories are pointwise, each $\mathcal{E}$-functor (5.1) is likewise cocontinuous, and so we may define a right adjoint $(-)^F$ just as in the unenriched case by taking:

$$G^F(X) = [\mathcal{C}^{\mathrm{op}}, \mathcal{E}](\mathcal{C}(-, X) \times F, G) \ . \qquad \square$$

Proposition 23. *For any small $\mathcal{E}$-category $\mathcal{C}$, the tangent category corresponding under Theorem 19 to the presheaf $\mathcal{E}$-category $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ is representable.*

Proof. This tangent category is the underlying ordinary category of $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ equipped with the tangent structure $T_n X = y_{W_n} \pitchfork X$. Since $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ is cartesian closed as an $\mathcal{E}$-category, its underlying category is also cartesian closed, and so we need only show that each functor T_n is given by an exponential. Now, since $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ is cocomplete as an $\mathcal{E}$-category, it admits all copowers; thus, as for any object $X \in [\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ the exponential $\mathcal{E}$-functor $X^{(-)} \colon [\mathcal{C}^{\mathrm{op}}, \mathcal{E}]^{\mathrm{op}} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ preserves limits, in particular powers, we have for each $E \in \mathcal{E}$ an isomorphism

$$X^{(E \cdot 1)} \cong E \pitchfork X^1 \cong E \pitchfork X \ ,$$

so that *any* power in $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$, and in particular each T_n, can be computed as an $\mathcal{E}$-enriched exponential. □

Combining this with the remarks that began this section, we obtain:

Theorem 24. *For any small tangent category $\mathcal{C}$, the $\mathcal{E}$-enriched Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ provides a full tangent-preserving embedding of $\mathcal{C}$ into a representable tangent category.*

6. An explicit presentation

To conclude the paper, we extract an explicit presentation of the representable tangent category $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ into which the preceding theorem embeds each small tangent category $\mathcal{C}$. Consider first the case where $\mathcal{C}$ is the terminal tangent category 1: now $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ is simply $\mathcal{E}$ itself *qua* $\mathcal{E}$-enriched category, and powers by objects of $\mathcal{E}$ are simply given by the internal hom of $\mathcal{E}$. So $\mathcal{E}$ is a representable tangent category with tangent functor

$$TX = X^{y_W} \cong X(W \otimes -) = X(T-) \tag{6.1}$$

where the isomorphism comes from the formula (4.1) for the internal hom in $[\mathcal{W}, \mathbf{Set}]$, which by Proposition 17 is equally the internal hom in $\mathcal{E}$. Of course, the representing object for this tangent structure is $y_W \in \mathcal{E}$.

Consider now the case of a general tangent category $\mathcal{C}$. Objects of $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ are $\mathcal{E}$-enriched functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{E}$, which are equally $\mathcal{E}$-enriched functors $\mathcal{C} \to \mathcal{E}^{\mathrm{op}}$. Since *qua* $\mathcal{E}$-category both $\mathcal{C}$ and $\mathcal{E}^{\mathrm{op}}$ admit powers by representables, we may by Remark 20 identify such $\mathcal{E}$-functors with lax tangent functors $\mathcal{C} \to \mathcal{E}^{\mathrm{op}}$; here, the tangent structure on $\mathcal{E}^{\mathrm{op}}$ is induced by the $\mathcal{E}$-enriched *copowers* of $\mathcal{E}$ and so given by $TX = y_W \times X$ (where the product here is taken in $\mathcal{E}$).

It follows that a lax tangent functor $\mathcal{C} \to \mathcal{E}^{\mathrm{op}}$ comprises an ordinary functor $H\colon \mathcal{C} \to \mathcal{E}^{\mathrm{op}}$ together with a transformation $\varphi\colon HT \Rightarrow y_W \times H(-)$ in $[\mathcal{C}, \mathcal{E}^{\mathrm{op}}]$ rendering commutative the diagrams in (2.2). This is equally a functor $H\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{E}$ together with a natural family of maps $y_W \times HC \to H(TC)$ in $\mathcal{E}$, or equally by adjointness, a natural family of maps

$$\varphi_C\colon HC \to H(TC)^{y_W} \cong H(TC)(T-)$$

in $\mathcal{E}$ satisfying suitable axioms. Now, giving $H\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{E}$ is in turn equivalent to giving a functor $H\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{W} \to \mathbf{Set}$ which preserves tangent limits in its second variable; and φ is now equally a family of maps

$$\varphi_{C,A}\colon H(C, A) \to H(TC, TA)$$

natural in $C \in \mathcal{C}$ and $A \in \mathcal{W}$ and rendering commutative those diagrams which correspond to the axioms in (2.2). All told, we see that see that objects of the $\mathcal{E}$-functor category $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ are equally well *tangent modules* $\mathcal{C} \to\!\!\!\!\!+ \mathcal{W}$ in the sense of the following definition:

Definition 25. A *tangent module* $\mathcal{C} \to\!\!\!\!\!+ \mathcal{D}$ between tangent categories comprises:

- A functor $X\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$ preserving tangent limits in its second variable;
- A family of maps $T\colon X(C, D) \to X(TC, TD)$ which are natural in C and D, and make the following diagrams commute for all $x \in X(C, D)$:

$$\begin{array}{ccc} C & \xrightarrow{x} & D \\ {\scriptstyle e_C}\downarrow & & \downarrow{\scriptstyle e_D} \\ TC & \xrightarrow{Tx} & TD \\ {\scriptstyle p_C}\downarrow & & \downarrow{\scriptstyle p_D} \\ C & \xrightarrow{x} & D \end{array} \qquad \begin{array}{ccc} TC \times_{p_C} TC & \xrightarrow{Tx \times_x Tx} & TD \times_{p_D} TD \\ {\scriptstyle m_C}\downarrow & & \downarrow{\scriptstyle m_D} \\ TC & \xrightarrow{Tx} & TD \\ {\scriptstyle \ell_C}\downarrow & & \downarrow{\scriptstyle \ell_D} \\ TTC & \xrightarrow{TTx} & TTD \end{array} \qquad \begin{array}{ccc} TTC & \xrightarrow{TTx} & TTD \\ {\scriptstyle c_C}\downarrow & & \downarrow{\scriptstyle c_D} \\ TTC & \xrightarrow{TTx} & TTD \end{array}.$$

Here, we use the evident notation for elements of the module X, and for the action on such elements by maps in $\mathcal{C}$ and $\mathcal{D}$. Note that, to construct the element top centre, we use X's preservation of tangent pullbacks in its second variable.

A *map of tangent modules* $f\colon X \to Y$ is a natural transformation $f\colon X \Rightarrow Y$ commuting with T_X and T_Y in the evident sense. We write $\mathbf{TMod}(\mathcal{C}, \mathcal{D})$ for the category of tangent modules from $\mathcal{C}$ to $\mathcal{D}$, and endow it with a tangent structure by defining TX to be the tangent module with components $(TX)(C, D) = X(C, TD)$ and with operation

$$T_{TX} = X(C, TD) \xrightarrow{T_X} X(TC, TTD) \xrightarrow{c_D \circ (-)} X(TC, TTD)\ .$$

The remaining data for the tangent structure on $\mathbf{TMod}(\mathcal{C}, \mathcal{D})$ is obtained from the corresponding data in $\mathcal{D}$ by postcomposition.

Remark 26. If $X\colon \mathcal{C} \rightarrow\!\!\!\!\!\rightarrow \mathcal{D}$ is a tangent module, then we obtain a new tangent category $\mathrm{coll}(X)$, the so-called *collage* [24] of X, whose objects are the disjoint union of those of $\mathcal{C}$ and $\mathcal{D}$, whose morphism-sets are defined by:

$$\mathrm{coll}(X)(C,C') = \mathcal{C}(C,C') \qquad \mathrm{coll}(X)(D,D') = \mathcal{C}(D,D')$$
$$\mathrm{coll}(X)(C,D) = X(C,D) \qquad \mathrm{coll}(X)(D,C) = \emptyset$$

for all $C,C' \in \mathcal{C}$ and $D,D' \in \mathcal{D}$, whose tangent functor is defined from the tangent functors of $\mathcal{C}$ and $\mathcal{D}$ together with the family of maps $X(C,D) \to X(TC,TD)$, and whose remaining data for the tangent structure is obtained from that in $\mathcal{C}$ and in $\mathcal{D}$. This $\mathrm{coll}(X)$ is a *bipartite* tangent category, in the sense that it admits a tangent functor to the arrow category $\mathbf{2}$ endowed with the trivial tangent structure. In fact, it is easy to see that tangent modules from $\mathcal{C}$ to $\mathcal{D}$ are the same as bipartite tangent categories $p\colon \mathcal{X} \to \mathbf{2}$ such that $p^{-1}(0) = \mathcal{C}$ and $p^{-1}(1) = \mathcal{D}$.

To give a universal characterisation of the collage of a tangent module, we would have to construct the *equipment* [28] of tangent categories, tangent functors and tangent profunctors; we leave consideration of this to future work.

Proposition 27. *For any tangent category $\mathcal{C}$, the underlying tangent category of the $\mathcal{E}$-category $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ is isomorphic to* $\mathbf{TMod}(\mathcal{C}, \mathcal{W})$.

Proof. The bijection on objects was verified above, and that on morphisms is equally straightforward. All that remains is to show that the tangent structures on $\mathbf{TMod}(\mathcal{C}, \mathcal{W})$ and on $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ coincide; which follows easily from the description above of the tangent structure on $\mathcal{E}$, and the fact that powers in a functor $\mathcal{E}$-category are computed pointwise. □

In particular, this result tells us that $\mathbf{TMod}(\mathcal{C}, \mathcal{W})$ is a representable tangent category; the representing object is by Proposition 23 the copower of the terminal object of $\mathbf{TMod}(\mathcal{C}, \mathcal{W})$ by $y_W \in \mathcal{E}$; this is the object $D \in \mathbf{TMod}(\mathcal{C}, \mathcal{W})$ given by

$$D(C,A) = \mathcal{W}(W,A) \tag{6.2}$$

and with $T_D\colon D(C,A) \to D(TC,TA)$ given (after some calculation) by

$$\begin{aligned} \mathcal{W}(W,A) &\to \mathcal{W}(W, W \otimes A) \\ f &\mapsto e_A \circ f \ . \end{aligned} \tag{6.3}$$

Finally, let us give a concrete characterisation of the action of the $\mathcal{E}$-enriched Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$. This sends $X \in \mathcal{C}$ to the $\mathcal{E}$-functor $\mathcal{C}(\text{–}, X)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{E}$, which corresponds to the tangent module $YX\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{W} \to \mathbf{Set}$ with

$$YX(C,A) = \mathcal{C}(C, A * X)$$

and with $T_{YX}\colon YX(C,A) \to YX(TC,TA)$ sending an element $f\colon C \to A * X$ of $YX(C,A)$ to the element

$$TC \xrightarrow{Tf} T(A * X) = W * (A * X) \xrightarrow{\cong} (W \otimes A) * X = TA * X$$

of $YX(TC,TA)$. Putting all the above together, we obtain the following more concrete form of the embedding theorem:

Theorem 28. *For any small tangent category $\mathcal{C}$, there is a full tangent embedding $Y : \mathcal{C} \to \mathbf{TMod}(\mathcal{C}, \mathcal{W})$ into the representable tangent category of tangent modules from $\mathcal{C}$ to $\mathcal{W}$.*

Having arrived at this concrete form of the embedding theorem, one might be tempted to dismantle the abstract scaffolding by which it was obtained. However, there are several reasons why this would be not only disingenuous but positively unhelpful. In the first instance, the concrete description is subtle enough that without the abstract justification it would appear entirely *ad hoc*. Secondly, without the general theory behind it, a detailed proof of Theorem 28 from first principles would be rather involved—requiring us to show by hand that $\mathbf{TMod}(\mathcal{C}, \mathcal{W})$ is a tangent category, that it is representable, and that $Y : \mathcal{C} \to \mathbf{TMod}(\mathcal{C}, \mathcal{W})$ is a fully faithful tangent functor.

Finally, the enriched-categorical viewpoint encourages us to look at tangent categories in a different way. For example, it is immediate from the enriched perspective that the functor $T : \mathcal{C} \to \mathcal{C}$ associated to any tangent category is in fact a tangent functor (since it is an $\mathcal{E}$-enriched power functor, and as such preserves $\mathcal{E}$-enriched powers); or that the 2-category of tangent categories and tangent functors admits all bilimits and bicolimits. As indicated in the introduction, we hope to exploit the full power of this viewpoint in forthcoming work.

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