On the Interval Routing of Chordal Rings

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Abstract

The Shortest-Path Interval Routing Scheme is an efficient strategy to code distributed routing algorithms in a compact way. Characterising networks which admit shortest-path Strict Interval Routing Scheme using one interval per edge (1-SIRS) is known to be NP-complete. In this paper, we study 1-SIRS for a popular class of networks, known as Chordal Rings.

We prove that for any chordal ring of degree 4 with a chord of length $k(>3)$, there exists an infinite set of $n$ such that the chordal ring $\langle 1, k \rangle_n$ of size $n$ does not admit a 1-SIRS, regardless of the node-labeling. This gives a negative answer to an open-question from Gavoille and extends the characterisation of node-labelings which allow a shortest path 1-SIRS in chordal rings.

Mainly, we study the natural cyclic node-labeling and derive an alternative node-labeling using isomorphic properties. First we show that there is an infinite class of chordal rings with chord $k$ of even length that do not have a shortest-path 1-SIRS. Second, we show the limitation of the cyclic node-labeling and its alternative representation when $k$ is odd. Finally, we conjecture that any chordal ring with shortest path 1-SIRS has a node-labeling isomorphic to a chordal ring with a cyclic node-labeling.

1. Introduction

In this paper, we are concerned with efficient schemes for routing in networks. Research activities focused on identifying classes of network topologies where the shortest path information at each node can be succinctly stored, assuming that suitably “short” labels can be assigned to nodes and links at preprocessing time. Such labels used to encode useful information about the network, with special regard to shortest paths. Among numerous strategies Interval Routing [10, 12, 19] have been proposed to derive space-efficient such routing schemes. Characterising networks which admit shortest-path Strict Interval Routing Scheme using one interval per edge (1-SIRS) is known to be NP-complete [5].

In this paper, we focus on studying such a routing scheme for a class of networks known as Chordal Rings. Some instances of the chordal ring are reminiscent of the torus, and some designs for redundant tori and redundant hypercubes are also chordal rings, e.g. [3].

The purpose of this paper is to essentially extend the partial characterisation of the Chordal Rings of degree four having a Strict Interval Routing Scheme (SIRS) providing a shortest path. In particular we present both positive and negative results on the existence of such SIRS. This answers an open question of Gavoille [10], and extends the previous results of Krizanc and Luccio [11] and Narayanan et al [16, 17]. We deduce that, for any chordal ring of chord of length $k(>3)$, there exists an infinite set of size $n$ such that $\langle 1, k \rangle_n$ does not admit a Shortest-path SIRS with one interval per edge.

In this introduction we first define Chordal Rings as a special class of Circulant Graphs, and then introduce the isomorphic properties necessary to analyse the node labeling of such graphs. We then briefly recall the Interval Routing Scheme definition.

In Section 2, we analyse different node-labeling and extend previous results. We study chordal rings with chords of even and odd length in Section 3 and Section 4 respectively. Finally, we conclude with a conjecture on the complete characterisation of shortest path SIRS scheme for chordal rings in Section 5.

1.1. Chordal Rings

One very important and common network topology is the ring structure (i.e., Hamiltonian cycle). Its simplicity and incremental extensibility is unfortunately challenged by a high vulnerability (low connectivity) and a high diameter (maximum distance). It is known that these drawbacks can be eliminated by adding bypass chords in a regular manner along the cycle. By suitably choosing the length of the chord between two nodes, one can achieve the appropriate property: e.g., low diameter, high connectivity, or efficient
routing. However, adding links must be as limited as possible to allow VLSI drawing or efficient switching mechanism at each node to implement a systematic scheme to efficiently route information (multiple routes of different length create non-trivial path selection and congestion problems). Multicomputers based on this design exist (e.g., [18]). For larger area networks, the geometric distance between nodes and the actual cost of wires forbid adding many links. To this end, four-regular Chordal Rings networks (also called Loop networks [2]) have been intensively studied to efficiently solve distributed problems (e.g. [13, 15]).

We recall that Chordal rings are particular cases of circulant graphs. An n-vertex circulant graph $G$ is a graph whose adjacency matrix $A = (a_{ij})_{i,j=1}^{n}$ is a circulant. That is the $i$th rows of $A$ is the cyclic shift of the first row by $i - 1$, $a_{ij} = a_{1,j-i+1}$, $i, j = 1, \ldots, n$.

Hereafter, subscripts are taken modulo $n$. We also assume that $a_{ij} = 0$, $i = 1, \ldots, n$. Therefore with every circulant graph one can associate a set of “chords” $S \subseteq \mathbb{Z}_n$ of the positions of non-zero entries of the first row of the adjacency matrix of the graph. Respectively we denote by $(S)_n$ the corresponding circulant graph.

1.2. The Ádám property

In a circulant graph $(S)_n$, the nodes are identified with $0, 1, \ldots, n-1$ and, if $j \in S, \lfloor n/2 \rfloor \geq j \geq 1$, then each node $i$ is connected by a chord of length $j$ to node $(i + j) \mod n$. (Note that this implies that there is also a chord from $i$ to $(i - j) \mod n$.) Following each chord of length $j \in S$, co-prime to $n$ (including $j = 1$), yields an Hamiltonian cycle.

We recall that two graphs $G_1, G_2$ are isomorphic, and write $G_1 \cong G_2$, if their adjacency matrices differ by a permutation of their rows and columns.

We say that two sets $S, T \subseteq \mathbb{Z}_n$ are proportional, and write $S \sim T$, if, for some integer $l$ with $\gcd(l, n) = 1$, $S = l \cdot T$ (where the multiplication is taken over $\mathbb{Z}_n$).

Obviously, $S \sim T$ implies $(S)_n \cong (T)_n$. For example in Figure 1, $S = \{1, 5\}$, $T = \{1, 9\}$, and $n = 23$, $(S)_n \cong (T)_n$ since $S \sim T$ (with $l = 5$).

Ádám [1] conjectured that the inverse statement is true as well. We say that a set $S \subseteq \mathbb{Z}_n$ has the Ádám property if for any other set $T \subseteq \mathbb{Z}_n$, the isomorphism $(S)_n \cong (T)_n$ implies the proportionality $S \sim T$. Thus the Ádám conjecture is equivalent to the statement that all sets $S \subseteq \mathbb{Z}_n$ have Ádám property.

It has been proven ([4, 7], see [14] for further references) that the conjecture holds for circulant graphs of degree 4: the isomorphism $(S)_n \cong (T)_n$ implies the proportionality $S \sim T$. However this property is not always true for degree larger than 4 (see [6] for counter-example).

![Figure 1. Ádám isomorphic Chordal Rings.](image)

Since we are only interested in Chordal Rings of degree 4, i.e., $(1, k)_n$ circulant graph, we can further specialise this result.

**Proposition 1** $(1, k)_n \cong (1, k')_n$, $k \neq k'$ iff $k = k'^{-1} \mod n$

**Proof** By the Ádám property, $(1, k)_n \cong (1, k')_n$ implies the proportionality $(1, k) \sim (1, k')$, that is there exists $l$ with $\gcd(l, n) = 1$, such that $(1, k) = l \cdot (1, k')$. Since $k \neq k'$, the only solution is $l = l = k' \mod n$ and $k = k' \mod n$, that is $k = l = k'^{-1}$, the inverse of $k'$ modulo $n$.

This can be seen easily by noting, that, if $\gcd(k, n) = 1$, the chords of length $k$ in $(1, k)_n$ form an Hamiltonian cycle as well, and therefore can be drawn along a cycle, while the edges of the previous ring now become chords of length $k^{-1}$, i.e., preserving the chordal ring definition.

Note that some Circulant Graphs $(a, b)_n$ are not isomorphic to a Chordal Ring $(1, k)_n$, if $\gcd(a, b) = 1$ but $\gcd(a, n) \neq 1$ and $\gcd(b, n) \neq 1$. We are not interested in such graphs in this paper.
1.3. Shortest-Path Strict Interval Routing Scheme

A simplistic routing scheme in chordal rings can be designed by using the Hamiltonian cycle. However, in models where the system requires a shortest path, the routing scheme may be difficult to devise depending on the length of the chords (and, as we prove below, sometimes is impossible to built in a compact way).

Interval Routing Scheme is based on representing the routing table at each node in a compact manner, by grouping the set of destination addresses that use the same outgoing link into (cyclic) intervals of consecutive addresses.

Definition 1 Let $G = (V,E)$ be a graph with $|V(G)| = n$. A pair $(L, I)$ is a $k$-Interval Routing Scheme on $G$ (k-IRS for short), iff $L$ consists of a bijection between $V(G)$ and $\{0, 1, \ldots, n - 1\}$, where $L(v)$ is the label of a vertex $v$, and $I$ is the assignment to each edge $e \in E(G)$ of an edge label $I(e)$ which is a set containing $k$ or fewer disjoint subintervals of the cyclic interval $[0, n - 1]$, such that:

1. for each $v$, the intervals associated with the outgoing edges form a partition of $[0, n - 1]$, and,

2. for each distinct vertex $u$ and $v$, there exists a sequence of vertices $(a_0, a_1, \ldots, a_t)$ such that $a_0 = u$, $a_t = v$ and for every $i \in \{1, \ldots, t\}$, $\{e \mid (a_i, a_{i-1}) \in I\}$.

A node $u$ transmits a message to a node $v$ by sending it through the edge labeled $I(e)$ such that $v \in I(e)$. Upon receipt of a message, the receiving node repeats the same process unless this node is $v$.

A $k$-Interval Routing Scheme is said to be strict (and denoted $k$-SIRS) if the intervals at $v$ are not allowed to contain the label of $v$ itself. Obviously, a non-strict $k$-IRS can always be interpreted as a $(k + 1)$-SIRS.

A $k$-Interval Routing Scheme is said to be a shortest-path scheme if all paths specified by $(L, I)$ are shortest paths in the graph. Note that this implies that there is at least one interval per edge since the shortest path forces it to contain the adjacent node of the edge. Note also that, by the partitioning condition (1) of the 1-SIRS definition, a shortest-path 1-SIRS provides a single shortest-path between any two nodes (this observation will be heavily used below). Throughout the rest of the paper, we denote by SIRS($G$)=$k$ the fact that a graph $G$ has a shortest path SIRS using at most $k$ intervals per edge, and $k$ is minimum.

Throughout the rest of the paper we only consider shortest-path 1-SIRS (see [9] for the general case).

In this paper, we mainly study node-labelings which allow a shortest path 1-SIRS in Chordal Rings of degree four.

2. On the cyclic node-labeling for chordal rings

The simplicity and uniformity of the cyclic node-labeling of Chordal Rings in which nodes are cyclically labeled only the ring (i.e., $L(i) = i$) makes it an attractive candidate to devise SIRS. The complexity of preprocessing and work involved in equipping this network with such a scheme is drastically reduced. However, as we outline in this Section, few of the Chordal Rings with cyclic node-labeling have a shortest path 1-SIRS.

On one hand, Krizanc and Luccio [11] have proved the following Theorem 1:

Theorem 1 [11] A chordal ring $(1, k)_n$ has a shortest path 1-SIRS with cyclic labeling $\iff n \mod k = 0$, $n \mod n - k = 0$, $n = sk + 1$, or $n = sk - 1$ and $s$ is odd, or $n = sk + 2$ and $s$ is even.

On the other hand, recently Naranayan and Opatrny [16] proved that for chordal rings with chords of odd length (and larger than 3), one can build one chordal ring of a particular size which has no shortest-path 1-SIRS.

Theorem 2 [16] Chordal rings $(1, 2d + 1)_n$ with $n = 2d^2 + 2d + 1$, $d > 1$, have no shortest-path 1-SIRS, regardless of the node labeling.

Krizanc and Luccio further proved in [11] that:

Corollary 1 [11] There exists an infinite class of chordal rings for which a shortest-path 1-SIRS exists, but is not cyclic.

Proof Sketch. The authors show that the nodes of $(1, 3)_n$ with $n = 29$ can be relabeled

$$L : V \rightarrow \{0, 10, 20, 1, 11, 21, 2, 12, 22, \ldots, 8, 18, 28, 9, 19\}$$

and therefore a shortest-path 1-SIRS can be given by $I(0,1) = [1,5]$, $I(0,10) = [6,15]$, $I(0,19) = [16,23]$, and $I(0,28) = [24,28]$ in node 0, and can be derived accordingly in other nodes by shifting the values.

However, as an immediate consequence of Proposition 1, we can remark that, with $n = 29$, $gcl(3, 29) = 1$ and thus $(1,10)_n \approx (1,3)_n$. Since SIRS$(1,10)_n = 1$ by Theorem 1, the node-labeling given in the proof for $(1,3)_n$ corresponds to the cyclic labeling of $(1,10)_n$.

Theorem 3 If a chordal ring $G(V, E) = (1, k)_n$ is such that $\{1, k\} \sim \{1, k'\}$ and $G'(V', E') = (1, k')_n$ is a chordal ring which admits a shortest path 1-SIRS with cyclic labeling, then the chordal ring $G$ admits a shortest path 1-SIRS which can be constructed locally in each node in $O(\log n)$ time.

\footnote{The Theorem was unfortunately mistyped in the published paper, however the proof was correctly published. Here, we give the correct Theorem.}
**Proof** By construction. By assumption, there exists a \((L', I')\) shortest path \(1\)-SIRS with cyclic labeling for \(G'(V', E') = (1, k')_n\) such that \(I'(0) = 0\) and
\[
\begin{align*}
I'(0, 1) &= [1, [k'/2]] \\
I'(0, n - 1) &= [n - [k'/2], (n - 1)] \\
I'(0, k') &= [[k'/2] + 1, [n/2]] \\
I'(0, n - k') &= [[n/2] - 1, n - [k'/2]].
\end{align*}
\]
The intervals of any node \(i, i \in \{1, \ldots, n - 1\}\) is defined accordingly by adding \(i\) (modulo \(n\)) to each value of the bounds of the intervals of \(I'(0)\).

Each node \(j\) of \(V\) can use the Euclidean algorithm to check locally that \(\gcd(n, k) = 1\) in \(O(\log n)\) time. If this is true, by the Ádám property and Proposition 1, the node deduces that there exists \(k' = k^{-1}\). It can then compute locally the value \(k' = k^{-1}\) in \(O(\log n)\). Each node can re-label itself \(L(j) = k' \cdot j\) and can compute the bounds of its intervals in the same way: for example, at the node 0, \(L(0) = 0\) and
\[
\begin{align*}
I(0, k' \cdot 1) &= [1, [k'/2]] \\
I(0, k' \cdot (n - 1)) &= [n - [k'/2], (n - 1)] \\
I(0, k' \cdot k) &= [[k'/2] + 1, [n/2]] \\
I(0, k' \cdot (n - k)) &= [[n/2] - 1, n - [k'/2]].
\end{align*}
\]
The intervals of other nodes \(j, j \in \{1, \ldots, n - 1\}\) is defined accordingly by shifting.

Obviously, computing \(k^{-1}(= k')\) requires \(O(\log n)\) time overall and the re-labeling can be done in constant time. \(\square\)

Using this Theorem, we can slightly extend the previous results of [11].

**Corollary 2** A chordal ring \((1, k)_n\) with \(n = sk - 1\), \(s\) is even and \(k\) odd, has a shortest path \(1\)-SIRS with a non-cyclic labeling.

**Proof** Since \(n = sk - 1\), we deduce that \(\gcd(n, k) = 1\) and that there exists \(k^{-1} = s\), an inverse of \(k\) modulo \(n\) (i.e., \(k \cdot k^{-1} = 1 \mod n\)). Thus, the chordal ring \((1, s)_n\) has a shortest path \(1\)-SIRS with a cyclic labeling (since \(n = sk - 1\) and \(k\) odd). Since \((1, k)_n \simeq (1, s)_n\), we can re-label \((1, k)_n\) accordingly. \(\square\)

We immediately obtain the following Corollary.

**Corollary 3** For every \(k \leq 3\), \(\text{SIRS}(1, k)_n = 1\).

**Proof** Obvious for \(k = 1\), since this is a ring. For \(k = 2\), only two cases may occur: \(n \mod k = 0\) or \(n = sk + 1\), both covered by Theorem 1. For \(k = 3\), only the case \(n = sk - 1\), \(s\) even, is not covered by Theorem 1. However, since \(k = 3\) is odd, we complete the proof using Corollary 2. \(\square\)

The previous corollary is not insignificant since by Theorem 3, we can actually prove that some chordal rings with chords of “large” length, but proportional to a chord of length less than 4, have a shortest-path \(1\)-SIRS as well (e.g., \((1, 10)_n\) with \(n = 23\), presented previously).

**Corollary 4** \(\text{SIRS}(1, k)_n = 1\) if \(\{1, k\} \sim \{1, k'\}\) with \(k' \leq 3\).

Using the Ádám property, the results can be extended to all circulant graphs of degree four.

**Corollary 5** \(\text{SIRS}(s, s')_n = 1\) if \(\{s, s'\} \sim \{1, k\}\) with \(k \leq 3\).

3. **Chordal Rings with chords of even length**

In this Section, we give a negative answer to an open-question from Gavoille [10] by showing that there is an infinite class of chordal rings with chord of even length such that \((1, k)_n\) does not admit a shortest-path \(1\)-SIRS, regardless of the node-labeling. This also extends the result of [16].

**Figure 2. Chordal Ring \((1, k)_n\) with \(n = ks + k/2, s \geq 3\) odd, \(k\) even.**

**Theorem 4** For every even \(k \geq 4\), \(\text{SIRS}(1, k)_n > 1\) for an infinite set of \(n\).

**Proof** We study the class \((1, k)_n\) with \(n = ks + k/2, s \geq 3\), odd, and \(k \geq 4\), even.

By contradiction, assume that a node-labeling exists such that shortest path \(\text{SIRS}(1, k)_n = 1\).
By the vertex transitivity property of circulant graphs \( \langle 1, k \rangle_n \) and we can assume, w.l.o.g., that the node \( L(0) = 0 \) in this labeling. For any node \( v \), we denote by \( I_{+1}(v), I_{-1}(v), I_{+k}(v), \) and \( I_{-k}(v) \), the intervals assigned to adjacent edges leading to nodes at distance \( -1, +1, +k, \) and \( -k \) (modulo \( n \)) respectively. It is easy to see that any graph of this class is connected as in Figure 2 (where only important nodes are shown: black nodes are at distance \( k / 2 \) from each other, and white nodes \( l \) and \( r \) are at distance \(-1 \) and \(+1\) respectively from node 0). By definition of the shortest path 1-SIRS labeling, and the fact that some nodes are connected through a unique shortest path, we must have from node 0:

\[
\{l, l'\} \in I_{+1}(0), \quad \{r, r'\} \in I_{+1}(0), \\
\{d, d', d''\} \in I_{+k}(0), \quad \{u, u', u''\} \in I_{-k}(0).
\]

From (r'):

\[
\{0, r\} \in I_{+1}(r'), \quad \{u\} \in I_{-1}(r'), \\
\{l', u''\} \in I_{+k}(r'), \quad \{d', d''\} \in I_{-k}(r').
\]

And from (u'):

\[
\{d'\} \in I_{+1}(u''), \quad \{d'\} \in I_{-1}(u''), \\
\{u', u, 0\} \in I_{+k}(u''), \quad \{l', r'\} \in I_{-k}(u'').
\]

By definition of SIRS, \( I_{+1}(v), I_{-1}(v), I_{+k}(v), \) and \( I_{-k}(v) \) are pairwise disjoint intervals and form a partition of \( V - \{v\} \). We deduce that no interval of the node 0 can “wrap-around” (i.e., no interval \([x, y]\) is cyclic with \( x > y \).

In the following, for brevity, we define the expression \( x \prec y \prec z \) to denote the fact that \( x < z \) and \( y < z \), but nothing is known between \( x \) and \( y \). We define \( z \nprec x, y \) accordingly.

Case 1: let us assume that \( 0 \prec l, l' \prec r, r' \).

By (u'), we know that, since \( \{0, r\} \in I_{+1}(r') \), \( I_{+1}(r') \) is the only interval from \( r' \) which can “wrap-around”. Since \( l' < r' \) by assumption, \( r \) has the largest value among labels shown in (r'), i.e., \([r, 0] \subseteq I_{+1}(r')\). Also, since \( u, u', d', d'' \) were represented in the intervals of (0), we obtain that \( I_{-1}(0) \) contains the largest values. We deduce from (r'):

\[
l', u'' < r' < r \leq (n - 1), (1) \\
d', d'' < r' < r \leq (n - 1), (2) \\
u < r' < r \leq (n - 1). (3)
\]

Identically, by (u') we know that, since \( \{u', u, 0\} \in I_{+k}(u''), I_{-k}(u'') \) is the only interval from \( u'' \) which can “wrap-around”. Since \( u', u < r' \leq (n - 1), \) and since \( d', d', l' \) were represented in the intervals of (0), we obtain that \( I_{-k}(0) \) contains the smallest values. From (u'):

\[
0 < u, u' < u'' < l', r', (4) \\
0 < u, u' < u'' < d', (5) \\
0 < u, u' < u'' < d''. (6)
\]

Combining Equations (1) and (4), we obtain:

\[
0 \prec u, u' \prec u'' < l' < r' < r \leq (n - 1) (7)
\]

Combining Equations (2) and (7), we obtain:

\[
0 < u, u' \prec u'' < d' \prec d'' < r' < r \leq (n - 1) (8)
\]

Indeed, because of the partitioning property, the interval \( I_{-k}(r') \) containing \( \{d', d''\} \) cannot intersect with the interval \( I_{+k}(r') \) containing \( \{l', u''\} \). However by combining Equations (4), (5) and (7), we obtain:

\[
0 < u, u' \prec u'' < d' \prec l' < r' < r \leq (n - 1) (9)
\]

Again because of the partition property, the interval \( I_{-1}(u'') \) containing \( \{d'\} \) cannot intersect with the interval \( I_{+1}(r') \) containing \( \{l', r'\} \). Equations (8) and (9) raise an ordering contradiction between \( l' \) and \( d' \).

Case 2: by symmetry, the case \( 0 \prec r, r' \prec l, l' \) can be proved similarly.

4. Chordal Rings with chords of odd length

Using the Ádám property, we can prove the following.

**Theorem 5** For any size of chord \( k \) odd and \( k > 3 \), there exists an infinite class of chordal rings \( \langle 1, k \rangle_n \) for which no isomorphic chordal ring has a shortest path 1-SIRS with cyclic labeling.

**Proof** We consider the class \( \langle 1, k \rangle_n \) with \( n = ks + 2 \), both \( s \) and \( k \) odd, and both \( s, k > 3 \). Since \( n = ks + 2 \) and \( k \) is odd,

\[
gcd(n, k) = gcd(ks + 2, k) = gcd(2, k) = 1. 
\]

Thus, there exists \( k^{-1} \) s.t. \( k \cdot k^{-1} \equiv 1 \mod n \). By Proposition 1, we deduce that \( \langle 1, k^{-1} \rangle_n \) is the only chordal ring isomorphic to \( \langle 1, k \rangle_n \). It implies that there exists \( p \) s.t.

\[
k \cdot k^{-1} = p \cdot n + 1 = psk + 2 + 1 = psk + 2p + 1. 
\]

We can obtain \( k^{-1} \) by choosing \( p = (k - 1) / 2: \)

\[
k^{-1} = ps + (2p + 1) / k = s(k - 1) / 2 + 1 
\]

We finally obtain

\[
n = ks + 2 = 2(k^{-1} - 1) / s + 1, s + 2 = 2k^{-1} + s. 
\]

By Theorem 1, and since \( s > 3 \), we can see that the only isomorphic chordal ring to \( \langle 1, k \rangle_n \) does not have a cyclic node-labeling with a shortest-path 1-SIRS.

Using the result of [16], which proves that shortest path SIRS(\( \langle 1, k \rangle_n > 1 \), \( k \) odd, for an infinite class of chordal rings (in their proof: \( k = 2d + 1 \) and \( n = 2d^2 + 2d + 1 \)), we deduce the following.

**Theorem 6** For every \( k > 3 \), there exists \( n \) such that SIRS(\( \langle 1, k \rangle_n > 1 \).
5. Conclusions and Open Problems

We have extended the characterisation of node-labelings which allow a shortest path 1-SIRS in Chordal Rings of degree 4, and introduced a node-labeling using isomorphic properties on the natural cyclic node-labeling. We proved that for any chordal ring of chord of degree \( k(> 3) \), there exists an infinite set of size \( n \) such that \( (1,k)_n \) does not admit a Shortest-path 1-SIRS. Also, we must emphasised the fact that there is a large class of chordal rings \( (1,k)_n \) with \( k = k^{-1} \) for which the Theorem 3 does not help and, thus, for which another scheme of node-labeling has yet to be found. In particular, this is the case of \( (1, 2d + 1)_n \), with \( n = 2d^2 + 2d + 1 \) given in [16]. This may explain why those graphs have no shortest-path 1-SIRS, regardless of the node labeling. Observing the vertex-transitivity property of circulant graphs, this yields the following conjecture.

Conjecture 1 If SIRS\((1, k)_n\)=1 then either \((1, k)_n\) or \((1, k^{-1})_n\) has a shortest path 1-SIRS with cyclic labeling.

This question is important since, if the answer is positive, we can immediately provide a complete characterisation of chordal rings of degree 4 that has a shortest-path 1-SIRS. Moreover, Conjecture 1 implies that determining whether SIRS\((1, k)_n\)=1 will take only \( O(\log n) \) time for any \( n \) (and if possible the labeling can be computed accordingly in constant time).

Clearly the combinatorial nature of Chordal Ring Networks requires a longer study to obtain a complete characterisation. We believe that our approach will be useful for studying IRS for chordal rings of degree larger than 4 and for extending the preliminary results obtained in [8].

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References