BISIMULATION VS TRACE EQUIVALENCE

1) Generative T-systems

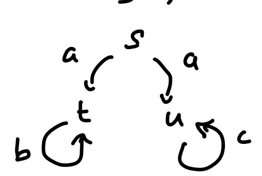
<u>Defn</u> Let A be a finite set (an <u>alphabet</u>) and let T be a monad on <u>Set</u>. A generative T-system w/ alphabet A is a set S (of states) t/w a function $\sigma : S \to T(A \times S)$. We write Gen_T for caty of such systems, where a map $(S, \sigma) \to (S', \sigma')$ is a f^n $f : S \to S'$ st $\sigma' \circ f = T(A \times f) \circ \sigma$.

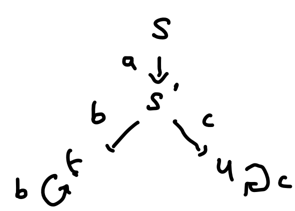
<u>Ex</u> When $T = \mathrm{id}$, get <u>deterministic</u> gen systems: comprises a set S of states, and a f^n $\sigma = (g, n) : S \to A \times S$, assigning to each state s, an output symbol $g(s)$, and a next state $n(s)$.

<u>Ex</u> When $T = \begin{cases} \mathcal{P}_f^+ \\ \mathcal{P}^+ \end{cases}$, we get $\begin{Bmatrix} \text{finitely branching} \\ \text{———} \end{Bmatrix}$, nonterminating, labelled trans systems.

Comprises a set S, t/w a relation $\sigma \subseteq S \times A \times S$, written as $s \xrightarrow{a} s'$ if $(s, a, s') \in \sigma$, such that for all $s \in S$, $\{(a, s') : s \xrightarrow{a} s'\}$ is $\begin{Bmatrix} \text{nonempty finite} \\ \text{nonempty} \end{Bmatrix}$.

<u>Eg</u>: $A = \{a, b, c\}$, we have lts's:

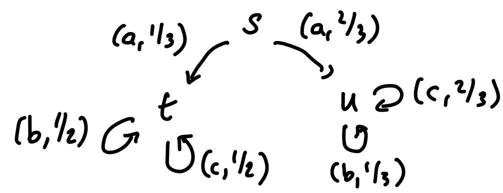


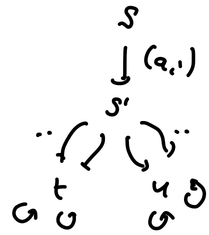


Ex When $T = \mathcal{D}$, the finitely supported prob. dist. monad, with

$$\mathcal{D}(X) = \{ \omega : X \to [0,1] : \mathrm{supp}(\omega) \text{ finite}, \ \sum_{x \in X} \omega(x) = 1 \},$$

we get probabilistic generative systems: set S w/ $\sigma : S \to \mathcal{D}(A \times S)$, giving for each state a prob. dist. over output symbols + next states, eg:





2) <u>Bisimulation equivalence</u>

For a deterministic gen. system $\sigma = (g, n) : S \to A \times S$, each state $s \in S$ has a corresponding <u>behaviour</u>:

$$(g(s), g(n(s)), g(n(n(s))), \ldots) \in A^{\mathbb{N}}.$$

We call states $s, s' \in S$ <u>bisimilar</u> if they have the same behaviour; equivalently, if they are related by a <u>bisimulation</u>: a eq. rel$^{\underline{n}}$ $\equiv \subseteq S \times S$ st

$$u \equiv v \implies g(u) = g(v) \quad \text{and} \quad n(u) \equiv n(v).$$

We can capture bisimilarity abstractly: indeed, $A^{\mathbb{N}}$ is the underlying set of the final determ. gen. system:

$$\alpha : A^{\mathbb{N}} \longrightarrow A \times A^{\mathbb{N}}$$
$$(a_0, a_1, \ldots) \longmapsto (a_0, (a_1, a_2, \ldots))$$

and two states are bisimilar if have same image under the ! map $\underline{S} \longrightarrow \underline{A}^{\mathbb{N}}$ in Gen_{id}.

<u>Defn</u> An <u>obj of behaviours</u> for gen. T-systems is a final object (Beh, β) in Gen_T. The <u>behavior</u> map of $\underline{S} \in \mathrm{Gen}_T$ is the ! map $\mathrm{beh} : \underline{S} \longrightarrow \underline{\mathrm{Beh}}$ in Gen_T. Two states $s, s' \in \underline{S}$ are <u>bisimilar</u> if $\mathrm{beh}(s) = \mathrm{beh}(s')$.

<u>Ex</u> When $T = id$, we get what we saw above.

<u>Ex</u> When $T = \mathcal{P}_f^+$, $s, s' \in \underline{S}$ are bisimilar iff related by a bisimulation on $\underline{S}$: an eq. rel $\equiv \subseteq S \times S$ st

① $u \equiv v$ and $u \xrightarrow{a} u'$ $\Rightarrow$ $\exists v'$ st $v \xrightarrow{a} v'$ and $u' \equiv v'$

② $u \equiv v$ and $v \xrightarrow{a} v'$ $\Rightarrow$ $\exists u'$ st $u \xrightarrow{a} u'$ and $u' \equiv v'$

<u>Eg</u>: in

$$\begin{tikzcd}
 & u & \\
s \arrow[ur, "c"] \arrow[d, "a"'] \arrow[drr, "a"] & & s' \arrow[ul, "c"'] \arrow[d, "a"] \\
t \arrow[loop below, "b"] & & t' \arrow[loop below, "b"]
\end{tikzcd}$$

have s, s' bisimilar
t, t' bisimilar

<u>Ex</u> When $T = \mathcal{D}$, $s, s' \in S$ are bisimilar iff related by some $\equiv \subseteq S \times S$ eq. rel. st:

$$u \equiv v \Rightarrow \sigma(u)(\{a\} \times C) = \sigma(v)(\{a\} \times C) \quad \text{for all } a \in A,\ C \in S/_{\equiv}$$

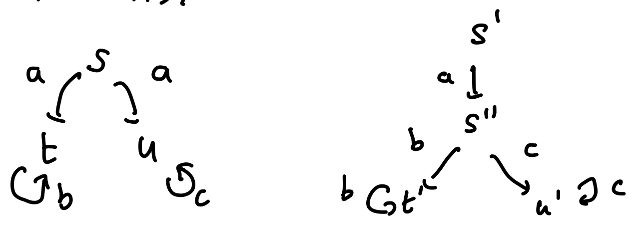
<u>Eg</u>: in:

$$\begin{tikzcd}
 & u & \\
s \arrow[ur, "{(c, \frac{1}{2})}"] \arrow[d, "{(a, \frac{1}{6})}"'] \arrow[drr, "{(a, \frac{1}{3})}"] & & s' \arrow[ul, "{(c, \frac{1}{2})}"'] \arrow[d, "{(a, \frac{1}{2})}"] \\
t \arrow[loop below, "{(b, 1)}"] & & t' \arrow[loop below, "{(b, 1)}"]
\end{tikzcd}$$

have s, s' bisimilar
t, t' bisimilar

3) TRACE EQUIVALENCE

Consider the lts:



The states s, s' are <u>not</u> bisimilar. But they are "the same" in a weaker sense, since both produce the same possible streams of values:

$$\{ab\dot{b}\cdots,\ ac\dot{c}\cdots\};$$

we say that s, s' are <u>trace equivalent</u>.

How can we capture this abstractly? One approach: Power – Turi (1999).

<u>Defn</u> An <u>A-ary comagma</u> in a cat'y $\mathcal{C}$ with copowers is an object $X \in \mathcal{C}$ t/w, $\xi : X \longrightarrow \underbrace{A \cdot X}_{= \sum_{a \in A} X}$.

Now note: a generative T-system $S \to T(A \times S)$ is <u>same as</u> an A-ary comagma $S \nrightarrow A \cdot S$ in $Kl(T)$. However, maps of gen. T-systems are functions; maps of A-ary comagmas in $Kl(T)$ are Kleisli maps.

P-T define an <u>object of traces</u> for gen T-systems to be a final A-ary comagma in $Kl(T)$, and two states to be <u>trace equiv</u> if identified by the ! map to this object of traces in $\mathcal{Kl}(T)$.

Issue: an object of traces need not exist! Eg: it exists for $T = \mathrm{id}$, $T = \mathcal{P}^+$, but not for $T = \mathcal{P}_f^+$, $T = \mathcal{D}$.

To fix this, we can look for a final A-ary comagma, not in $\mathcal{Kl}(T)$, but in $EM(T)$.

<u>Defn</u> An <u>object of traces</u> for gen. T-systems is a final A-ary comagma $(Tr, \tau : Tr \to A \cdot Tr)$ in $EM(T)$.

<u>Ex</u> When $T = \mathrm{Id}$, $Tr = A^{\mathbb{N}}$.

<u>Ex</u> When $T = \mathcal{P}_f^+$, Tr = non-empty closed sets in $A^{\mathbb{N}}$, seen as $\mathcal{P}_f^+$-algebra under union, and comagma structure:

$$\underbrace{V^+(A^{\mathbb{N}})}_{\text{ne closed sets.}} \xrightarrow{V^+(\alpha)} V^+(A \times A^{\mathbb{N}}) \cong A \cdot V^+(A^{\mathbb{N}})$$

<u>Ex</u> When $T = \mathcal{D}$, Tr = Borel probability dists on $A^{\mathbb{N}}$ seen as $\mathcal{D}$-alg under convex combination.

To assign a trace to an element of a gen. T-system, need:

Defn Given $S \xrightarrow{\sigma} T(A\times S)$ a gen T-system, the associated A-ary comagma in $EM(T)$ is $F^T(S) = (T(S), \mu_S)$ w/ comagma structure:

$$\sigma^\# = F^T(S) \xrightarrow{F^T\sigma} F^T(T(A\times S)) \xrightarrow{\mu} F^T(A\times S) \cong A \bullet F^T(S)$$

Given (S,σ), the trace map $tr: S \to Tr$ is the precomposite of the ! homomorphism $(F^T(S), \sigma^\#) \longrightarrow (Tr, \tau)$ with $\eta: S \to T(S)$. Two states s, s' are trace equivalent if $tr(s) = tr(s')$.

Ex • when $T = id$, trace = behaviour.

• when $T = \mathcal{P}_f^+$, the trace of $s \in S$ is the closed set

$$tr(s) = \{\vec{a} \in A^{\mathbb{N}} : \exists \vec{s} \in S^{\mathbb{N}} \text{ st } s \overset{a_0}{\rightsquigarrow} s_0 \overset{a_1}{\rightsquigarrow} s_1 \overset{a_2}{\rightsquigarrow} s_2 \rightsquigarrow \cdots\}$$

• when $T = \mathcal{D}$, the trace of $s \in S$ is the prob distn on $A^{\mathbb{N}}$:

$$tr(s)(A^{\mathbb{N}}) = 1$$
$$tr(s)(a_0 \cdots a_n A^{\mathbb{N}}) = \sum_{t \in S} \sigma(s)(a_0, t) \times tr(t)(a_1 \cdots a_n A^{\mathbb{N}}).$$

Mention:

• Another interesting example: $T = T_B$ = free monad on a B-ary operation. A gen T_B-system is an automaton for turning B-streams into A-streams. Eg: $A = B = \{0,1\}$, have $\sigma: S \longrightarrow T_B(A\times S)$ given by

$$s \longmapsto \begin{array}{ccc} (1,t) & & (0,s) \\ & {}_{0}\!\searrow \ \swarrow\!{}_{1} & \\ & \cdot & \end{array}$$

$$t \longmapsto \begin{array}{ccc} (0,t) & & (1,t) \\ & {}_{0}\!\searrow \ \swarrow\!{}_{1} & \\ & \cdot & \end{array}$$

here the state s implements the "+1" operation on binary streams, eg:

$$1110011\cdots \longmapsto 0001011\cdots$$

In this case, two states are bisimilar if $\exists \equiv \subseteq S\times S$ st

$u \equiv v \Rightarrow \overbrace{\sigma(u),\ \sigma(v)}^{\in T_B(A\times S)}$ have same projection onto $T_B(A)$,

and $T_B(\equiv)$-related proj's on $T_B(S)$.

OTOH: object of traces is set of cts $f^{\underline{n}}$s $B^{\mathbb{N}} \to A^{\mathbb{N}}$,
and two states are trace equiv. if they encode same cts $f^{\underline{n}}$.