COFREE COCOMMUTATIVE COALGEBRAS + ABSTRACT DIFFERENTIATION

1) Cofree cocommutative coalgebras

Let k be a comm. ring; a <u>cocomm. k-coalg</u> is a k-module C together with k-lin. maps

$$\Delta : C \longrightarrow C \otimes C \qquad \varepsilon : C \longrightarrow k$$

+ coassociativity, counitality, cocommutativity.

Given a k-module V, a <u>cofree</u> $\overbrace{\text{cocomm. coalg}}^{\text{cocoalg}}$ on V is a cocoalg C t/w a linear map $e : C \longrightarrow V$ st: if D is a cocoalg, and $f : D \longrightarrow V$ a k-linear map, $\exists!$ factorisation of f through e via a coalg homom. $\bar{f} : D \longrightarrow C$.

$$\begin{tikzcd} D \arrow[r, dashed, "\exists!\,\bar{f}\ \text{coalg homom.}"] \arrow[dr, bend right, "\forall f"'] & C \arrow[d, "e"] \\ & V. \end{tikzcd} \qquad \circledast$$

Cofree cocoalg always exist. Describing them explicitly is hard!
Easy in one case — when k is an alg. closed field of char 0.

<u>PROP</u> If k an alg. closed field of char 0, the cofree cocoalg on V is given by

$$QV = \bigoplus_{v \in V} \mathrm{Sym}(V).$$

free symm. k-alg on V

Let's write $\langle v_1, \dots, v_n \rangle_w \in QV$ for pure tensor $v_1 \otimes \cdots \otimes v_n \in \mathrm{Sym}(V)$ inside the w-summand of $\bigoplus$.

Proof QV a cocoalg via:

$$\begin{aligned} \varepsilon_v : QV &\longrightarrow k \\ \langle \, \rangle_v &\longmapsto 1 \\ \langle v_1, \dots, v_n \rangle_v &\longmapsto 0 \quad \text{if } n \geq 1 \end{aligned} \tag{1}$$

$1 \in \mathrm{Sym}(V)$ in v-summand $\dashrightarrow \langle \, \rangle_v$

$$\begin{aligned} \Delta_v : QV &\longrightarrow QV \otimes QV \\ \langle v_1, \dots, v_n \rangle_v &\longmapsto \sum_{I \subseteq [n]} \langle v_I \rangle_v \otimes \langle v_{[n]-I} \rangle_v \end{aligned} \tag{2}$$

where $[n] = \{1, \dots, n\}$

and if $I = \{i_1, \dots, i_k\} \subseteq [n]$, then $\langle v_I \rangle_v = \langle v_{i_1}, \dots, v_{i_k} \rangle_v$

The map

$$\begin{aligned} e_v : QV &\longrightarrow V \\ \langle \, \rangle_v &\longmapsto v \\ \langle v_1 \rangle_v &\longmapsto v_1 \\ \langle v_1, \dots, v_n \rangle_v &\longmapsto 0 \quad \text{if } n \geq 2 \end{aligned} \tag{3}$$

How to get liftings as in ⊛? Start from cocoalg D and $f : D \to V$.

<u>FACT 1</u>: $D = \bigoplus_{i \in I} D_i$ where each D_i is a cocoalg with a unique <u>grouplike</u> element g (ie $\Delta(g) = g \otimes g$).

So wlog it's enough to find liftings of $f : D \to V$ when D has a !

grouplike element g. In this case: first let's define

$$\Delta_0 = \varepsilon, \quad \Delta_{n+1} = (\Delta_n \otimes 1) \circ \Delta \quad \text{for } n \geq 1 \qquad \text{"n-ary comult."}.$$

and now take

$D = kg \oplus \ker \varepsilon$

$$\begin{aligned} \bar{f} : D &\longrightarrow QV \\ d &\longmapsto \sum_{n \geq 0} \langle f^{\otimes n} \circ (1 - g \cdot \varepsilon)^{\otimes n} \circ \Delta_n \rangle_g \end{aligned}$$

Looks like an ∞ sum, but actually it's always finite (FACT 2). $\square$

When k not alg. closed field of char 0, facts 1+2 may fail, so QV isn't cofree cocoalg. But it still exists, and is still a cocoalg; so can ask: does it have some characterisation. Yes! Goal: explain what this is.

2) The comonad structure of Q

Def^n A comonad on a cat$_y$ $\mathcal{C}$ is a functor $Q: \mathcal{C} \to \mathcal{C}$ t/w. $e: Q \Rightarrow id_{\mathcal{C}}$ and $d: Q \Rightarrow QQ$, plus coass + counit axioms.
A comodule over a comonad Q is some $X \in \mathcal{C}$ t/w $X \xrightarrow{\xi} QX$ satisfying two axioms:

$$\begin{tikzcd} X \arrow[r, "\xi"] \arrow[dr, "1"'] & QX \arrow[d, "e_X"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\xi"] \arrow[d, "\xi"'] & QX \arrow[d, "Q\xi"] \\ QX \arrow[r, "d_X"'] & QQX \end{tikzcd} \qquad \text{commute.}$$

Example If C is a coalgebra, then $C \otimes (-)$ is a comonad on k-Mod;

a $C \otimes (-)$ - comodule is a C-comodule.

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When k alg closed of char 0, we get a comonad structure on $Q : k\text{-Mod} \to k\text{-Mod}$ cofree cocoalg as follows. $e : Q \Rightarrow \mathrm{id}$ has components $QV \xrightarrow{e_V} V$ as above; $d : Q \Rightarrow QQ$ has components obtain using cofreeness:

$$
\begin{tikzcd}
QV \arrow[r, dashed, "\overline{\mathrm{id}} := d_V"] \arrow[dr, "\mathrm{id}"'] & Q(QV) \arrow[d, "e_{QV}"] \\
& QV
\end{tikzcd}
.
$$

Explicitly:
$$
\begin{aligned}
d_V : QV &\longrightarrow QQV \\
\langle v_1, \dots, v_n \rangle_V &\longmapsto \sum_{[n] = A_1 | \cdots | A_k} \langle \langle v_{A_1} \rangle_V, \dots, \langle v_{A_k} \rangle_V \rangle_{\langle \rangle_V}
\end{aligned}
\qquad (4).
$$

This gives a comonad $Q : k\text{-Mod} \to k\text{-Mod}$, whose comodules are cocoalgs.

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When k is not of this form, we still have a "cofree cocoalg" comonad... but we also have a different comonad Q given by above formulae.

Prop The formulae (3), (4) endow $QV = \bigoplus_{v \in V} \mathrm{Sym}(V)$ with comonad structure always.

3) The sym. monoidal comonad structure of Q

Defn A comonad Q on a sym. mon. caty $\mathcal{C}$ is a sym. monoidal comonad if $Q : \mathcal{C} \to \mathcal{C}$ equipped with symm. monoidal (lax) structure:

$$\left(m_k : k \to Qk \qquad m_\otimes : QV \otimes QW \longrightarrow Q(V \otimes W),\right.$$

+ axioms), and $d: Q \Rightarrow QQ$, $e: Q \Rightarrow \mathrm{id}$ are monoidal nat transfs.

Ex If C is a comm. bialgebra, then the comonad $C \otimes (-)$ is sym. monoidal via

$$m_\otimes : ((C \otimes V) \otimes (C \otimes W)) \xrightarrow{\sigma} (C \otimes C) \otimes (V \otimes W) \xrightarrow{\mu \otimes 1} C \otimes (V \otimes W)$$

$$m_k : \quad \cdots\cdot$$

Point: sym. monoidal structure on Q $\leftrightsquigarrow$ liftings of sym. monoidal structure of $\mathcal{C}$ to Q-Comod.

In ptic, if k alg closed of char 0, Q-comod = k-cocoalg, and $\otimes$ of cocoalgs is another cocoalg! So $Q : k\text{-Mod} \to k\text{-Mod}$ is sym. monoidal. Explicitly:

$$\begin{aligned} m_k : k &\longrightarrow Qk \\ 1 &\longmapsto \langle\,\rangle, \end{aligned} \qquad (5)$$

$$m_\otimes : QV \otimes QW \longrightarrow Q(V \otimes W) \qquad (6)$$

$$\langle v_1, \dots, v_n \rangle_V \otimes \langle w_1, \dots, w_m \rangle_W \longmapsto \sum_{\substack{I \subseteq [n] \\ J \subseteq [m] \\ \theta : I \cong J}} \langle \dots, \underset{\substack{\uparrow \\ i \in I}}{v_i \otimes w_{\theta(i)}}, \dots, \underset{\substack{\uparrow \\ i' \notin I}}{v_{i'} \otimes w}, \dots, \underset{\substack{\uparrow \\ j' \notin \mathrm{im}\,\theta}}{v \otimes w_{j'}}, \dots \rangle_{V \otimes W}$$

PROP Even when k is not of the nice form, (5) and (6) make Q into a

Sym. mon. comonad.

4) The differential comonad structure of Q

Defn A monoidal coalgebra modality on a sym. mon. caty $\mathcal{C}$ is a sym monoidal comonad Q st each QV is a cocoalg + axioms.

So what we have so far is that $Q: k\text{-Mod} \to k\text{-Mod}$ is a monoidal coalg. modality (for any k).

Defn A monoidal coalgebra modality on a k-linear sym. mon. caty $\mathcal{C}$ is called a monoidal differential modality if endowed with a "deriving transformation"

$$\partial: QV \otimes V \longrightarrow QV$$

satisfying five axioms (product rule, chain rule, additive rule).

Prop Over any k, Q is a monoidal differential modality, where

$$\begin{aligned} \partial: QV\otimes V &\longrightarrow QV \\ \langle v_1,\dots,v_n\rangle_v \otimes w &\longmapsto \langle v_1,\dots,v_n,w\rangle_v \end{aligned} \qquad (7)$$

Thm (G. - Lemay) (1)-(7) make $QV = \bigoplus_{v\in V} \mathrm{Sym}(V)$ into the initial monoidal differential modality on k-Mod.

What's this got to do with differentiation?

Defn Let Q be a comonad on $\mathcal{C}$. The coKleisli category of $\mathcal{C}$, $\mathrm{coK}\ell(Q)$ has:

- same objs as $\mathcal{C}$;
- maps $A \rightsquigarrow B$ are maps $QA \longrightarrow B$ in $\mathcal{C}$;
- composition of $\dfrac{A \rightsquigarrow B \qquad B \rightsquigarrow C}{QA \xrightarrow[f]{} B \qquad QB \xrightarrow[g]{} C}$ is

$$\frac{A \rightsquigarrow C}{QA \xrightarrow[d_A]{} QQA \xrightarrow[Qf]{} QB \xrightarrow[g]{} C}$$

Prop If Q is a monoidal diff modality, then $\mathrm{coK}\ell(Q)$ is a cartesian closed category equipped with a differentiation operator:

$$A \xrightarrow{f} B \qquad \rightsquigarrow \qquad A \times A \xrightarrow{Df} B$$

$$'' \quad (a, a') \longmapsto (\nabla f)(a)\; a'$$

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What is $\mathrm{coKl}(Q)$ in our example?

- objects are k-modules
- maps $V \rightsquigarrow W$ are maps $QV \longrightarrow W$ in $k.\mathrm{Mod}$, ie:

$$QV = \bigoplus_{v \in V} \mathrm{Sym}(V)$$

$$\left(f^{(0)} : V \longrightarrow W,\ f^{(1)} : V \times V \longrightarrow W,\ f^{(2)} : V \times V \times V \longrightarrow W,\ \ldots \right)$$

where each $f^{(n)}$ is sym. multilinear in last n args (but not first)

Think of these as "formal smooth maps" : $f^{(0)}$ is a function f

$f^{(1)}$ is $(v,w) \mapsto \nabla f(v)\, w$

$f^{(2)}$ is $\cdots\cdots \quad H \quad \cdots$

In this case, Df is given by

$$Df^{(n)}(v_0, w_0, \ldots, v_n, w_n) = f^{(n+1)}(\vec{v}, w_0) + \sum_{i=1}^{n} f^{(n)}(v_0, \ldots, v_{i-1}, w_i, v_{i+1}, \ldots, v_n)$$